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Abstract

In this paper we have given conditions on exponential polynomials $P_n(s)$ of Dirichlet type to be attained the equality between each of two pairs of bounds, called essential bounds, $a_{P_n(s)}$, ρ_N and $b_{P_n(s)}$, ρ_0 associated with $P_n(s)$. The reciprocal question has been also treated. The bounds $a_{P_n(s)}$, $b_{P_n(s)}$ are defined as the end-points of the minimal closed and bounded real interval $I = [a_{P_n(s)}, b_{P_n(s)}]$ such that all the zeros of $P_n(s)$ are contained in the strip $I \times \mathbb{R}$ of the complex plane $\mathbb{C}$. The bounds ρ_N, ρ_0 are defined as the unique real solutions of Henry equations of $P_n(s)$. Some applications to the partial sums of the Riemann zeta function have been also showed.

Keywords (separated by '-')	Dirichlet polynomials - Zeros of exponential polynomials - Diophantine and rational dependence - Zeros of partial sums of the Riemann zeta function
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ORIGINAL PAPER

Essential bounds of Dirichlet polynomials

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Abstract
In this paper we have given conditions on exponential polynomials $P_n(s)$ of Dirichlet type to be attained the equality between each of two pairs of bounds, called essential bounds, $a_{P_n(s)}$, ρ_N and $b_{P_n(s)}$, ρ_0 associated with $P_n(s)$. The reciprocal question has been also treated. The bounds $a_{P_n(s)}$, $b_{P_n(s)}$ are defined as the end-points of the minimal closed and bounded real interval $I = [a_{P_n(s)}, b_{P_n(s)}]$ such that all the zeros of $P_n(s)$ are contained in the strip $I \times \mathbb{R}$ of the complex plane $\mathbb{C}$. The bounds ρ_N, ρ_0 are defined as the unique real solutions of Henry equations of $P_n(s)$. Some applications to the partial sums of the Riemann zeta function have been also showed.

Keywords Dirichlet polynomials · Zeros of exponential polynomials · Diophantine and rational dependence · Zeros of partial sums of the Riemann zeta function

Mathematics Subject Classification Primary 30B50 · 11M41; Secondary 30D05

1 Introduction

An integer $N \geq 1$, non-null complex numbers α_j and positive real numbers $\lambda_1 < \cdots < \lambda_N$ define an exponential polynomial of the form

$$P(s) = 1 + \sum_{j=1}^{N} \alpha_j e^{-s\lambda_j}, \quad s := \sigma + it \in \mathbb{C}, \tag{1.1}$$

where α_j are called the coefficients and λ_j the exponents, or frequencies, of $P(s)$. An immediate property is satisfied:

$$\lim_{\sigma \to +\infty} P(s) = \lim_{\sigma \to -\infty} Q(s) = 1, \tag{1.2}$$

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where $Q(s) := \alpha_N^{-1} e^{s\lambda_N} P(s)$. On the other hand, any non-constant exponential polynomial has infinitely many zeros as a consequence of Hadamard's Factorization Theorem [2, p. 151] or from Pólya's Theorem (see [12, p. 71]). Therefore, since $P(s)$ and $Q(s)$ have exactly the same zeros, by (1.2) it follows the existence of a constant $A > 0$ such that $P(s) \neq 0$ for all s with $\Re s = \sigma$ such that $|\sigma| > A$. This means that the two half planes $\{s : \sigma < -A\}$, $\{s : \sigma > A\}$ are zero-free for $P(s)$. Consequently for any $P(s)$ of the form (1.1) there exist two real numbers $a_{P_n(s)}, b_{P_n(s)}$ defined as

$$a_{P(s)} := \inf\{\sigma : P(s) = 0\}, \quad b_{P(s)} := \sup\{\sigma : P(s) = 0\}. \tag{1.3}$$

On the other hand, given an exponential polynomial $P(s)$ we have the equations, with ρ as unknown,

$$|\alpha_N| e^{-\rho\lambda_N} = 1 + \sum_{j=1}^{N-1} |\alpha_j| e^{-\rho\lambda_j}, \quad 1 = \sum_{j=1}^{N} |\alpha_j| e^{-\rho\lambda_j}, \tag{1.4}$$

called Henry's equations [7]. By Pólya Criterium [13, p. 46], each equation of (1.4) has a unique real solution denoted by ρ_N and ρ_0, respectively. Therefore to an exponential polynomial $P(s)$ of the form (1.1) we can associated the numbers $a_{P(s)}, b_{P(s)}$, defined in (1.3), and ρ_N, ρ_0, defined in (1.4). These four numbers will be named essential bounds associated with $P(s)$.

An elementary analysis of the real functions

$$f(\rho) := |\alpha_N| e^{-\rho\lambda_N} - \left(1 + \sum_{j=1}^{N-1} |\alpha_j| e^{-\rho\lambda_j}\right), \quad g(\rho) := 1 - \sum_{j=1}^{N} |\alpha_j| e^{-\rho\lambda_j},$$

whose unique real zeros are ρ_N and ρ_0 respectively, proves that there is no zero of $P(s)$ at the left of ρ_N neither at the right of ρ_0. That is, if s is a zero of $P(s)$, necessarily $\rho_N \leq \Re s \leq \rho_0$. Therefore $\rho_N \leq a_{P(s)}$ and $b_{P(s)} \leq \rho_0$. Since it is always true that $a_{P(s)} \leq b_{P(s)}$, the essential bounds of any exponential polynomial $P(s)$ of the form (1.1) are related by the inequalities

$$\rho_N \leq a_{P(s)} \leq b_{P(s)} \leq \rho_0. \tag{1.5}$$

Furthermore, noticing that any non-constant exponential polynomial $P(s)$ has infinitely many zeros, the closure of the real parts of its zeros

$$R_{P(s)} := \overline{\{\sigma : P(s) = 0\}}, \tag{1.6}$$

is a non-empty-set. From (1.5), it follows that

$$R_{P(s)} \subset [a_{P(s)}, b_{P(s)}] \subset [\rho_N, \rho_0]. \tag{1.7}$$

It is evident that if $N = 1$ (the trivial case) the two Henry's equations are equal, so the numbers ρ_N and ρ_0 coincide. Then, from (1.5), the four bounds are equal and the exponential polynomial $P(s)$ has infinitely many zeros aligned. Consequently, in order to avoid the trivial case, from now on, we will assume that $N > 1$ in the expression that defines an exponential polynomial $P(s)$ of the form (1.1). That is, we will consider exponential polynomials with at least three non-null terms.

For a given $P(s)$ of the form (1.1) it would not be too much difficult to obtain computationally the values of ρ_N, ρ_0 by means of Henry's equations (1.4). However we could not say the same for finding an analytical expression for ρ_N, ρ_0, as well as for the numbers $a_{P(s)}$, $b_{P(s)}$, when $N \to \infty$. Furthermore, even in the case to have analytical expressions of those

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numbers, as usually they are asymptotic, the differences between the bounds ρ_N, $a_{P(s)}$ and ρ_0, $b_{P(s)}$, if there are, are really not easy to determine. For instance, for the special type of Dirichlet polynomials $\zeta_n(s) := \sum_{j=1}^{n} j^{-s}$, i.e., the nth partial sums of the series $\sum_{j=1}^{\infty} j^{-s}$, $\sigma > 1$, that defines the Riemann zeta function, we have the estimates

$$b_{\zeta_n(s)} = 1 + \left(\frac{4}{\pi} - 1 + o(1)\right) \frac{\log\log n}{\log n}, \quad n \to \infty,$$

and

$$a_{\zeta_n(s)} = -\frac{\log 2}{\log(\frac{n-1}{n-2})} + \Delta_n, \quad \limsup_{n\to\infty} |\Delta_n| \le \log 2, \quad n > 2,$$

found in 2001 [8] and 2015 [10], respectively. By comparing the previous bounds $a_{\zeta_n(s)}$, $b_{\zeta_n(s)}$ with the solutions of (1.4) for ρ_N, ρ_0, we can see that computationally ρ_N, $a_{\zeta_n(s)}$ and ρ_0, $b_{\zeta_n(s)}$ are indistinguishable when n is large. The difficulty to settle the equality or not between ρ_0 and $b_{\zeta_n(s)}$ is specially hard in this case. Regarding to ρ_N and $a_{\zeta_n(s)}$, to appreciate such difference the value of $|\rho_N - a_{\zeta_n(s)}|$ would have to be greater than $\log 2$, but usually it does not occur, as we can see in [10].

