Accepted Manuscript

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PII: S1544-6123(19)30074-1
DOI: https://doi.org/10.1016/j.frl.2019.04.037
Reference: FRL 1181

To appear in: *Finance Research Letters*

Received date: 21 January 2019
Revised date: 8 April 2019
Accepted date: 28 April 2019

Please cite this article as: Beatriz Acereda, Angel Leon, Juan Mora, Estimating the Expected Shortfall of Cryptocurrencies: An Evaluation Based on Backtesting, *Finance Research Letters* (2019), doi: https://doi.org/10.1016/j.frl.2019.04.037

Highlights

- Estimating the expected shortfall of Bitcoin with normal or Student-t is not adequate
- Rolling-window backtest yields the best results for generalized Student-t and NGARCH
- With other cryptocurrencies, heavy-tailed distributions outperform the normal

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Abstract

We estimate the Expected Shortfall (ES) of four major cryptocurrencies using various error distributions and GARCH-type models for conditional variance. Our aim is to examine which distributions perform better and to check what component of the specification plays a more important role in estimating ES. We evaluate the performance of the estimations using a rolling-window backtesting technique. Our results highlight the importance of estimating the ES of Bitcoin using a generalized GARCH model and a non-normal error distribution with at least two parameters. Though the results for other cryptocurrencies are less clear-cut, heavy-tailed distributions continue to outperform the normal distribution.

Keywords: expected shortfall; backtesting; cryptocurrencies.

JEL Codes: C22, C58, G1.

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Abstract

We estimate the Expected Shortfall (ES) of four major cryptocurrencies using various error distributions and GARCH-type models for conditional variance. Our aim is to examine which distributions perform better and to check what component of the specification plays a more important role in estimating ES. We evaluate the performance of the estimations using a rolling-window backtesting technique. Our results highlight the importance of estimating the ES of Bitcoin using a generalized GARCH model and a non-normal error distribution with at least two parameters. Though the results for other cryptocurrencies are less clear-cut, heavy-tailed distributions continue to outperform the normal distribution.

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1. INTRODUCTION

The statistical behavior of cryptocurrency returns has recently been analyzed in several papers. The literature on cryptocurrency returns examines whether these series behave as asset, currency or commodity series (Dyhrberg, 2016; Baur et al., 2018), their informational efficiency (Urquhart, 2016; Sensoy, 2018), the main features of their volatility (Katsiampa, 2017; Baur and Dimpfl, 2018), and the relationships between them and other financial assets (Corbet et al., 2018; Feng et al., 2018), among other issues. In this note, we analyze the estimation of risk measures with cryptocurrency returns; specifically, the estimation of the expected shortfall (ES) of these series. While Value-at-Risk (VaR) had long been considered the standard risk measure in the financial literature, it is not coherent (Acerbi and Tasche 2002). By contrast, ES is a coherent measure of risk, and in addition, deals with extreme losses better than VaR. It is perhaps for these reasons that when the Basel Committee on Banking Supervision (BCBS) published revised standards for the minimum capital requirements for market risk in 2013 and 2016, their key proposal shifted the quantitative risk metrics system from VaR to ES for models-based analyses, and lowered the confidence level from 99% to 97.5% (BCBS, 2013; BCBS, 2016). As a consequence, ES is gradually replacing VaR as the standard risk measure.

Some recent papers have estimated the VaR and ES of cryptocurrency returns. Among others, Gkillas and Katsiampa (2018) perform this estimation assuming a generalized Pareto distribution for the marginal distributions, Stavroyiannis (2018) and Feng et al. (2018) assume an ARMA model and GARCH-type models for the conditional mean and variance and a parametric distribution for the standardized errors, and Ardia et al. (2018) extend the analysis by incorporating regime changes. Although we also assume parametric models for the conditional mean and variance and the error distribution, the aim of our exercise is different. Given that ES estimates require estimating the average of the values below the VaR, the parametric assumption about the distribution of the error may play a key role in the performance of the resulting ES estimates. Our aim is to examine how important this assumption is when analyzing cryptocurrency returns. The error distribution we use in our econometric specification is the asymmetric Student-t (AST) distribution proposed by Zhu and Galbraith (2010), which is an extension of Hansen's skewed-t distribution. Unlike Hansen's distribution, which has two parameters, the AST has three: one which controls the overall skewness and two which control the right and left tails separately, thus allowing for greater flexibility. By comparing the results of the AST distribution to those obtained with the distributions nested in it, we will be able to assess whether the additional complexity of using a three-parameter distribution is worthwhile. In our specifications, we also use various GARCH-type models for the conditional variance to examine how our results are affected by this choice and check which component of the parametric specification plays a more important role.

