The projections of the zeros of exponential polynomials with complex frequencies

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Abstract

In this paper we study the global distribution of zeros of exponential polynomials with complex coefficients and frequencies. For any $P(z)$ in some class of such polynomials, we show that the closure of the projection of $P^{-1}(0)$ on a certain line is a finite union of disjoint segments. We describe this set, in particular we discuss the case where it consists of one segment.

1 Introduction

For each integer $n \geq 2$, let

$$(1.1) \qquad P(z) = m_1 e^{\alpha_1 z} + m_2 e^{\alpha_2 z} + \ldots + m_n e^{\alpha_n z}, \ z \in \mathbb{C},$$

be an exponential polynomial with non-zero complex coefficients $m_1, m_2, \ldots, m_n$ and distinct complex frequencies $\alpha_1, \alpha_2, \ldots, \alpha_n$. It is known that $P(z)$

is an entire function having infinitely many zeros located in certain specific regions of the complex plane. The distribution of the zeros of such a polynomial has been extensively studied during the twentieth century, mostly in the first third of that century, due to its relation with some general problems of the expansion theory for linear differential equations. See, for example, [6], [9], [10], [15], [16] and [17].

If $Q(z) = m_1 e^{\alpha_1 z} + m_2 e^{\alpha_2 z} + \ldots + m_n e^{\alpha_n z}$ is an exponential polynomial of type (1.1) with real frequencies $\alpha_1 < \alpha_2 < \ldots < \alpha_n$, it is not difficult to prove that the numbers

$$a_Q := \inf\left\{\operatorname{Re} z : Q(z) = 0\right\} \quad \text{and} \quad b_Q := \sup\left\{\operatorname{Re} z : Q(z) = 0\right\}$$

are both finite, and so the zeros of $Q(z)$ lie in the vertical strip bounded by the lines $a_Q + i\mathbb{R}$ and $b_Q + i\mathbb{R}$. These bounds allow us to define an interval $I_Q := [a_Q, b_Q]$, called the critical interval, which contains the closure of the set of the real projections of the zeros of $Q(z)$, denoted by

$$(1.2) \qquad\qquad R_Q := \overline{\left\{\operatorname{Re} z : Q(z) = 0\right\}}.$$

To our best knowledge, the first work on the existence of zeros of $Q(z)$ (with real frequencies) arbitrarily close to any line contained in certain substrips of its critical strip was made by Moreno in 1973 [8, Main theorem]. It is also necessary to cite Avellar and Hale [1, Theorem 3.1], who introduced in 1980 a criterion to decide whether a real number is in the set R_Q. The study of density properties of the real projections of the zeros of exponential polynomials with real frequencies has recently become a topic of increasing interest (see for example [2, 3, 4, 5, 7, 11, 12, 13, 14]).

In this paper, we first analyse in Section 2 the global position of the zeros of the general class of exponential polynomials of type (1.1). The rest of the paper is devoted to generalize the procedure used in [7] in order to extract similar results in a more general case (specially in the case where exponential polynomials satisfy conditions presented in Notation 3.4). In short, for this type of polynomials, our paper provides a description of the set R_P defined as the closure of the projection of $P^{-1}(0)$ on a certain line (see Proposition 3.2 and Theorem 3.3) and, in fact, its main results can be summarized by:

i) The set R_P is the union of a finite amount of disjoint nondegenerate line segments (see Theorem 3.3);

ii) If the coefficients of $P(z)$ have the same absolute value, then the set R_P consists of one segment (see Corollary 3.5).

2 The global position of the zeros of exponential polynomials

The distribution of the set of zeros of exponential polynomials with polynomial coefficients and complex frequencies was studied by several authors such as Pólya [9, p. 286] or Langer [6, Theorem 8]. If $P(z) = m_1 e^{\alpha_1 z} + m_2 e^{\alpha_2 z} + \ldots + m_n e^{\alpha_n z}$ is an exponential polynomial, let Z_P denote the set of its zeros, and let C_P denote the convex hull $\text{conv}\{\overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$. Then the zeros of $P(z)$ lie in half-strips in the directions of the exterior normals to C_P. For completeness we will provide the proof of this result. First, we consider some particular cases.