In the same way that is relevant the abscissa of convergence for a Dirichlet series [1, p. 165] (very interesting works on this subject and generalizations can be seen, for instance, in [4,5]), it is also relevant the essential bounds for a Dirichlet polynomial. In a recent article [14] both notions, i.e., the abscissa of convergence of an ordinary Dirichlet series, whose coefficients α_n are defined by $\alpha_n := f(n)$ (f denotes a multiplicative function [1, p. 138]) and the essential bound $b_{P_N(s)}$ corresponding to the Nth partial sum $P_N(s)$ of the given series, have been related. Important results have been obtained in the aforementioned paper such as an analytical expression for $b_{P_N(s)}$ that generalizes the obtained for $b_{\zeta_n(s)}$ in [8] by Montgomery and Vaughan.

In the present paper, for a given exponential $P(s)$ of the form (1.1) we have treated the problem of the equality between the bounds ρ_N, $a_{P(s)}$ and ρ_0, $b_{P(s)}$. Our study has been focused on a class of exponential polynomials that we have called Dirichlet polynomials because they are partial sums of ordinary Dirichlet series [1, p. 161]. That is, we have considered the class of normalized exponential polynomials of the form

$$P_n(s) := 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}, n > 2, \quad \alpha_j \ge 0, \alpha_j \alpha_{n-1} \ne 0 \text{ for some } j < n-1. \quad (1.8)$$

To be more concrete, in the present paper we have found the conditions that must be imposed on a Dirichlet polynomial $P_n(s)$ to have either

$$\rho_N = a_{P_n(s)} \quad (1.9)$$

or

$$\rho_0 = b_{P_n(s)}. \quad (1.10)$$

The converse question is relevant and it also has been studied. Throughout the manuscript we have demonstrated that the aforementioned conditions are linked to the notions of diophantine and rational dependence (see below for a precise definition). We will show that the exponential polynomials of the class of prime Dirichlet polynomials (see below Definition 2.1) satisfy the property (1.9), whereas the exponential polynomials of the class of strict prime Dirichlet polynomials (see below Definition 2.2) satisfy both properties (1.9) and (1.10). To prove the main results of the paper we have used analytical-arithmetical techniques. For instance, it

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has been used a characterization of the set (1.6) (see [3, Theorem 3.1], [11, Theorem 1] that is crucial to prove Theorem 3.1 and Theorem 3.2. This theorem generalizes [9, Theorem 3] (see below Example 3.2). Another important result that we have handled has been Bohr's equivalence theorem [1, Theorem 8.16] used to prove Theorem 4.1. Among the applications to the partial sums of the Riemann zeta function (see the Sect. 5 in the manuscript), we have obtained a characterization of the set of all prime numbers by means of those partial sums $\zeta_n(s)$ that satisfy the property $\rho_N = a_{\zeta_n(s)}$.

2 Preliminaries

Firstly we introduce two remarkable classes of Dirichlet polynomials.

Definition 2.1 A Dirichlet polynomial $P_n(s)$ of the form (1.8) is said to be a prime Dirichlet polynomial if and only if n is a prime number. The class of all prime Dirichlet polynomials will be denoted as $\mathcal{P}$.

For instance, the Dirichlet polynomials $P_5(s) := 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s}$, $P_7(s) := 1 + 2^{-s} + 4^{-s} + 7^{-s}$ are in the class $\mathcal{P}$.

A special subclass of $\mathcal{P}$ is the following.

Definition 2.2 A Dirichlet polynomial $P_n(s)$ of the form (1.8) is said to be a strict prime Dirichlet polynomial if and only if n is a prime number and $\alpha_j \neq 0$ for $j = p-1$, $\alpha_j = 0$ for all $j \neq p-1$, for all primes p with $p \leq n$. The class of all strict prime Dirichlet polynomials will be denoted as $\mathcal{P}_{\text{st}}$.

For instance, the prime Dirichlet polynomials $Q_3(s) := 1 + 2^{-s} + 3^{-s}$, $Q_5(s) := 1 + 2^{-s} + 3^{-s} + 5^{-s}$ are in the class $\mathcal{P}_{\text{st}}$.

In order to use some results already known on exponential polynomials, in the next result we prove that the class of Dirichlet polynomials is a subclass of a more general class of exponential polynomials that can be written in the form

$$P(s) := 1 + \sum_{j=1}^{N} \alpha_j e^{-s\gamma_j \cdot r}, \quad \alpha_j \in \mathbb{R}, \alpha_{n-1} \neq 0, \quad N > 1, \tag{2.1}$$

where $\gamma_j \cdot r$ represents the inner product of $\gamma_j = (\gamma_{j_1}, \gamma_{j_2}, \ldots, \gamma_{j_M})$, non-null vectors of $\mathbb{R}^M$, $M \geq 1$, distinct, with non-negative integers components, by a vector $r = (r_1, r_2, \ldots, r_M)$ of $\mathbb{R}^M$ with positive rationally independent components (i.e., the equation $\sum_{k=1}^{M} \epsilon_k r_k = 0$, with $\epsilon_k \in \mathbb{Q}$, implies that $\epsilon_k = 0$ for all $k = 1, \ldots, M$). Observe that any exponential polynomial $P(s)$ of the form (2.1) is in turn a particular case of exponential polynomial of the form (1.1) by increasingly ordering the exponents $\lambda_j := \gamma_j \cdot r$, $1 \leq j \leq N$.

Lemma 2.1 *Any Dirichlet polynomial of the form* (1.8) *can be expressed in the form* (2.1) *with* $M > 1$.

Proof Let $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$ be a Dirichlet polynomial of the form (1.8). Then we have

$$\frac{1}{(1+j)^s} = e^{-s\log(1+j)} \quad \text{for every } j = 1, \ldots, n-1. \tag{2.2}$$

For a given integer $n > 2$, consider all prime numbers not greater than n, denoted as $p_1 < p_2 < \cdots < p_{k_n} \leq n$. Hence $p_1 = 2$, $p_2 = 3$, etc., and p_{k_n} is the last prime not exceeding n,

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where k_n represents the number of primes p such that $p \leq n$, so $k_n \geq 2$. Take $M = k_n$ and define

$$r := (\log p_1, \log p_2, \ldots, \log p_{k_n}). \tag{2.3}$$

It is immediate to check that the numbers $\log p_1$, $\log p_2 \ldots, \log p_{k_n}$ are rationally independent. By virtue of Fundamental Theorem of Arithmetic write

$$j+1 = p_1^{m_{j_1}} \cdot p_2^{m_{j_2}} \cdots p_{k_n}^{m_{j_{k_n}}} \quad \text{for every } j = 1, \ldots, n-1$$

and then

$$\log(j+1) = m_{j_1} \log p_1 + m_{j_2} \log p_2 + \cdots + m_{j_{k_n}} \log p_{k_n}. \tag{2.4}$$