The accuracy of the various ES estimates is measured using a "backtesting" technique with rolling-window data. The purpose of backtesting techniques is to check whether or not realized losses are in line with forecasts. Many backtesting techniques have been proposed to evaluate the validity of VaR estimates (see, e.g., Ziggel et al., 2014). Although the literature on backtesting techniques for ES is somewhat scarce, it is worth mentioning the Monte Carlo hypothesis test of Acerbi and Szekely (2014) and the test based on cumulative violations of Du and Escanciano (2017). However, here we use the test devised in Kratz et al. (2018), which is particularly easy to implement in a dynamic context. In essence, this is a multinomial test of VaR exceptions at several levels below the pre-specified level for ES, as we describe below.

Our paper is structured as follows. Section 2 presents the econometric methodology. Section 3 describes the data, reports the results and discusses their implications. Section 4 concludes.

2. ECONOMETRIC METHODOLOGY

We assume that returns r_t satisfy

$$r_t = \mu_t + \sigma_t z_t,$$

where $\mu_t = \mathbb{E}[r_t \mid \Omega_{t-1}]$, $\sigma_t = \left(\mathbb{E}[(r_t - \mu_t)^2 \mid \Omega_{t-1}]\right)^{1/2}$, Ω_{t-1} is the set of information available up to $t-1$, and $\{z_t\}$ is a sequence of i.i.d. zero-mean unit-variance random variables. We assume an AR(1) model for the conditional mean $\mu_t = c + \phi r_{t-1}$, and we use various GARCH-type models for the the conditional variance. In addition to the GARCH model, we consider the component GARCH (CGARCH), which produces the best fit for Bitcoin returns according to Katsiampa (2017); the non-linear GARCH (NGARCH), as in Zhu and Galbraith (2011); and the threshold GARCH (TGARCH) of Zakoïan (1994), which specifies the standard deviation instead of the variance. All these GARCH-type models are described in Table 1.

TABLE 1. GARCH-type models

$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t$

	Volatility equation	Second-order stationarity condition
GARCH	$\sigma_t^2 = \omega + \beta_1\varepsilon_{t-1}^2 + \beta_2\sigma_{t-1}^2$	$\beta_1 + \beta_2 < 1$
TGARCH	$\sigma_t = \omega + (\beta_1 + \delta\mathbb{I}\{\varepsilon_{t-1} < 0\})\,\lvert\varepsilon_{t-1}\rvert + \beta_2\sigma_{t-1}$	$\beta_1\mathbb{E}[\lvert z_t\rvert] + \beta_2 + \delta\mathbb{E}[\mathbb{I}\{z_t < 0\}\,\lvert z_t\rvert] < 1^*$
NGARCH	$\sigma_t^2 = \omega + \beta_1(\varepsilon_{t-1} - \delta\sigma_{t-1})^2 + \beta_2\sigma_{t-1}^2$	$\beta_1(1+\delta^2) + \beta_2 < 1$
CGARCH	$\begin{cases} \sigma_t^2 = q_t + \beta_1(\varepsilon_{t-1}^2 - q_{t-1}) + \beta_2(\sigma_{t-1}^2 - q_{t-1}), \\ \text{and } q_t = \omega + \rho(q_{t-1} - \omega) + \theta(\varepsilon_{t-1}^2 - \sigma_{t-1}^2) \end{cases}$	$\beta_1 + \beta_2 < 1$ and $\rho < 1$

