Topological & Uniform Generalization of Densificable Spaces

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Abstract

In 1890 Giusepe Peano discovered the first Peano curve, a surjective curve from an interval to a square. In 199? Gaspar Mora introduce the generalitation α-dense curves which weak the surjective condition in metric spaces. In this poster I present new generalitation and the relation between them in topological and uniform spaces.

1. Introduction

In a metric space (X, d) we say that a set A is densificable if

$$\forall \epsilon > 0 \exists \gamma \in \mathcal{C}(I, A) : \forall x \in A : \gamma(I) \cap B(x, \epsilon) \neq \emptyset$$

We make the logical generalitation to quasi-uniform spaces $(X, \mathcal{U})$, replacing the balls of radii ϵ by the entourage, i.e. A is densificable if

$$\forall U \in \mathcal{U} \exists \gamma \in \mathcal{C}(I, A) \forall x \in A : \gamma(I) \cap U(x) \neq \emptyset$$

where $U(x) = \{y \in X : (x, y) \in U\}$.
This generalitation are as good as we can expect; a metric space is densificable if and only if the uniform space associated is densificable.

2. Topological space generalitation

In topological spaces we have not a good generalization because densification is not a topological property.

Example 1 *The set $]0, 1[$ is densificable with the euclidean distance but not densificable with the distance $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ and both metric have the same topological space.*

Knowing with lack of fortune, we consider the following definition for topological spaces

Definition 1 *Let X be a topological space we say that:*

1. *X is sequence densificable if*

$$\exists \{\gamma_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(I, X) \forall x \in X, \forall D \in \mathcal{E}(x), \exists n_0 \in \mathbb{N} \forall n \geq n_0, \gamma_n(I) \cap D \neq \emptyset$$

   and we say that the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ densify X.

2. *X is sequence strong densificable if*

$$\exists \{\gamma_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(I, X) \forall x \in X, \exists n_0 \in \mathbb{N} \forall n \geq n_0, x \in \gamma_n(I)$$

   and we say that the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ strong densify X.

3. *X is finite densificable if*

$$\forall n \in \mathbb{N}, \forall \{A_j\}_{j=0}^{n} \subseteq \mathcal{T}, \exists \gamma \in \mathcal{C}(I, X) : \forall j \in \{0, 1, \ldots, n\}, \gamma(I) \cap A_j \neq \emptyset$$

4. *X is weakly densificable if there exists a quasiuniformity $\mathcal{D}$ such that the associated topology is $\mathcal{T}$ and $(X, \mathcal{D})$ is densificable,*
5. *X is strongly densificable if, for all quasiuniformity $\mathcal{D}$ with associated topology $\mathcal{T}$, $(X, \mathcal{D})$ is densificable.*

All these generalitation except the strong sequencely densificable have the following property in X metric space