Let

(2.1) $\qquad Q(z) = m_1 e^{\alpha_1 z} + m_2 e^{\alpha_2 z} + \ldots + m_n e^{\alpha_n z}, \ z \in \mathbb{C},$

where $n \geq 2$, be an exponential polynomial with non-zero complex coefficients $m_1, m_2, \ldots, m_n$ and distinct real frequencies $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Thus $Q(z)$ has the same set of zeros as

$$Q_1(z) = \frac{Q(z)}{m_1 e^{\alpha_1 z}} = 1 + \frac{m_2}{m_1} e^{(\alpha_2 - \alpha_1)z} + \ldots + \frac{m_n}{m_1} e^{(\alpha_n - \alpha_1)z},$$

which is an exponential polynomial with increasing positive frequencies $0 < \alpha_2 - \alpha_1 < \ldots < \alpha_n - \alpha_1$. Since

$$\lim_{x \to -\infty} Q_1(x + iy) = 1, \quad \lim_{x \to +\infty} \frac{Q_1(x + iy)}{\frac{m_n}{m_1} e^{(\alpha_n - \alpha_1)(x+iy)}} = 1,$$

for any value of y, there exist $x_1 < 0 < x_2$ such that

$$|Q_1(z) - 1| < 1, \text{ for all } z \text{ with } \operatorname{Re} z \leq x_1,$$

and

$$\left| \frac{Q_1(z)}{\frac{m_n}{m_1} e^{(\alpha_n - \alpha_1)(x+iy)}} - 1 \right| < 1, \text{ for all } z \text{ with } \operatorname{Re} z \geq x_2.$$

Hence $Q_1(z)$ has no zero neither in the half-plane $\operatorname{Re} z \leq x_1$ nor in the half-plane $\operatorname{Re} z \geq x_2$. Consequently all the zeros of $Q(z)$, which are the same as zeros of $Q_1(z)$, are situated in the vertical strip $\{z \in \mathbb{C} : x_1 < \operatorname{Re} z < x_2\}$.

Let

$$R(z) = m_1 e^{\beta_1 z} + m_2 e^{\beta_2 z} + \ldots + m_n e^{\beta_n z}, \ n \geq 2,$$

be an exponential polynomial with non-zero complex coefficients $m_1, m_2, \ldots, m_n$ and with distinct complex frequencies $\beta_1, \beta_2, \ldots, \beta_n$ whose principal arguments are equal. Thus it can be written in the form

$$R(z) = m_1 e^{\alpha_1 e^{i\phi} z} + m_2 e^{\alpha_2 e^{i\phi} z} + \ldots + m_n e^{\alpha_n e^{i\phi} z}$$

for some $\phi \in \mathbb{R}$ and $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Then the exponential polynomial $Q(z) = R(e^{-i\phi}z) = m_1 e^{\alpha_1 z} + \ldots + m_n e^{\alpha_n z}$ is of type (2.1) and therefore there exist $x_1 < 0 < x_2$ such that

$$(2.2) \qquad\qquad Z_Q \subset \{z \in \mathbb{C} : x_1 < \operatorname{Re} z < x_2\}.$$

More so, it is clear that $Z_R = e^{-i\phi} Z_Q$. In fact, if $z_0 \in Z_R$ then

$$z_0 e^{i\phi} = \operatorname{Re} z_0 \cdot \cos\phi - \operatorname{Im} z_0 \cdot \sin\phi + i(\operatorname{Im} z_0 \cdot \cos\phi + \operatorname{Re} z_0 \cdot \sin\phi) \in Z_Q.$$

From (2.2) we have that $x_1 < \operatorname{Re}(z_0 e^{i\phi}) < x_2$. Hence all the zeros of $R(z)$ are in the strip

$$\{z \in \mathbb{C} : x_1 < \operatorname{Re} z \cdot \cos\phi - \operatorname{Im} z \cdot \sin\phi < x_2\} = \{z \in \mathbb{C} : x_1 < \operatorname{Re}(z e^{i\phi}) < x_2\}.$$

More generally, let

$$(2.3) \qquad S(z) = m_1 e^{\gamma_1 z} + m_2 e^{\gamma_2 z} + \ldots + m_n e^{\gamma_n z}, \ n \geq 2,$$

be an exponential polynomial with non-zero complex coefficients $m_1, m_2, \ldots,$ m_n and with distinct aligned (or co-linear) frequencies $\gamma_1, \gamma_2, \ldots, \gamma_n$. Without loss of generality we may assume that there exists $\varphi \in [0, \pi)$ such that the frequencies of $S(z)$ can be written in the form

$$\gamma_j = \gamma_1 + \alpha_j e^{i\varphi}, \ j = 1, 2, \ldots, n,$$

with $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Hence

$$S(z) = m_1 e^{\gamma_1 z}\left(1 + \frac{m_2}{m_1} e^{\alpha_2 e^{i\varphi} z} + \ldots + \frac{m_n}{m_1} e^{\alpha_n e^{i\varphi} z}\right).$$