Now we define

$$\gamma_j = (\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{jM}) := (m_{j_1}, m_{j_2}, \ldots, m_{j_{k_n}}),\ j = 1, \ldots, n-1. \tag{2.5}$$

In particular if $j+1$ is prime, i.e. if $j+1 = p_m$ with $m \in \{1, 2, \ldots, k_n\}$, then the vector γ_j is given by

$$\gamma_j = (0, \ldots, 1, \ldots 0), \quad \text{where 1 is the } m\text{th component of } \gamma_j. \tag{2.6}$$

Then, from (2.5) and taking into account the definition of the vector r, it follows that $\gamma_j \cdot r = \log(1+j)$ for every $j = 1, \ldots, n-1$. Therefore by taking $N = n-1$ (see (2.1), noticing (2.2), we get

$$P_n(s) = 1 + \sum_{j=1}^{n-1} \alpha_j e^{-s\log(1+j)} = 1 + \sum_{j=1}^{N} \alpha_j e^{-s\gamma_j \cdot r}.$$

It means that $P_n(s)$ can be written in the form (2.1), where the vector r is given by (2.3) and the vectors γ_j by (2.5). Then the lemma follows. □

3 The property $\rho_N = a_{P_n(s)}$ in the class of Dirichlet polynomials

In this section we will analyse the property $\rho_N = a_{P_n(s)}$ for the class of Dirichlet polynomials (see (1.8)). For this, firstly we note that if some coefficient α_j of $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$ is equal to 0, by removing the corresponding term, we can write $P_n(s)$ under the form

$$P_n(s) = 1 + \sum_{j=1}^{m} \beta_j e^{-s\log n_j} \quad \text{with } 0 < \beta_j, m \geq 2, \tag{3.1}$$

where $\beta_j = \alpha_j$ for the $\alpha_j \neq 0$, and positive integers $2 \leq n_1 < n_2 < \cdots < n_m = n$.

We introduce the following concepts.

Definition 3.1 Given an integer $n > 2$ and integers $2 \leq n_1 < n_2 < \cdots < n_k$, we will say that $\log n$ is diophantinally dependent on $\log n_j$, $1 \leq j \leq k$, if and only if there are integers δ_j, $1 \leq j \leq k$, such that

$$\log n = \sum_{j=1}^{k} \delta_j \log n_j. \tag{3.2}$$

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Definition 3.2 Given an integer $n > 2$ and integers $2 \leq n_1 < n_2 < \cdots < n_k$, we will say that $\log n$ is rationally dependent on $\log n_j$, $1 \leq j \leq k$, if and only if there are rationals ϵ_j, $1 \leq j \leq k$, such that

$$\log n = \sum_{j=1}^{k} \epsilon_j \log n_j.$$

Observe that if $\log n$ is diophantinally dependent on $\log n_j$, $1 \leq j \leq k$, then in particular $\log n$ is rationally dependent on $\log n_j$, $1 \leq j \leq k$. The converse is not true. Indeed, noticing

$$\log 36 = \frac{2}{3} \log 8 + \log 9, \tag{3.3}$$

we can see that $\log 36$ is rationally dependent on $\log 8$ and $\log 9$, but it is not diophantinally dependent.

It is worth to stress that $\log n$ can be rationally independent of $\log n_j$, $1 \leq j \leq k$, but the set $\{\log n, \log n_j, 1 \leq j \leq k\}$ can be rationally dependent. For instance, $\log 2250$ is rationally independent of $\log 30$ and $\log 900$. Indeed, assume there are rationals ϵ_1, ϵ_2 such that $\log 2250 = \epsilon_1 \log 30 + \epsilon_2 \log 900$. Then, since $30 = 2 \cdot 3 \cdot 5$, $900 = 2^2 \cdot 3^2 \cdot 5^2$ and $2250 = 2 \cdot 3^2 \cdot 5^3$ we have

$$\log 2250 = \log 2 + 2 \log 3 + 3 \log 5 = (\epsilon_1 + 2\epsilon_2)(\log 2 + \log 3 + \log 5). \tag{3.4}$$

But $\log 2$, $\log 3$ and $\log 5$ are rationally independent because 2, 3 and 5 are primes. Then by identifying the coefficients of $\log 2$, $\log 3$ and $\log 5$ in both sides of (3.4), we are led to a contradiction. Therefore $\log 2250$ is rationally independent of $\log 30$ and $\log 900$. However, the numbers $\{\log 2250, \log 30, \log 900\}$ are rationally dependent. Indeed, the equation

$$A \log 2250 + B \log 30 + C \log 900 = 0$$

has non-null solutions, it is satisfied for $A = 0$, $B = 1$ and $C = -1/2$.

The next result requires a characterization of the set defined in (1.6) (see for instance [3, Theorem 3.1], [11, Theorem 1]).

Theorem 3.1 *Given an integer $n > 2$, if a Dirichlet polynomial $P_n(s)$ written in the form* (3.1) *satisfies the property $\rho_N = a_{P_n(s)}$, then $\log n$ is diophantinally independent of $\log n_j$, $1 \leq j < m$.*

Proof Let $P_n(s)$ be a Dirichlet polynomial of the form (3.1) that satisfies the property $\rho_N = a_{P_n(s)}$. Since it is always true that $a_{P_n(s)} \in R_{P_n(s)}$ (see (1.3) and (1.6)), then $\rho_N \in R_{P_n(s)}$. By applying [3, Theorem 3.1], there exists a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_M) \in \mathbb{R}^M$, $M \leq k_n$ (see the proof of Lemma 2.1), such that

$$1 + \sum_{j=1}^{m} \beta_j e^{-\rho_N \log n_j} e^{i\gamma_j \cdot \theta} = 0, \tag{3.5}$$

where γ_j are the vectors defined in (2.5) and $\gamma_j \cdot \theta$ denotes the inner product of γ_j by θ. From (3.5) and noticing $n_m = n$, it follows that

$$|\beta_m e^{-\rho_N \log n} e^{i\gamma_m \cdot \theta}| = \left| 1 + \sum_{j=1}^{m-1} \beta_j e^{-\rho_N \log n_j} e^{i\gamma_j \cdot \theta} \right|. \tag{3.6}$$

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Now we write the right hand side of (3.6) as

$$
\begin{aligned}
&\left|\frac{1}{m-1}+\beta_1 e^{-\rho_N \log n_1} e^{i\gamma_1\cdot\theta}+\cdots+\frac{1}{m-1}+\beta_{m-1} e^{-\rho_N \log n_{m-1}} e^{i\gamma_{m-1}\cdot\theta}\right| \\
&\quad\le \left|\frac{1}{m-1}+\beta_1 e^{-\rho_N \log n_1} e^{i\gamma_1\cdot\theta}\right|+\cdots \\
&\quad\quad+\left|\frac{1}{m-1}+\beta_{m-1} e^{-\rho_N \log n_{m-1}} e^{i\gamma_{m-1}\cdot\theta}\right| \\
&\quad\le \frac{1}{m-1}+\beta_1 e^{-\rho_N \log n_1}+\cdots+\frac{1}{m-1}+\beta_{m-1} e^{-\rho_N \log n_{m-1}}.
\end{aligned}
\tag{3.7}
$$

Then, from (3.6) and (3.7), we deduce that

$$
\beta_m e^{-\rho_N \log n} \le 1+\sum_{j=1}^{m-1} \beta_j e^{-\rho_N \log n_j}. \tag{3.8}
$$

But, taking into account the definition of ρ_N (see Eq. (1.4)), the inequality in (3.8) becomes an equality. Now we recall the property:

Given $z, w \in \mathbb{C}$ with $zw \neq 0$, one has $|z+w| = |z|+|w|$ if and only if there exists $\lambda > 0$ such that $w = \lambda z$.