X is metrically densificable iff X is topologically densificable

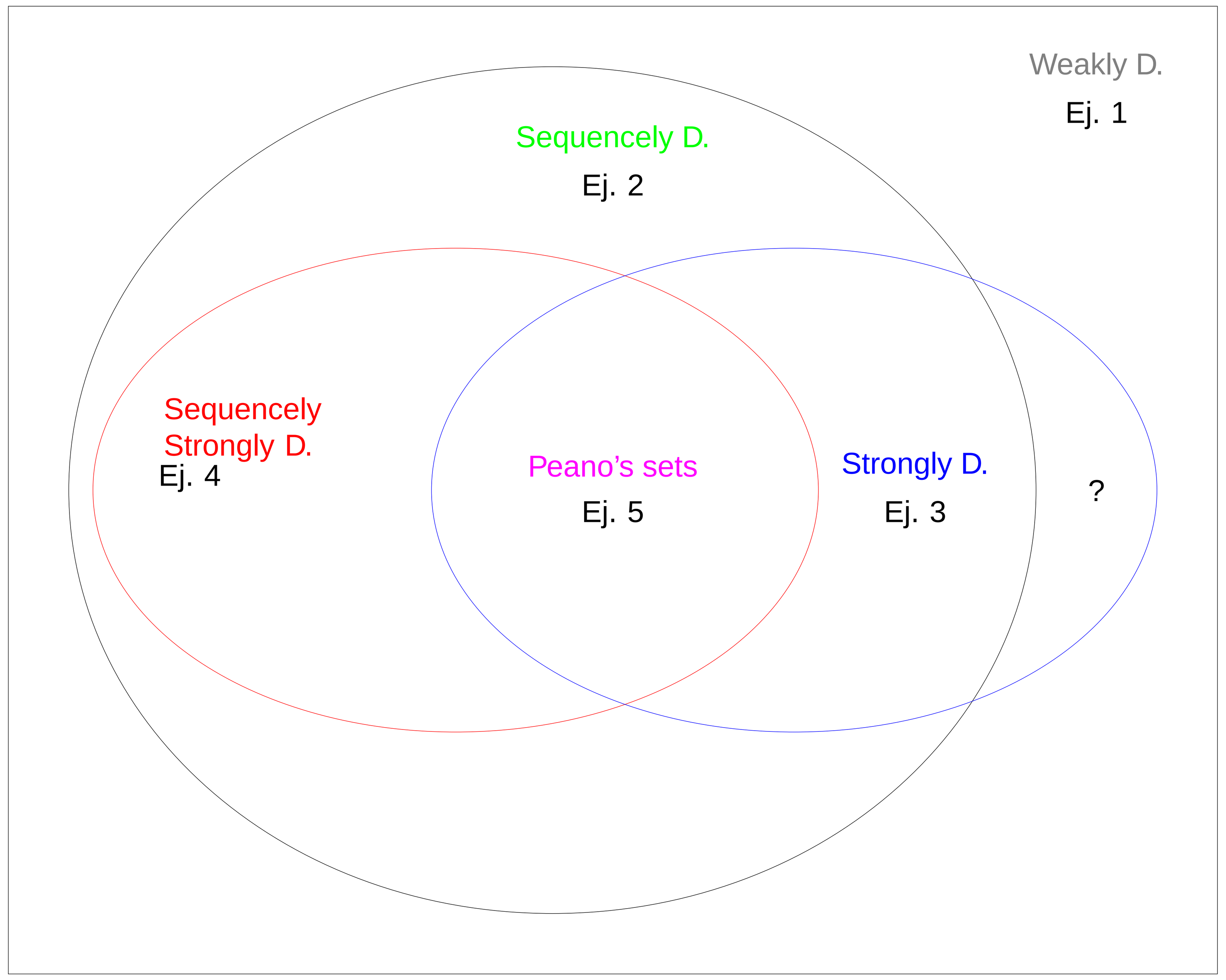

Figure 1: *Relation between densificable sets*

3. Relations

We state the result related with the relation between the definitions

Theorem 1 *Let X be a topological space:*

1. *X is finitely densificable iff it is weakly densificable,*
2. *X is sequencely densificable then X is weakly densificable, moreover if X is 2AN then X is weakly densificable iff X is sequencely densificable,*
3. *X is sequencely strongly densificable then X is sequencely densificable,*
4. *X is strongly densificable then X is weakly densificable,*
5. *X is Peano set then X is sequencely strongly densificable and strongly densificable.*

Theorem 2 *Let X be a topological space, X is sequencely strongly densificable iff X is the image of $[0, 1[$.*

Theorem 3 *Let X be a topological space, X is strongly densificable then X is compact.*

Theorem 4 *X is a Peano set iff X is strongly densificable and sequencely strongly densificable.*

We can see the relation in the figure.

4. Examples

Let see example of topological set for each section of the figure.

1. $(I^I, \| \circ \|_\infty)$.
   This space is weakly densificable but not sequencely densificable.