Notice also that the exponential polynomial $Q(z) = S(z e^{-i\varphi})$ is so that $Z_S = e^{-i\varphi} Z_Q$. Moreover, since the set of zeros of $Q(z)$ coincides with that of the exponential polynomial $1 + \frac{m_2}{m_1} e^{\alpha_2 z} + \ldots + \frac{m_n}{m_1} e^{\alpha_n z}$, which is of type (2.1), there exist $x_1 < 0 < x_2$ such that $Z_Q \subset \{z \in \mathbb{C} : x_1 < \operatorname{Re} z < x_2\}$. Consequently, if $z_0 \in Z_S$ then $z_0 e^{i\varphi} \in Z_Q$ and $x_1 < \operatorname{Re}(z_0 e^{i\varphi}) < x_2$. That means that all the zeros of $S(z)$ are in the strip
(2.4)
$$\{z \in \mathbb{C} : x_1 < \operatorname{Re}(z e^{i\varphi}) < x_2\} = \{z \in \mathbb{C} : x_1 < \operatorname{Re} z \cdot \cos\varphi - \operatorname{Im} z \cdot \sin\varphi < x_2\}.$$

We next deal with the general case where we will divide the plane into a finite number of sectors and, subsequently, we will apply the methods above to each of them.

Before going to this, let us recall the following elementary remark that will be taken into account: if $V = (a, b)$ and $U = (\cos \gamma, \sin \gamma)$ are two vectors in the plane, with $a, b, \gamma \in \mathbb{R}$, so that $0 \leq \theta \leq \frac{\pi}{2}$ is the angle between them, then the length of the orthogonal projection of V on the direction of U is $\|V\| \cos \theta$, and thus the scalar product of U and V coincides with it.

Theorem 2.1. *Let*

$$P(z) = m_1 e^{\alpha_1 z} + \ldots + m_n e^{\alpha_n z}, \ \ n \geq 2,$$

be an exponential polynomial of type (1.1) and let $C_P = \mathrm{conv}\{\overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$. Then all the zeros of $P(z)$ lie in half-strips in the directions of the exterior normals to C_P.

Proof. Let $l_1, l_2, \ldots, l_q$ be the sides of C_P taken in counterclockwise succession with any one as the initial side and, for each $r = 1, 2, \ldots, q$, let $\overline{\alpha_{r,1}}, \overline{\alpha_{r,2}}, \ldots, \overline{\alpha_{r,h_r}}$ be the complex conjugates of the frequencies of $P(z)$ which lie on the side l_r, the succession being again in the counterclockwise sense around the polygon C_P. First notice that it is possible that some complex conjugates of the frequencies, say $\overline{\alpha_{0,1}}, \overline{\alpha_{0,2}}, \ldots, \overline{\alpha_{0,h_0}}$, lie within the polygon C_P. Finally, we denote as $m_{r,i}$ the coefficient of $P(z)$ associated with $\alpha_{r,i}$. Given $r \in \{1, 2, \ldots, q\}$, let γ be a real number such that $w_{r-1} + \varepsilon \leq \gamma < w_r + \varepsilon$ where w_r is the inclination angle of the outer normal to the side l_r and $\varepsilon > 0$ is sufficiently small. (Considering all the possible values of r, note that the set of these sectors completely fills out the complex plane). If $z = |z|e^{i\gamma}$ and $\alpha_s = a_s + ib_s \in \mathbb{C}$, it is worth noting that

$$\mathrm{Re}\left(\alpha_s e^{i\gamma}\right) = \mathrm{Re}\left((a_s + ib_s)(\cos \gamma + i \sin \gamma)\right) = a_s \cos \gamma - b_s \sin \gamma = \mathrm{Re}\left(\overline{\alpha_s} e^{-i\gamma}\right)$$

and

$$(2.5) \qquad \left|e^{\alpha_s z}\right| = \left|e^{(a_s + ib_s)|z|e^{i\gamma}}\right| = e^{|z|(a_s \cos \gamma - b_s \sin \gamma)} = e^{|z| \, \mathrm{Re}\left(\overline{\alpha_s} e^{-i\gamma}\right)}.$$