Then, noticing the above property, in each summand of (3.7), necessarily it must be $e^{i\gamma_j\cdot\theta} > 0$ for each $1 \le j \le m-1$. Since $|e^{i\gamma_j\cdot\theta}| = 1$, it means that

$$
e^{i\gamma_j\cdot\theta} = 1 \text{ for all } 1 \le j \le m-1, \text{ so } \gamma_j\cdot\theta = \pi l_j, l_j \in \mathbb{Z}, l_j \text{ even}. \tag{3.9}
$$

Therefore, substituting in (3.5), it follows $e^{i\gamma_m\cdot\theta} < 0$. Consequently,

$$
e^{i\gamma_m\cdot\theta} = -1, \quad \text{so } \gamma_m\cdot\theta = \pi l_m, l_m \in \mathbb{Z}, l_m \text{ odd}. \tag{3.10}
$$

Let $\{q_1, q_2, \ldots, q_M\}$ be the minimal set of ordered prime numbers that are necessary to obtain the prime factorization of the numbers $\{n_1, n_2, \ldots, n_{m-1}, n\}$. Then, from (2.3), the vector $r := (\log q_1, \log q_2, \ldots, \log q_M)$. Noticing the expression (3.1), for each $1 \le j \le m$, the vector $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jM})$ (see (2.4), (2.5)) is such that

$$
\log n_j = \sum_{k=1}^{M} \gamma_{jk} \log q_k, \quad \text{with } \gamma_{jk} \ge 0 \text{ integers.}
$$

Then the above equality can be written as

$$
\log n_j = \gamma_j \cdot r, \quad \text{for all } 1 \le j \le m. \tag{3.11}
$$

Assume $\log n$ is not diophantinally independent of $\log n_j$, $1 \le j \le m-1$. It means that there are integers $(\delta_j)_{j=1}^{m-1}$ such that $\log n = \sum_{j=1}^{m-1} \delta_j \log n_j$. Then, since $n_m = n$, by (3.11), we can write

$$
\log n = \gamma_m \cdot r = \sum_{j=1}^{m-1} \delta_j (\gamma_j \cdot r).
$$

Therefore,

$$
\left(\gamma_m - \sum_{j=1}^{m-1} \delta_j \gamma_j\right) \cdot r = 0. \tag{3.12}
$$

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

Since $\{q_k : 1 \leq k \leq M\}$ are primes, the set $\{\log q_k : 1 \leq k \leq M\}$ is rationally independent, so, from (3.12), we infer that $\gamma_m = \sum_{j=1}^{m-1} \delta_j \gamma_j$. Then, by multiplying by θ, we have $\gamma_m \cdot \theta = \sum_{j=1}^{m-1} \delta_j (\gamma_j \cdot \theta)$. Now, by dividing by π, we are led to the following contradiction

$$\frac{1}{\pi}(\gamma_m \cdot \theta) = \sum_{j=1}^{m-1} \frac{\delta_j}{\pi}(\gamma_j \cdot \theta). \tag{3.13}$$

Indeed, because (3.9), the right hand side of (3.13) is an even integer whereas, from (3.10), the left hand side is odd. This completes the proof. □

The next example proves that the converse of the previous theorem is not true in general.

Example 3.1 The Dirichlet polynomial

$$P_{36}(s) := 1 + 2 \cdot 8^{-s} + 2 \cdot 9^{-s} + \frac{35}{36} \cdot 36^{-s}$$

does not satisfy the property $\rho_N = a_{P_{36}(s)}$. However $\log 36$ is diophantinally independent of $\log 8$ and $\log 9$.

Indeed, in (3.3) we have seen that $\log 36$ is diophantinally independent of $\log 8$ and $\log 9$. From (1.4), ρ_N is the unique real solution of the equation

$$\frac{35}{36} \cdot 36^{-\rho} = 1 + 2 \cdot 8^{-\rho} + 2 \cdot 9^{-\rho}, \tag{3.14}$$

that clearly is satisfied for $\rho = -1$, so $\rho_N = -1$. Assume $\rho_N = a_{P_{36}(s)}$, so $-1 = a_{P_{36}(s)}$. Since $\{2, 3\}$ is the minimal set of ordered prime numbers that are necessary to obtain the prime factorization of $\{8, 9, 36\}$, the vectors γ_j (see (2.5)) are $\gamma_1 = (3, 0)$, $\gamma_2 = (0, 2)$ and $\gamma_3 = (2, 2)$. Then, since always it is true that $a_{P_{36}(s)} \in R_{P_{36}(s)}$ (see (1.3) and (1.6)), from [3, Theorem 3.1] there exists a vector $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ such that

$$1 + 2 \cdot 8e^{i\gamma_1 \cdot \theta} + 2 \cdot 9e^{i\gamma_2 \cdot \theta} + \frac{35}{36} \cdot 36e^{i\gamma_3 \cdot \theta} = 0, \tag{3.15}$$

where $\gamma_j \cdot \theta$, $j = 1, 2, 3$, denotes the inner product in $\mathbb{R}^2$. From (3.15), we have

$$|35e^{i\gamma_3 \cdot \theta}| = |1 + 16e^{i\gamma_1 \cdot \theta} + 18e^{i\gamma_2 \cdot \theta}|.$$

Therefore

$$35 = |1 + 16e^{i\gamma_1 \cdot \theta} + 18e^{i\gamma_2 \cdot \theta}|,$$

which means that we are in a particular case of (3.6). Then, we get $\gamma_j \cdot \theta = \pi l_j$, $l_j \in \mathbb{Z}$, where l_j is even for $j = 1, 2$ (see (3.9)). Consequently $e^{i\gamma_1 \cdot \theta} = e^{i\gamma_2 \cdot \theta} = 1$. By substituting in (3.15), it follows that

$$e^{i\gamma_3 \cdot \theta} = -1. \tag{3.16}$$

Nevertheless, $\gamma_1 \cdot \theta = 3\theta_1 = \pi l_1$, $\gamma_2 \cdot \theta = 2\theta_2 = \pi l_2$. Then

$$\gamma_3 \cdot \theta = 2\theta_1 + 2\theta_2 = \frac{2}{3}\pi l_1 + \pi l_2, \text{ with } l_j \text{ even for } j = 1, 2,$$

so, $e^{i\gamma_3 \cdot \theta} = e^{i\frac{2}{3}\pi l_1}.e^{i\pi l_2} = e^{i\frac{2}{3}\pi l_1}$. But it is immediate that $e^{i\frac{2}{3}\pi l_1} \neq -1$ for any l_1 even integer. This contradicts (3.16). Consequently, $\rho_N \neq a_{P_{36}(s)}$.

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

To guarantee the validity of the converse of the above result we need a more restrictive condition that of the diophantine independence. Such condition is exactly the rational independence.