*If the distribution of z_t is normal, then $\mathbb{E}[\mathbb{I}\{z_t < 0\}\,\lvert z_t\rvert] = (2\pi)^{-1/2}$ and $\mathbb{E}[\lvert z_t\rvert] = (2\pi)^{1/2}$. If the distribution of z_t is AST, then $\mathbb{E}[\mathbb{I}\{z_t < 0\}\,\lvert z_t\rvert] = 2\gamma\gamma^*(\upsilon_1/\pi)^{1/2}\Gamma((\upsilon_1-1)/2)/\Gamma(\upsilon_1/2)$ and $\mathbb{E}[\lvert z_t\rvert] = 2\gamma\gamma^*(\upsilon_1/\pi)^{1/2}\Gamma((\upsilon_1-1)/2)/\Gamma(\upsilon_1/2) + 2(1-\gamma)(1-\gamma^*)(\upsilon_2/\pi)^{1/2}\Gamma((\upsilon_2-1)/2)/\Gamma(\upsilon_2/2)$.

The distribution of z_t plays a key role in ES estimates. The most general distribution that we consider is the standardized version of the AST distribution. This is a distribution with three parameters $\gamma \in (0,1)$, $\upsilon_1 > 2$, $\upsilon_2 > 2$. Its probability density function (pdf) is given in the Appendix. Parameter γ controls the overall skewness, whereas parameters υ_1 and υ_2 control the heaviness of the left and right tails, respectively. We also consider three alternative distributions nested in the standardized AST distribution: the standardized version of Hansen's skewed-t distribution, which is obtained when $\upsilon_1 = \upsilon_2$; the "relatively symmetric" version of the standardized AST distribution (sym-AST) that is obtained when $\gamma = \frac{1}{2}$ (in this case the ratio of the left deviation to the total deviation is $\frac{1}{2}$; see Zhu and Galbraith, 2010); and the standardized version of the usual Student-t distribution, which is obtained when $\upsilon_1 = \upsilon_2$ and $\gamma = \frac{1}{2}$. As a benchmark, we also consider the standard normal distribution, which is also asymptotically nested in the standardized AST distribution.

The risk measure of the returns r_t that we consider is the conditional ES, which is defined as $\mathrm{ES}_t(\alpha) = \mathbb{E}[r_t \mid r_t \leq \mathrm{VaR}_t(\alpha),\ \Omega_{t-1}]$, where $\alpha \in (0, 1)$ is a pre-specified value, $\mathrm{VaR}_t(\alpha) = F_{r_t}^{-1}(\alpha \mid \Omega_{t-1})$ is the Value-at-Risk, and $F_{r_t}(\cdot \mid \Omega_{t-1})$ is the conditional cumulative distribution function (cdf) of r_t given Ω_{t-1}. The analytical expressions for $\mathrm{VaR}_t(\alpha)$ and $\mathrm{ES}_t(\alpha)$ that are obtained with our assumptions are given in the Appendix. The backtesting procedure that we use is based on Kratz et al. (2018). The authors suggest checking the appropriateness of an $\mathrm{ES}_t(\alpha)$ estimation procedure by comparing the estimates of $\mathrm{VaR}_t(\alpha_j)$ and r_t for a sufficiently large number N of levels $\{\alpha_j\}_{j=1}^N$, where $0 < \alpha_1 < \cdots < \alpha_N \leq \alpha$. If the estimation procedure is adequate, then the proportion of times that r_t falls between the estimates of $\mathrm{VaR}_t(\alpha_{j-1})$ and $\mathrm{VaR}_t(\alpha_j)$ should be approximately equal to $\alpha_j - \alpha_{j-1}$, and this can be the basis for a multinomial test. More specifically, we proceed as follows. Given the sequence of returns $\{r_t\}_{t=1}^T$, we first use $\{r_t\}_{t=1}^{T_0}$ to estimate the model parameters, where T_0 is fixed in advance, with $1 < T_0 < T$. Using these estimates, we obtain $\widehat{\mathrm{VaR}}_{T_0+1}(\alpha_j)$ by replacing the unknown parameters with their estimates in the formula for $\mathrm{VaR}_{T_0+1}(\alpha_j)$ given in the Appendix, where we consider $\alpha_j = \frac{j}{N}\alpha$ for $j = 1, ..., N$, and N is also fixed in advance. This process is then repeated $T - T_0$ times using a rolling window (i.e., the $\mathrm{VaR}_{T_0+i}(\alpha_j)$ estimate is obtained using the returns $\{r_t\}_{t=i}^{T_0+i-1}$) to obtain the sequence of VaR estimates $\{\widehat{\mathrm{VaR}}_{T_0+i}(\alpha_j)\}_{i=1}^{T-T_0}$. Finally, for $j = 1, ..., N$ we compute