Moreover, $a_s \cos \gamma - b_s \sin \gamma$ can be expressed as the scalar product of the vectors $\overline{\alpha_s} = (a_s, -b_s)$ and $e^{i\gamma} = (\cos \gamma, \sin \gamma)$. So, by the remark before the theorem, if the term $\mathrm{Re}\left(\overline{\alpha_s} e^{-i\gamma}\right)$ is positive, it is represented geometrically by the absolute value of the projection of $(a_s, -b_s)$ on the direction of $e^{i\gamma}$. Now, from this geometrical interpretation, and since $w_{r-1} + \varepsilon \leq \gamma < w_r + \varepsilon$, we have

$$(2.6) \quad \mathrm{Re}\left(\overline{\alpha_{s,k}} e^{-i\gamma}\right) < \mathrm{Re}\left(\overline{\alpha_{r,1}} e^{-i\gamma}\right), \ \text{with } s \neq r \text{ and } k \in \{1, 2, \ldots, h_s\}.$$

Note that if $\operatorname{Re}\left(\overline{\alpha_{s,k}}e^{-i\gamma}\right) < 0$ for some $s \neq r$ and $k \in \{1, 2, \ldots, h_s\}$, inequality (2.6) is trivially true. Also, if $\gamma = w_r$ it is clear that $\operatorname{Re}\left(\overline{\alpha_{r,1}}e^{-i\gamma}\right) = \operatorname{Re}\left(\overline{\alpha_{r,2}}e^{-i\gamma}\right) = \ldots = \operatorname{Re}\left(\overline{\alpha_{r,h_r}}e^{-i\gamma}\right)$. Now take $z = |z|e^{i\gamma} \in \mathbb{C}$, then

$$P(z) = m_1 e^{\alpha_1 z} + m_2 e^{\alpha_2 z} + \ldots + m_n e^{\alpha_n z} = \sum_{k=1}^{h_r} m_{r,k} e^{\alpha_{r,k} z} + \sum_{s=0, s \neq r}^{q} \sum_{k=1}^{h_s} m_{s,k} e^{\alpha_{s,k} z}$$

and, by dividing by $m_{r,1} e^{\alpha_{r,1} z}$, the set of zeros of $P(z)$ is the same as the set of solutions of the equation

$$(2.7) \qquad 1 + \sum_{k=2}^{h_r} \frac{m_{r,k}}{m_{r,1}} e^{(\alpha_{r,k} - \alpha_{r,1})z} + \sum_{s=0, s \neq r}^{q} \sum_{k=1}^{h_s} \frac{m_{s,k}}{m_{r,1}} e^{(\alpha_{s,k} - \alpha_{r,1})z} = 0.$$

Let $s \in \{0, 1, \ldots, q\}$ with $s \neq r$ and $k \in \{1, 2, \ldots, h_s\}$. From (2.5), we have

$$(2.8) \qquad \left| e^{(\alpha_{s,k} - \alpha_{r,1})z} \right| = e^{|z|\left(\operatorname{Re}(\overline{\alpha_{s,k}}e^{-i\gamma}) - \operatorname{Re}(\overline{\alpha_{r,1}}e^{-i\gamma})\right)}.$$

Therefore, we deduce from (2.6) and (2.8) that

$$\lim_{|z| \to \infty} e^{|z|\left(\operatorname{Re}(\overline{\alpha_{s,k}}e^{-i\gamma}) - \operatorname{Re}(\overline{\alpha_{r,1}}e^{-i\gamma})\right)} = 0.$$

Consequently, noticing (2.7), all but finitely many of the zeros of $P(z)$ are determined by the exponential polynomial $1 + \sum_{k=2}^{h_r} \frac{m_{r,k}}{m_{r,1}} e^{(\alpha_{r,k} - \alpha_{r,1})z}$, whose zeros are the same as those of $S_r(z) = \sum_{k=1}^{h_r} m_{r,k} e^{\alpha_{r,k} z}$. Finally, since the frequencies of $S_r(z)$ are aligned (their conjugates lie on the side l_r), we deduce from (2.4) that the zeros of $S_r(z)$ are in a half-strip that is parallel to the normal to the side l_r, i.e. the one which points in the same direction as w_r. Consequently, by repeating this process for each $r = 1, 2, \ldots, q$, we obtain the result. $\qquad \square$