Theorem 3.2 *Given an integer $n > 2$. If in a Dirichlet polynomial $P_n(s)$ written in the form* (3.1)*, $\log n$ is rationally independent of $\log n_j$, $1 \leq j < m$, then $P_n(s)$ satisfies the property $\rho_N = a_{P_n(s)}$.*

Proof Let $\{q_1, q_2, \ldots, q_M\}$ be the minimal set of ordered primes numbers that it is necessary to obtain the prime factorization of $\{n_1, n_2, \ldots, n_{m-1}, n_m = n\}$. Then there are integers $\delta_{jk} \geq 0$, $1 \leq j \leq m$, $1 \leq k \leq M$, such that we can write

$$n_j = q_1^{\delta_{j1}} \cdot q_2^{\delta_{j2}} \cdots q_M^{\delta_{jM}},$$

so the vectors γ_j (see (2.5)) are

$$\gamma_j = (\delta_{j1}, \ldots, \delta_{jM}), \quad 1 \leq j \leq m.$$

Consequently

$$\log n_j = \sum_{k=1}^{M} \delta_{jk} \log q_k, \quad 1 \leq j \leq m. \tag{3.17}$$

Hence, since $\log n$ is rationally independent of $\log n_j$, $1 \leq j \leq m-1$, the vector γ_m is rationally independent of the vectors γ_j, $1 \leq j \leq m-1$. First we claim that $M \geq 2$, i.e., we need at least two primes $\{q_{k_1}, q_{k_2}\}$ to obtain the prime factorization of n_j, $1 \leq j \leq m$. Otherwise, assume all $n_j = q_{k_1}^{\delta_{jk_1}}$, $1 \leq j \leq m$ for some $1 \leq k_1 \leq M$. Since $n_j \geq 2$ for all $1 \leq j \leq m$ (see (3.1)), necessarily $\delta_{jk_1} > 0$ for all $1 \leq j \leq m$. Then we can write

$$\log n = \delta_{mk_1} \log q_{k_1} = \delta_{mk_1} \frac{\delta_{1k_1}}{\delta_{1k_1}} \log q_1 = \frac{\delta_{mk_1}}{\delta_{1k_1}} \log n_1 + 0 \log n_2 + \cdots + 0 \log n_{m-1}$$

that contradicts the fact of $\log n$ is rationally independent of $\log n_j$, $1 \leq j \leq m-1$.

With the aim to exhibit the reasoning of the proof, we start by proving the theorem for $m = 2$ (the minimum value for m). We claim that the system $\gamma_1 \cdot \theta = 0$, $\gamma_2 \cdot \theta = \pi$, has a solution, where the unknown is the vector $\theta = (\theta_1, \ldots, \theta_M)$. Indeed, by writing the system of the usual form

$$\left. \begin{array}{r} \delta_{11}\theta_1 + \cdots + \delta_{1M}\theta_M = 0 \\ \delta_{21}\theta_1 + \cdots + \delta_{2M}\theta_M = \pi \end{array} \right\}, \tag{3.18}$$

it is immediate that

$$\operatorname{rank} \begin{pmatrix} \delta_{11} & \ldots & \delta_{1M} \\ \delta_{21} & \ldots & \delta_{2M} \end{pmatrix}_{2\times M} = 2,$$

because the vector $\gamma_2 = (\delta_{21}, \ldots, \delta_{2M})$ is rationally independent of $\gamma_1 = (\delta_{11}, \ldots, \delta_{1M})$. Hence

$$\operatorname{rank} \begin{pmatrix} \delta_{11} & \ldots & \delta_{1M} \\ \delta_{21} & \ldots & \delta_{2M} \end{pmatrix}_{2\times M} = \operatorname{rank} \begin{pmatrix} \delta_{11} & \ldots & \delta_{1M} & 0 \\ \delta_{21} & \ldots & \delta_{2M} & \pi \end{pmatrix},$$

and then, by Rouché–Frobenius Theorem, the system (3.18) has a solution. It means that there exists a vector θ such that $\gamma_1 \cdot \theta = 0$, $\gamma_2 \cdot \theta = \pi$. Then $e^{i\gamma_1 \cdot \theta} = 1$, $e^{i\gamma_2 \cdot \theta} = -1$, i.e., it fulfills (3.9) and (3.10). Consequently, for $m = 2$, (3.5) follows. Then by applying [3,

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

Theorem 3.1], $\rho_N \in R_{P_n(s)}$ (see (1.6)). Now, by using (1.5), we have $\rho_N = a_{P_n(s)}$. Therefore the theorem is true for $m = 2$. Now we study the general case $M \geq 2$, $m > 2$.

Since $\log n$ is rationally independent of $\log n_j$, $1 \leq j < m$, the system $\log n = \sum_{j=1}^{m-1} X_j \log n_j$ has no solution in $\mathbb{Q}$. Hence, by using (3.17), Rouché–Frobenius Theorem says us that the matrices

$$\begin{pmatrix} \delta_{11} & \delta_{21} & \dots & \delta_{m-1,1} \\ \delta_{12} & \delta_{22} & \dots & \delta_{m-1,2} \\ \dots & \dots & \dots & \dots \\ \delta_{1M} & \delta_{2M} & \dots & \delta_{m-1,M} \end{pmatrix}, \begin{pmatrix} \delta_{11} & \delta_{21} & \dots & \delta_{m-1,1} & \delta_{m1} \\ \delta_{12} & \delta_{22} & \dots & \delta_{m-1,2} & \delta_{m2} \\ \dots & \dots & \dots & \dots & \dots \\ \delta_{1M} & \delta_{2M} & \dots & \delta_{m-1,M} & \delta_{m,M} \end{pmatrix}, \tag{3.19}$$

have different rank. Denote $A_{M\times(m-1)}$ and $B_{M\times m}$ the matrices of (3.19), respectively. Then if $R := \text{rank}A_{M\times(m-1)}$, necessarily $\text{rank}B_{M\times m} = R + 1$ because $B_{M\times m}$ has one column more than $A_{M\times(m-1)}$. Now we consider the system $\gamma_1 \cdot \theta = 0, \gamma_2 \cdot \theta = 0, \dots, \gamma_m \cdot \theta = \pi$, written as

$$\left. \begin{array}{l} \delta_{11}\theta_1 + \dots + \delta_{1M}\theta_M = 0 \\ \dots \\ \delta_{m1}\theta_1 + \dots + \delta_{mM}\theta_M = \pi \end{array} \right\}. \tag{3.20}$$

We claim that (3.20) has solution. To show that, it is enough to prove that the matrices

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1M} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2M} \\ \dots & \dots & \dots & \dots \\ \delta_{m1} & \delta_{m2} & \dots & \delta_{mM} \end{pmatrix}, \begin{pmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1M} & 0 \\ \delta_{21} & \delta_{22} & \dots & \delta_{2M} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \delta_{m1} & \delta_{m2} & \dots & \delta_{mM} & \pi \end{pmatrix} \tag{3.21}$$

have equal rank. Indeed, denote as $C_{m\times M}$ and $D_{m\times(M+1)}$ the matrices of (3.21), respectively. Now we observe that $C_{m\times M}$ is the transposed matrix of $B_{M\times m}$, so $\text{rank}C_{m\times M} = \text{rank}B_{M\times m} = R + 1$. Since $D_{m\times(M+1)}$ has one column more than $C_{m\times M}$, the maximum value of $\text{rank}D_{m\times(M+1)}$ is $R + 2$. However it is not possible. Indeed, first consider the case $m = M+1$. In this case, we have $\det D_{m\times(M+1)} = \pi \det A'_{(m-1)\times M} = \pi \det A_{M\times(m-1)}$, where $A'_{(m-1)\times M}$ denotes the transposed matrix of $A_{M\times(m-1)}$. Hence, if $\text{rank}D_{m\times(M+1)} = R + 2$, necessarily $\det D_{m\times(M+1)} \neq 0$, so $\det A_{M\times(m-1)} \neq 0$, and then $R = \text{rank}A_{M\times(m-1)} = M = m - 1$. But, on the other hand,