$$\widehat{p}_j = \frac{1}{T - T_0} \sum_{i=1}^{T-T_0} \mathbb{I}\{\widehat{\mathrm{VaR}}_{T_0+i}(\alpha_{j-1}) < r_{T_0+i} \leq \widehat{\mathrm{VaR}}_{T_0+i}(\alpha_j)\},$$

where $\widehat{\mathrm{VaR}}_{T_0+i}(\alpha_0) \equiv -\infty$ and $\widehat{p}_0 \equiv 1 - \sum_{j=1}^{N} \widehat{p}_j$, and then the statistic

$$\widehat{S}_N = 2(T - T_0) \sum_{\substack{j=0 \\ \widehat{p}_j \neq 0}}^{N} \widehat{p}_j \ln \frac{\widehat{p}_j}{p_j},$$

where $p_1 \equiv \alpha_1$, $p_j \equiv \alpha_j - \alpha_{j-1}$ for $j = 2, ..., N$ and $p_0 \equiv 1 - \sum_{j=1}^{N} p_j$. If the $\mathrm{ES}_t(\alpha)$ estimation procedure is adequate, then the asymptotic distribution of $\widehat{S}_N$ is χ^2_N.Hence, the p-value of the backtesting procedure is $\Pr\{\widehat{S}_N > \chi^2_N\}$. Additionally, when two alternative specifications for the expected shortfall $\mathrm{ES}_t(\alpha)$ are not rejected, the p-values of this backtesting procedure can be used to compare the appropriateness of the two specifications. The Monte Carlo results reported in Table 6 of Kratz et al. (2018) indicate that this testing procedure performs reasonably well in terms of size and power when $T = 1000$, $T_0 = 500$ and $N = 4$ or 8.

3. DATA AND RESULTS

First, we analyze the Bitcoin data. We use the daily Bitcoin closing prices from 18 July 2010 to 31 July 2018, which were downloaded from coindesk.com. Returns r_t are calculated by taking the natural logarithm of the ratio of two consecutive prices. Tables 2A and 2B report the maximum-likelihood (ML) estimates assuming a standard normal distribution for the errors z_t and the standardized AST distribution. The ML estimations incorporate the second-order stationarity restrictions specified in Table 1 together with the non-negativity restrictions $\omega > 0$, $\beta_1 > 0$, $\beta_2 > 0$ (in all cases), $\beta_1 + \delta > 0$ (in the TGARCH) and $\rho - (\beta_1 + \beta_2) > 0$, $\theta > 0$, $\beta_2 - \theta > 0$ (in the CGARCH). All estimations are performed in Matlab using the command 'fmincon'.