   Let $(I^I, \mathcal{T})$ be the funtional space of all the function from I to I with the supremum norm ($\|f\| = \sup\{f(t) : t \in I\}$).
   If $\{\gamma_n\}_{n \in \mathbb{Z}^+}$ densify the space then we define the function

$$\begin{aligned} F : I &\rightarrow I \\ s &\mapsto \begin{cases} \mathrm{Dec}(\gamma_n(4(s - \frac{3}{2^{n+1}})(s) + \frac{1}{2}) & \text{if } s \in= [\frac{3}{2^{n+1}}, \frac{1}{2^{n-1}}] \\ 0 & \text{in other case} \end{cases} \end{aligned}$$

   For all $n \in \mathbb{Z}^+$,

$$d(\gamma_n(t), F) \geq d(\gamma_n(t)(\frac{3+t}{4}), F(\frac{3+t}{4})) = \frac{1}{2}$$

   and the intersection of the ball $B(F, \frac{1}{3})$ with any of the images of the curves is empty and therefore $\{\gamma_n\}_{n \in \mathbb{Z}^+}$ not densify and I^I is not sequencely densificable.

2. $(\{(t, \sin\frac{1}{t}) : t \in]]0, 1]\} \cup \{0\} \times [-1, 1], \mathcal{T}_U)$
   This space is sequencely densificable but not either sequencely strongly densificable or strongly densificable.

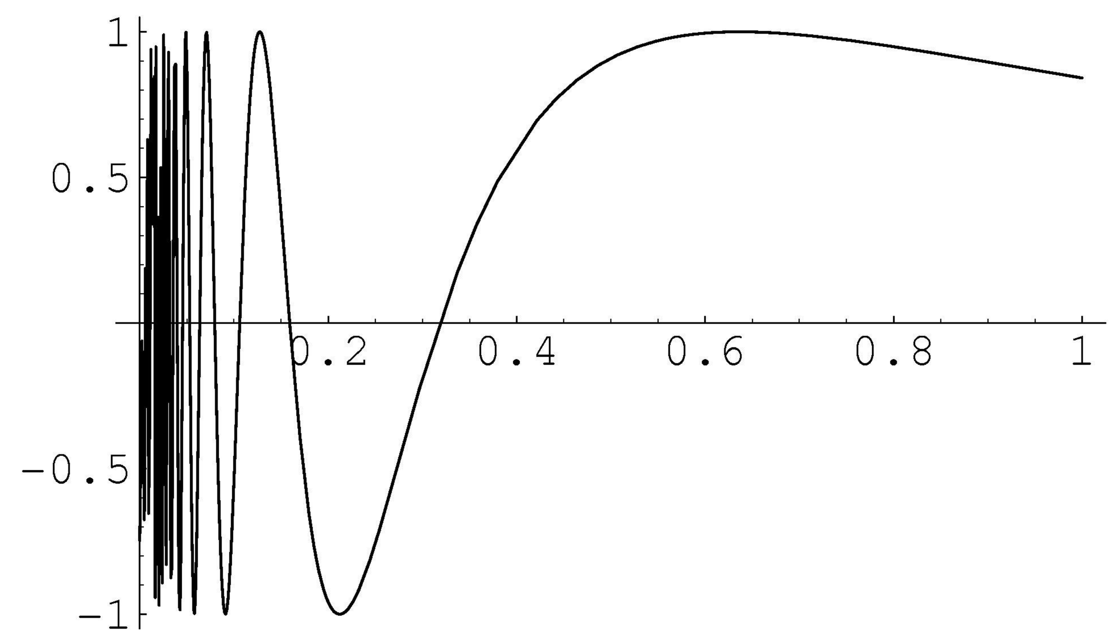

Figure 2: *Sequencely densificable but not neither sequencely strongly densificable or strongly densificable*

3. $(X, \mathcal{T}_t)$ topological trivial space with $Card(X) > \mathfrak{c}$.
   This space is strongly densificable but not sequencely strongly densificable.
4. $([0, 1[, \mathcal{T}_U)$.
   This space is sequencely strongly densificable but not strongly densificable.
5. $([0, 1], \mathcal{T}_U)$
   Peano set.

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