$$\text{rank}B_{M\times m} = R + 1 \leq \min\{M, m\} \leq M = R,$$

which is absurdum. Consequently $m \neq M + 1$. Assume $m < M + 1$. Since $R := \text{rank}A_{M\times(m-1)} \leq \min\{M, m-1\} = m - 1$, in $A_{M\times(m-1)}$ there are R rationally independent columns. In $B_{M\times m}$ there are $R + 1$ rationally independent columns, being the last one of those. Hence in $C_{m\times M}$ there are $R+1$ rationally independent rows, being the last one of those. Consequently in $D_{m\times(M+1)}$ there are exactly $R + 1$ rationally independent rows because if a new row were rationally independent of the others, then $\text{rank}D_{m\times(M+1)} = R + 2$. However, as the last component of such new row is a 0, it would mean that the $\text{rank}B_{M\times m} = R+2$. But this is a contradiction since $\text{rank}B_{M\times m} = R + 1$. Finally, we analyse the case $m > M + 1$. In this case $R := \text{rank}A_{M\times(m-1)} \leq M$ and then in $A_{M\times(m-1)}$ there are R rationally independent rows. In $B_{M\times m}$ there are $R + 1$ rationally independent rows, being the last one of those. Hence in C we have $R + 1$ rationally independent columns, being the last column one of those. If $\text{rank}D_{m\times(M+1)} = R + 2$, it would mean that the last column contributes to the number of rationally independent columns. Hence, there is a square submatrix of $D_{m\times(M+1)}$, say $E_{(R+2)\times(R+2)}$, that contains the last column, such that $\det E_{(R+2)\times(R+2)} \neq 0$. Since the last column is of the form

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \pi \end{pmatrix},$$

we have $\det E_{(R+2)\times(R+2)} = \pi \det F_{(R+1)\times(R+1)}$, where $F_{(R+1)\times(R+1)}$ is the submatrix of E obtained by deleting the last column and the last row of $E_{(R+2)\times(R+2)}$. Hence $F_{(R+1)\times(R+1)}$ is a submatrix of $A'_{(m-1)\times M}$ with

$$\det F_{(R+1)\times(R+1)} \neq 0.$$

Then, it means that $\operatorname{rank} A'_{(m-1)\times M} \geq R+1$. But this a contradiction because $\operatorname{rank} A'_{(m-1)\times M} = \operatorname{rank} A_{M\times(m-1)} = R$. Consequently (3.20) has solution. That is, there exists a vector θ such that $\gamma_1 \cdot \theta = 0, \gamma_2 \cdot \theta = 0, \ldots, \gamma_m \cdot \theta = \pi$. Then the conditions (3.9) and (3.10) are fulfilled. Consequently (3.5) follows. Hence by applying [3, Theorem 3.1], $\rho_N \in R_{P_n(s)}$ (see (1.6)) and it implies that $\rho_N = a_{P_n(s)}$. Now the proof is completed. □

In [9, Theorem 3] was proved that if the exponents (the $\log n_j$ in (3.1)) of an exponential polynomial are rationally independents, then the property $\rho_N = a_{P_n(s)}$ follows. However, in Theorem 3.2, for assuring that $\rho_N = a_{P_n(s)}$, it is only needed that $\log n$ does not depend rationally on $\{\log n_j : 1 \leq j \leq m-1\}$. This condition is less restrictive than the rational independence of the set $\{\log n_j : 1 \leq j \leq m\}$ as we have seen in a preceding example (see (3.4)). Therefore the previous Theorem 3.2 generalizes [9, Theorem 3] such as we point out by means of the following example:

Example 3.2 Consider the Dirichlet polynomial (see (3.4))

$$P_{2250}(s) := 1 + 30^{-s} + 900^{-s} + \frac{931}{2250} 2250^{-s}.$$

Since $\log 2250$ does not depend rationally on $\{\log 30, \log 900\}$, the Theorem 3.2 implies that $P_{2250}(s)$ satisfies the property $\rho_N = a_{P_{2250}(s)} = -1$ (it can be also checked that (3.5) is fulfilled for $\theta = (\pi, \pi, 0)$ and $\gamma_1 = (1, 1, 1)$, $\gamma_2 = (2, 2, 2)$, $\gamma_3 = (1, 2, 3)$). However, as we saw, the set $\{\log 30, \log 900, \log 2250\}$ is rationally dependent. Consequently, the property $\rho_N = a_{P_{2250}(s)}$ would not be able deduced from [9, Theorem 3].

In the next result we prove that the exponential polynomials of the class of prime Dirichlet polynomials satisfy the property $\rho_N = a_{P_n(s)}$.

Theorem 3.3 *Any prime Dirichlet polynomial $P_n(s)$ satisfies the property $\rho_N = a_{P_n(s)}$.*

Proof We know that any prime Dirichlet Polynomial $P_n(s)$ can be written under the form (3.1) as

$$P_n(s) = 1 + \sum_{j=1}^{m} \beta_j e^{-s \log n_j}, \quad 0 < \beta_j, m \geq 2,$$

with positive integers $2 \leq n_1 < n_2 < \cdots < n_m = n$ with n prime. Let $\{q_1, q_2, \ldots, q_L\}$ be the minimal set of ordered prime numbers that it is necessary to obtain the prime factorization of the numbers of the set $\{n_1, n_2, \ldots, n_{m-1}\}$. Then there are integers $\delta_{jk} \geq 0$, $1 \leq j \leq m-1$, $1 \leq k \leq L$, such that we can write $n_j = q_1^{\delta_{j1}} q_2^{\delta_{j2}} \ldots q_L^{\delta_{jL}}$. Hence

$$\log n_j = \sum_{k=1}^{L} \delta_{jk} \log q_k. \tag{3.22}$$

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

Now, by assuming that $\log n$ is rationally dependent of $\log n_j$, $1 \leq j \leq m-1$, there are rationals ϵ_j such that

$$\log n = \sum_{j=1}^{m-1} \epsilon_j \log n_j.$$

Consequently, from (3.22),

$$\log n = \sum_{j=1}^{m-1} \epsilon_j \sum_{k=1}^{L} \delta_{jk} \log q_k. \tag{3.23}$$

Since n is prime and $n > n_j$ for all $1 \leq j \leq m-1$, it follows that $n \notin \{q_1, q_2, \ldots, q_L\}$. Taking into account that the logarithms of a set of prime numbers define a rationally independent set, the coefficient of each logarithm of the left hand side of (3.23) must be equal to the coefficient of the same logarithm of the right hand side one. But this leads us a contradiction because the coefficient of $\log n$ in the left hand side of (3.23) is 1 whereas the coefficient of $\log n$ in the right hand side is 0. This means that $\log n$ is rationally independent of $\log n_j$, $1 \leq j \leq m-1$. Then by applying the previous theorem, $P_n(s)$ satisfies the property $\rho_N = a_{P_n(s)}$. □

4 A characterization of Dirichlet polynomials in terms of $\rho_0 = b_{P_n(s)}$

In the next result we characterize the class of Dirichlet polynomials that satisfy the property $\rho_0 = b_{P_n(s)}$.

Theorem 4.1 *Given an integer $n > 2$, a Dirichlet polynomial*

$$P_n(s) = 1 + \sum_{j=p-1} \frac{\alpha_j}{(j+1)^s} + \alpha_{n-1} n^{-s}, \quad \alpha_j > 0, \quad \alpha_{n-1} > 0, \tag{4.1}$$

for all primes p with $p \leq n$, satisfies the property $\rho_0 = b_{P_n(s)}$ if and only $m_1 + m_2 + \cdots + m_{k_n}$ is odd, where $p_1^{m_1} \cdot p_2^{m_2} \cdots p_{k_n}^{m_{k_n}}$ is the prime factorization of n and p_{k_n} being the last prime not exceeding n.