TABLE 2A. Estimation results for Bitcoin returns, normal distribution

	GARCH	**NGARCH**	**TGARCH**	**CGARCH**
Const (mean), c	0.0019	0.0019	0.0034	0.0019
	(0.0007)	(0.0007)	(0.0005)	(0.0007)
AR(1), ϕ	0.0615	0.0614	0.0487	0.0531
	(0.0215)	(0.0220)	(0.0179)	(0.0218)
Const (GARCH), ω	0.000093	0.000093	0.003105	0.429692
	(0.000004)	(0.000004)	(0.000103)	(3.368011)
ARCH, β_1	0.2297	0.2299	0.2722	0.1754
	(0.0083)	(0.0083)	(0.0096)	(0.0070)
GARCH, β_2	0.7702	0.7700	0.7595	0.7606
	(0.0056)	(0.0056)	(0.0055)	(0.0087)
NGARCH/TGARCH, δ		0.0050	-0.0049	
		(0.0231)	(0.0102)	
CGARCH, ρ				0.999985
				(0.000112)
CGARCH, θ				0.0474
				(0.0042)
Log-likelihood	4995.9106	4995.9200	5004.5959	5048.1529
AIC	-3.4010	-3.4003	-3.4062	-3.4352
BIC	-3.3908	-3.3881	-3.3940	-3.4209

The estimation period is from 18 July 2010 to 31 July 2018. Standard errors are reported in parentheses.

TABLE 2B: Estimation results for Bitcoin returns, AST distribution

	GARCH	**NGARCH**	**TGARCH**	**CGARCH**
Const (mean), c	0.0018	0.0018	0.0020	0.0018
	(0.0005)	(0.0005)	(0.0005)	(0.0005)
AR(1), ϕ	−0.0135	−0.0179	−0.0136	−0.0130
	(0.0170)	(0.0167)	(0.0163)	(0.0183)
Const (GARCH), ω	0.000038	0.000035	0.002357	1.174229
	(0.000007)	(0.000006)	(0.000287)	(23.55694)
ARCH, β_1	0.1998	0.1899	0.2234	0.1059
	(0.0229)	(0.0224)	(0.0196)	(0.0308)
GARCH, β_2	0.8001	0.8008	0.7926	0.1885
	(0.0133)	(0.0134)	(0.0148)	(0.2287)
NGARCH/TGARCH, δ		−0.2195	−0.0442	
		(0.0602)	(0.0189)	
CGARCH, ρ				0.999972
				(0.000560)
CGARCH, θ				0.1884
				(0.0146)
AST, γ	0.4817	0.4797	0.4856	0.4823
	(0.0186)	(0.0186)	(0.0184)	(0.0184)
AST, υ_1	2.9184	2.8480	3.1865	2.9360
	(0.2609)	(0.2471)	(0.2842)	(0.1938)
AST, υ_2	3.7613	3.8999	4.2670	3.7957
	(0.3907)	(0.4257)	(0.4508)	(0.3968)
Log-likelihood	5490.5179	5496.2239	5475.6407	5496.9376
AIC	−3.7360	−3.7392	−3.7251	−3.7389
BIC	−3.7196	−3.7208	−3.7068	−3.7186

The estimation period is from 18 July 2010 to 31 July 2018. Standard errors are reported in parentheses.

The parameter estimates reported in Table 2A are similar to those found in the literature. It should also be noted that the two information criteria select the CGARCH model, as in Katsiampa (2017). In comparing the results of Tables 2A and 2B, it can be seen that there are no substantial differences in the estimates of the parameters of conditional mean and variance in most cases. However, there are substantial improvements in the fit of the models, which lead to much higher information criteria values. Both now select the NGARCH model, although there are only minor differences between GARCH, NGARCH, and CGARCH. As can be seen in Table 2B, the null hypothesis $\gamma = \frac{1}{2}$ is not rejected in any case; however, the estimates of υ_1 and υ_2 are fairly different, and when testing the null hypothesis $\upsilon_1 = \upsilon_2$ the asymptotic p-values are around 0.10.

Table 3 shows the asymptotic p-values that are obtained when using the multinomial backtesting procedure. The upper part of the table reports the results for Bitcoin. In this case the sample size is $T = 2935$, and we implement the procedure with $N = 8$ and $T_0 = 1465$; thus, the estimation window is always approximately one half of the total sample. The ES confidence level α is 2.5%, as the BCBS suggests. The results in the upper part of Table 3 show that a normal specification is rejected in all cases. Additionally, the TGARCH specifications also yield poor results, with all their p-values below 0.10. The specifications with a GARCH model and non-normal errors are not rejected at the 5% significance level, but better results are obtained when using either the NGARCH or the CGARCH model for the conditional variance and a distribution with two or three parameters for z_t. The highest p-value is obtained when using the NGARCH model with the sym-AST distribution. Hence, it seems that using a two-parameter distribution might be sufficient to appropriately capture the ES of the series.

TABLE 3. p-values of the multinomial backtest

		normal	t	**skewed** t	**sym-AST**	**AST**
Bitcoin						
	GARCH	0.0015***	0.1873	0.1513	0.0921*	0.0829*
	NGARCH	0.0011***	0.0255**	0.1652	**0.3104**	0.2685
	TGARCH	0.0131**	0.0109**	0.0832*	0.0733*	0.0675*
	CGARCH	0.0000***	0.0501*	0.2117	0.1939	0.1962
Litecoin						
	GARCH	0.0011***	0.8388	0.5596	0.9130	0.5596
	NGARCH	0.0039***	0.7040	0.4847	0.5871	0.4030
	TGARCH	0.0097***	0.6580	0.4695	0.4003	0.3125
	CGARCH	0.0107**	0.6845	0.6415	**0.9230**	0.6415
Ripple						
	GARCH	0.1964	0.2935	**0.8012**	0.3032	0.2421
	NGARCH	0.0015***	0.4901	0.2964	0.2872	0.7122
	TGARCH	0.0663*	0.6138	0.7101	0.2279	0.2898
	CGARCH	0.0385**	0.2819	0.3363	0.7843	0.5944
Ethereum						
	GARCH	0.1780	0.8653	0.0441**	0.5725	0.4120
	NGARCH	0.1256	0.9170	0.6016	0.3626	0.2982
	TGARCH	0.1480	0.5966	0.5177	0.5725	0.4615
	CGARCH	0.0111**	**0.9474**	0.6435	0.2125	0.4573

In all cases, the standardized version of the distribution is used. One, two and three asterisks indicate that the null hypothesis of correct specification is rejected at the 0.10, 0.05 and 0.01 significance levels, respectively. The highest p-value for each cryptocurrency is shown in bold.

We also analyze data for three other well-known cryptocurrencies: Litecoin, Ripple and Ethereum. We use daily closing prices downloaded from coinmarketcap.com from the first available date to 31 July 2018. The multinomial backtesting procedure is implemented as above. The sample sizes of the return series are $T = 1920$, 1822 and 1089 for Litecoin, Ripple and Ethereum, respectively, and $T_0 = [T/2]$ in all cases. The p-values of the multinomial backtest are also reported in Table 3. We observe that the normal specification continues to perform poorly, but the results with non-normal distributions are satisfactory: all but one of the p-values are greater than 0.20 (and most of them are greater than 0.40). Thus, almost all the non-normal specifications appropriately capture the ES of these alternative cryptocurrencies. And, interestingly, the specifications with more parameters do not outperform those with fewer parameters. For example, a GARCH model for the conditional variance with a standardized Student-t distribution for z_t yields satisfactory results. Therefore, the incorporation of more complex specifications does not seem to pay off when working with these three cryptocurrencies.

4. CONCLUSIONS

Measuring investment risks related to cryptocurrency returns is of great importance. This paper employs parametric models for conditional variance and error distribution in order to analyze how important these specifications are when estimating the ES of cryptocurrency returns. The appropriateness of the specifications is examined using the multinomial backtesting procedure devised in Kratz et al. (2018). Our results show that it is crucial to estimate the ES of Bitcoin return series using a non-normal error distribution with at least two parameters and an extension of the GARCH model, such as the NGARCH or the CGARCH. Though the results for other cryptocurrencies are less clear-cut, heavy-tailed distributions continue to produce systematically better results than the normal distribution. This should be taken into account when the ES is used (following the BCBS recommendations) to assess the risk related to cryptocurrency returns.