Proof We first prove the sufficiency. Consider the exponential polynomial

$$Q_n(s) := 1 - \sum_{j=p-1} \frac{\alpha_j}{(j+1)^s} - \alpha_{n-1} n^{-s}.$$

Define a function $f : \mathbb{N} \to \mathbb{C}$ as

$$f(m) = (f(p_1))^{l_1} (f(p_2))^{l_2} \cdots (f(p_{k_m}))^{l_{k_m}}, m \in \mathbb{N}, m > 1, f(1) = 1, \tag{4.2}$$

where $f(p) = -1$ for any p prime and ${p_1}^{l_1} {p_2}^{l_2} \ldots p_{k_m}^{l_{k_m}}$ being the prime factorization of m. It is immediate that f is a completely multiplicative function [1, p. 138]. Then, because of $m_1 + m_2 + \cdots + m_{k_n}$ is odd, from (4.2), we have

$$f(n) = (-1)^{m_1+m_2+\cdots+m_{k_n}} = -1.$$

This proves that $Q_n(s)$ is an exponential polynomial that is Bohr equivalent to $P_n(s)$ (see for instance [1, Theorem 8.12]). By (1.4), ρ_0 satisfies

$$1 = \alpha_{n-1} e^{-\rho_0 \log n} + \sum_{j=p-1} \alpha_j e^{-\rho_0 \log(j+1)}, \tag{4.3}$$

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

so ρ_0 is a zero of $Q_n(s)$. Then by Bohr's equivalence theorem [1, Theorem 8.16], there exists a zero of $P_n(s)$ in every strip $S_\delta := \{s : \rho_0 - \delta \leq \Re s < \rho_0\}$, for arbitrary $\delta > 0$. It means that $\sup\{\Re s : P_n(s) = 0\} \geq \rho_0$. But $b_{P_n(s)} := \sup\{\Re s : P_n(s) = 0\}$ (see (1.3)), so $b_{P_n(s)} \geq \rho_0$. Then, from (1.5), we infer that $\rho_0 = b_{P_n(s)}$. Consequently the sufficiency follows.

Reciprocally, let $P_n(s)$ be a Dirichlet polynomial of the form (4.1) satisfying $\rho_0 = b_{P_n(s)}$. If n is prime, $n = p_{k_n}$ and then the prime factorization of n coincides with p_{k_n}, so $m_l = 0$ for all $l \neq k_n$ and $m_l = 1$ for $l = k_n$. Hence $m_1 + m_2 + \cdots + m_{k_n} = 1$ and the necessity follows in this case. We assume n is composite. Since $\rho_0 = b_{P_n(s)}$ and $b_{P_n(s)} \in R_{P_n(s)}$, from [3, Theorem 3.1], there exists a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_M)$ with $M = k_n$ such that

$$1 + \alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \sum_{j=p-1} \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta} = 0, \tag{4.4}$$

where γ_j, $j = 1, \ldots, n-1$, are the vectors defined in (2.5) and p runs on the prime numbers less than n. From (4.4) we have

$$1 = \left| \alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \sum_{j=p-1} \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta} \right|. \tag{4.5}$$

Taking into account that in the right hand side of (4.5) there are $k_n + 1$ summands, we put

$$\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} = k_n \frac{1}{k_n} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta},$$

and write (4.5) as

$$1 = \left| \alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \frac{1}{k_n} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} \right.$$
$$\left. + \sum_{\substack{j=p-1 \\ j>1}} \left(\frac{1}{k_n} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta} \right) \right|. \tag{4.6}$$

For instance, for $n = 8$, the number of primes $p < 8$, denoted as k_8, is 4, namely, 2, 3, 5 and 7. Hence the right hand side of (4.5) has 5 summands and then, for certain $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, it can be expressed as

$$1 = \left| \alpha_7 e^{-\rho_0 \log 8} e^{i\gamma_7\cdot\theta} + \frac{1}{4} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \frac{1}{4} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} \right.$$
$$+ \alpha_2 e^{-\rho_0 \log 3} e^{i\gamma_2\cdot\theta} + \frac{1}{4} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_4 e^{-\rho_0 \log 5} e^{i\gamma_4\cdot\theta}$$
$$\left. + \frac{1}{4} \alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_6 e^{-\rho_0 \log 7} e^{i\gamma_6\cdot\theta} \right|.$$

Here, according to the prime factorization of the numbers 2, 3, 5, 7 and 8, respectively, the vectors γ_{p-1}, with p prime less than 8, and γ_{n-1}, for $n = 8$ (see (2.6)) are

$$\gamma_1 = (1, 0, 0, 0),\ \gamma_2 = (0, 1, 0, 0),\ \gamma_4 = (0, 0, 1, 0),\ \gamma_6 = (0, 0, 0, 1),\ \gamma_7 = (3, 0, 0, 0).$$

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

Then, from (4.6), we have

$$1 \leq \left|\alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta}\right| + \sum_{\substack{j=p-1\\ j>1}} \left|\frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta}\right|. \tag{4.7}$$

Now, as in Theorem 3.1, we use the property that for any $z, w \in \mathbb{C}$ with $zw \neq 0$, we have $|z+w| = |z| + |w|$ if and only if there exists $\lambda > 0$ such that $w = \lambda z$. If there is either a summand of the form

$$\left|\frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta}\right|, \quad \text{for } j = p-1, j > 1,$$

or the summand

$$\left|\alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta}\right|$$

such that one has

$$\left|\frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta} + \alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta}\right| < \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} + \alpha_j e^{-\rho_0 \log(j+1)}$$

or

$$\left|\alpha_{n-1} e^{-\rho_0 \log n} e^{i\gamma_{n-1}\cdot\theta} + \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta}\right| < \alpha_{n-1} e^{-\rho_0 \log n} + \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2},$$

then, noticing (4.3), we are led to the following contradiction:

$$1 < \sum_{j=p-1} \alpha_j e^{-\rho_0 \log(j+1)} + \alpha_{n-1} e^{-\rho_0 \log n} = 1.$$

Therefore, for every $j = p-1 > 1$, $j = n-1$, there exists $\lambda_j > 0$ such that

$$\alpha_j e^{-\rho_0 \log(j+1)} e^{i\gamma_j\cdot\theta} = \lambda_j \frac{1}{k_n}\alpha_1 e^{-\rho_0 \log 2} e^{i\gamma_1\cdot\theta}.$$

Since $\frac{1}{k_n}\alpha_j > 0$ for all $j = p-1 > 1$ and $j = n-1$, it means that $e^{i(\gamma_j - \gamma_1)\cdot\theta} > 0$, so $e^{i(\gamma_j - \gamma_1)\cdot\theta} = 1$. Hence

$$\gamma_j \cdot \theta = \gamma_1 \cdot \theta + 2\pi l_j, l_j \in \mathbb{Z}, \quad \text{for all } j = p-1 > 1 \text{ and } j = n-1. \tag{4.8}$$

Then $e^{i\gamma_j\cdot\theta} = e^{i\gamma_1\cdot\theta}$ for $j = p-1$ and $j = n-1$, so by substituting it in (4.4) we have

$$e^{i\gamma_1\cdot\theta}\left(\alpha_{n-1} e^{-\rho_0 \log n} + \sum_{j=p-1} \alpha_j e^{-\rho_0 \log(j+1)}\right) = -1.$$

Now, noticing (4.3), we deduce that $e^{i\gamma_1\cdot\theta} = -1$. Then, $\gamma_1 \cdot \theta = \pi t_1$, with $t_1 \in \mathbb{Z}$ odd. Therefore, from (4.8), we get

$$\gamma_j \cdot \theta = \pi t_j, \quad t_j \in \mathbb{Z} \text{ odd for all } j = p-1, j = n-1. \tag{4.9}$$