Our analysis could be extended in at least two ways. First, although alternative models for the conditional variance and the error distribution could be used, the flexibility of the parametric models used here leads us to think that the conclusions of our exercise would remain the same. Second, it would be interesting to extend the econometric specification allowing for jumps (see, e.g., Laurent et al., 2016) and/or regime changes (Ardia et al., 2018), especially when analyzing Bitcoin data, whose return series is relatively long. We intend to investigate more in this direction in future research.

APPENDIX: FORMULAS FOR AST DISTRIBUTION

The pdf of the standardized AST distribution is:

$$f_{\text{S-AST}}(z) = \begin{cases} \frac{\gamma s^*}{\gamma^*} f_{\upsilon_1}\left(\frac{m^*+s^*z}{2\gamma^*}\right) & \text{if } z \leq -m^*/s^*, \\ \frac{(1-\gamma)s^*}{1-\gamma^*} f_{\upsilon_2}\left(\frac{m^*+s^*z}{2(1-\gamma^*)}\right) & \text{if } z > -m^*/s^*; \end{cases}$$

where $f_\upsilon(\cdot)$ denotes the pdf of the usual (non-standardized) Student-t distribution with υ degrees of freedom, and

$$\gamma^* = \left(1 + \frac{(1-\gamma)f_{\upsilon_2}(0)}{\gamma f_{\upsilon_1}(0)}\right)^{-1},$$

$$m^* = 4\left(-\frac{\upsilon_1}{\upsilon_1 - 1}\gamma\gamma^* f_{\upsilon_1}(0) + \frac{\upsilon_2}{\upsilon_2 - 1}(1-\gamma)(1-\gamma^*)f_{\upsilon_2}(0)\right),$$

$$s^* = \left(4\gamma\gamma^{*2}\frac{\upsilon_1}{\upsilon_1 - 2} + 4(1-\gamma)(1-\gamma^*)^2\frac{\upsilon_2}{\upsilon_2 - 2} - m^{*2}\right)^{1/2}.$$

If $r_t = \mu_t + \sigma_t z_t$, and the distribution of z_t is the standardized AST, then $\text{VaR}_t(\alpha)$ and $\text{ES}_t(\alpha)$ can be expressed analytically as follows:

$$\text{VaR}_t(\alpha) = \begin{cases} \mu_t + \frac{\sigma_t}{s^*}\left(2\gamma^* F_{\upsilon_1}^{-1}\left(\frac{\alpha}{2\gamma}\right) - m^*\right) & \text{if } \alpha \leq \gamma, \\ \mu_t + \frac{\sigma_t}{s^*}\left(-2(1-\gamma^*) F_{\upsilon_2}^{-1}\left(\frac{1-\alpha}{2(1-\gamma)}\right) - m^*\right) & \text{if } \alpha > \gamma; \end{cases}$$

where $F_\upsilon^{-1}(\cdot)$ denotes the inverse of the cdf of the usual (non-standardized) Student-t distribution with υ degrees of freedom, and

$$\mathrm{ES}_t(\alpha)=\begin{cases}\mu_t+\frac{\sigma_t}{s^*}\left(-\frac{4\gamma\gamma^*\upsilon_1}{\alpha(\upsilon_1-1)}g_{\upsilon_1}(\frac{\alpha}{2\gamma})-m^*\right) & \text{if } \alpha\leq\gamma,\\ \mu_t+\frac{\sigma_t}{s^*}\left(-\frac{4\gamma\gamma^*\upsilon_1}{\alpha(\upsilon_1-1)}f_{\upsilon_1}(0)-\frac{4(1-\gamma)(1-\gamma^*)\upsilon_2}{\alpha(\upsilon_2-1)}\{g_{\upsilon_2}(\frac{1-\alpha}{2(1-\gamma)})-f_{\upsilon_2}(0)\}-m^*\right) & \text{if } \alpha>\gamma;\end{cases}$$

where

$$g_\upsilon(\cdot)=(1+\frac{F_\upsilon^{-1}(\cdot)^2}{\upsilon})f_\upsilon(F_\upsilon^{-1}(\cdot)).$$

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