From (2.6) the vectors γ_j corresponding to each prime $p < n$, with $j = p-1$, are of the form

$$\gamma_1 = (1, 0, \ldots 0), \gamma_2 = (0, 1, 0, \ldots 0), \ldots, \gamma_{k_n - 1} = (0, 0, \ldots, 0, 1),$$

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

whereas the vector γ_{n-1} corresponding to n, according to its prime factorization, is

$$\gamma_{n-1} = (m_1, m_2, \ldots m_{k_n}).$$

Therefore, it follows that

$$\gamma_j \cdot \theta = \theta_j,\ j = p-1 \quad \text{and} \quad \gamma_{n-1} \cdot \theta = m_1\theta_1 + m_2\theta_2 + \cdots + m_{k_n}\theta_{k_n}. \tag{4.10}$$

Then, from (4.9), (4.10) and (4.8), we infer

$$\begin{aligned} \pi t_{n-1} &= \gamma_{n-1} \cdot \theta \\ &= m_1\theta_1 + m_2\theta_2 + \cdots + m_{k_n}\theta_{k_n} = m_1\theta_1 + m_2(\theta_1 + 2\pi l_2) + \cdots \\ &\quad + m_{k_n}(\theta_1 + 2\pi l_{k_n}) = (m_1 + m_2 + \ldots + m_{k_n})\theta_1 + 2\pi l, \quad \text{for some} \quad l \in \mathbb{Z}. \end{aligned} \tag{4.11}$$

By (4.9) and (4.10), we have $\gamma_1 \cdot \theta = \pi t_1 = \theta_1$ with t_1 odd. Now, by substituting θ_1 in (4.11) and dividing that expression by π, we get

$$t_{n-1} = (m_1 + m_2 + \cdots + m_{k_n})t_1 + 2l, \quad l \in \mathbb{Z}. \tag{4.12}$$

Then, since t_1 and t_{n-1} are odd, the relation (4.12) implies that necessarily $m_1 + m_2 + \cdots + m_{k_n}$ is odd. Consequently the theorem follows. □

In the next result we prove that all the exponential polynomials of the class $\mathcal{P}_{st}$ (see Definition 2.2) satisfy the properties (1.9) and (1.10).

Theorem 4.2 *Any strict prime Dirichlet polynomial satisfies the properties* $\rho_N = a_{P_n(s)}$ *and* $\rho_0 = b_{P_n(s)}$.

Proof Let $P_n(s)$ be a strict prime Dirichlet polynomial. Then in particular $P_n(s)$ is a prime Dirichlet polynomial, so by Theorem 3.3, $P_n(s)$ satisfies the property $\rho_N = a_{P_n(s)}$. On the other hand, $P_n(s)$ is of the form (4.1) and since n is prime,$n = p_{k_n}$ (see the statement of Theorem 4.1). Hence the prime factorization of n is p_{k_n} and then $m_1 = m_2 = \cdots = m_{k_n-1} = 0$ and $m_{k_n} = 1$. Therefore, $m_1 + m_2 + \cdots + m_{k_n} = 1$, so odd. Then, Theorem 4.1 applies and consequently $P_n(s)$ fulfills the property $\rho_0 = b_{P_n(s)}$. □

5 Applications

As an application of the previous results we obtain a characterization of the set of prime numbers by means of the partial sums of the Riemann zeta function that satisfy the property $\rho_N = a_{\zeta_n(s)}$.

Theorem 5.1 *A positive integer n is prime if and only if the partial sum of the Riemann zeta function* $\zeta_n(s) := \sum_{j=1}^{n} j^{-s}$ *satisfies the property* $\rho_N = a_{\zeta_n(s)}$.

Proof If $n = 2$, the zeros of $\zeta_2(s)$ are given by the formula $s_k = \frac{(2k+1)\pi i}{\log 2}$, $k \in \mathbb{Z}$. Then all the zeros of $\zeta_2(s)$ are imaginary, so $a_{\zeta_2(s)} = 0$ (see (1.3)), and ρ_N is the unique real solution of the equation $2^{-\rho} = 1$, so $\rho_N = 0$ (see (1.4)). Therefore $\rho_N = a_{\zeta_2(s)}$ and then the necessity follows for $n = 2$. Assume $n > 2$ is prime. Let $\zeta_n(s)$ be a partial sum with n prime. Then $\zeta_n(s)$ is a prime Dirichlet polynomial (see Definition 2.1). Therefore, by applying Theorem 3.3,

SPI Journal: 13398 Article No.: 1045 TYPESET DISK LE CP Disp.:2021/4/11 Pages: 17 Layout: Small

$\zeta_n(s)$ satisfies the property $\rho_N = a_{\zeta_n(s)}$. Reciprocally, suppose that $\zeta_n(s)$ satisfies the property $\rho_N = a_{\zeta_n(s)}$. We write

$$\zeta_n(s) = 1 + \sum_{j=1}^{n-1} (j+1)^{-s} = 1 + \sum_{j=1}^{n-1} e^{-s\log(j+1)}.$$

Hence $\zeta_n(s)$ is a Dirichlet polynomial of the form (3.1) with $\beta_j = 1$ and $n_j = j+1$, for each $1 \le j \le n-1 = m$, and $n_m = n$. Then, by applying Theorem 3.1, $\log n$ is diophantinally independent of $\{\log n_j : 1 \le j < m\}$. This means that n is prime. Indeed, assume n is composite. Then $n = p_1^{m_1}.p_2^{m_2} \ldots p_{k_n}^{m_{k_n}}$, where $p_1, p_2, \ldots, p_{k_n}$ are the prime numbers less than n and $m_1, m_2, \ldots, m_{k_n}$ are non negative integers with at least one of them, say m_l, with $m_l > 0$. Then

$$\log n = m_1 \log p_1 + m_2 \log p_2 + \cdots + m_{k_n} \log p_{k_n}. \tag{5.1}$$

Since $n_j = j+1$ and $1 \le j \le n-1$, noticing n is composite, one has $p_{k_n} < n$ and then $p_{k_n} \le n-1$. Therefore

$$\{p_1, p_2, \ldots, p_{k_n}\} \subset A := \{d \in \mathbb{N} : 1 < d \le n-1\}.$$

Then the expression (5.1) can be written as

$$\log n = \sum_{j=1}^{k_n} m_j \log p_j + \sum_{d \text{ composite } \in A} 0 \log d. \tag{5.2}$$

But it is clear that (5.2) is a contradiction because it would mean that $\log n$ is diophantinally dependent on $\{\log n_j : 1 \le j < m\}$. Then n is prime and consequently the theorem follows. □

Remark 5.1 We point out that the necessity of the above theorem could be also obtained as a consequence of a result contained in [6, Proposition 5]. Indeed, there it was proved that for n prime one has

$$\sup\{\sigma : G_{n-1}(\sigma) = n^\sigma\} = \sup\{\sigma : G_n(s) = 0\}, \tag{5.3}$$

where $G_n(s) := 1 + 2^s + \cdots + n^s$. By Pólya Criterium [13, p. 46], $G_{n-1}(\sigma) = n^\sigma$ has only one real solution. Then, noticing that the nth partial sum of the Riemann zeta function, $\zeta_n(s) := \sum_{j=1}^n j^{-s}$, is such that $\zeta_n(-s) = G_n(s)$ for all $s \in \mathbb{C}$, from equality (5.3), it follows that $-\rho_N = -a_{\zeta_n(s)}$, so $\rho_N = a_{\zeta_n(s)}$.

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