Example 2.2. Consider the exponential polynomial

$$\begin{aligned} P(z) = (2+i)e^{2z} + (-1+i)e^{(1.38 - 1.9i)z} + 3e^{(3 - 3.08i)z} + \\ + 3ie^{(4.62 - 1.9i)z} + (1+2i)e^{4z} + e^{3z} + ie^{(3-i)z}, \end{aligned}$$

whose frequencies are $\alpha_1 = 2$, $\alpha_2 = 1.38 - 1.9i$, $\alpha_3 = 3 - 3.08i$, $\alpha_4 = 4.62 - 1.9i$, $\alpha_5 = 4$, $\alpha_6 = 3$, $\alpha_7 = 3 - i$. Figure 1 presents the convex hull C_P of the set $\{\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}, \overline{\alpha_6}, \overline{\alpha_7}\}$, as well as the distribution of those zeros of $P(z)$ whose real and imaginary part is between -30 and 30. As one can observe, it follows the description given in Theorem 2.1.
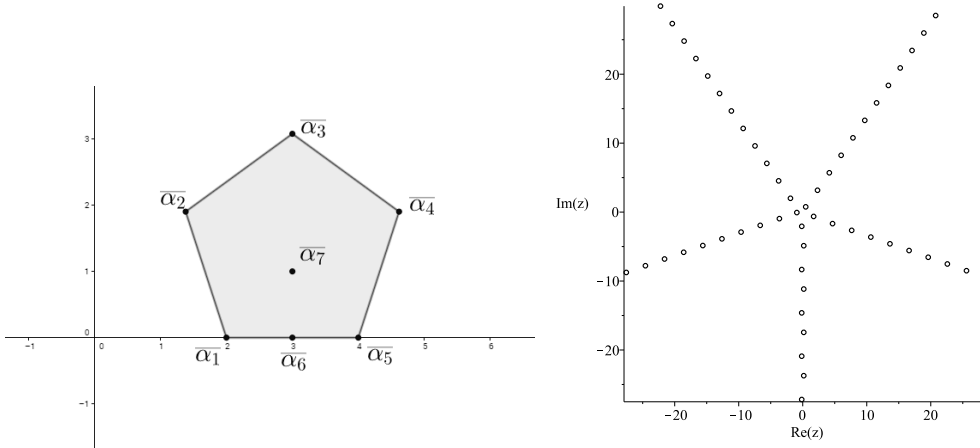
Figure 1: The convex hull C_P and the zeros of $P(z)$, respectively, in Example 2.2

3 Projections of the zeros of exponential polynomials

By Theorem 2.1, given an exponential polynomial $P(z)$ of type (1.1), the study of the properties of its zeros must be carried out in each one of the half-strips where the zeros of $P(z)$ are located. So, let

$$S(z) = m_1 e^{\gamma_1 z} + m_2 e^{\gamma_2 z} + \ldots + m_n e^{\gamma_n z}, \ n \geq 2,$$

be an exponential polynomial of type (2.3) with distinct co-linear frequencies $\gamma_1, \gamma_2, \ldots, \gamma_n$. Hence, without loss of generality, there exists $\varphi \in [0, \pi)$ such that $S(z)$ can be written as

$$(3.1) \qquad m_1 e^{(\gamma_1 + \alpha_1 e^{i\varphi})z} + m_2 e^{(\gamma_1 + \alpha_2 e^{i\varphi})z} + \ldots + m_n e^{(\gamma_1 + \alpha_n e^{i\varphi})z}, \ n \geq 2,$$

for some $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n$.

In this section we are interested in the set

$$(3.2) \qquad\qquad R_S = \overline{\{e^{-i\varphi}\mathrm{Re}(ze^{i\varphi}) : \ S(z) = 0\}},$$

which is the closure of the set of the projections of the zeros of $S(z)$ on the straight line $L_{-\varphi}$ which goes through the origin and has $e^{-i\varphi}$ as director vector. Geometrically, given a straight line $L_{-\varphi+\frac{\pi}{2}}^{x_0}$ which has $e^{-i(\varphi-\frac{\pi}{2})}$ as director vector and that goes through the point $x_0 e^{-i\varphi}$, with $x_0 \in \mathbb{R}$, the point $x_0 e^{-i\varphi}$ belongs to the set R_S when $S(z)$ possesses zeros arbitrarily close to the straight line $L_{-\varphi+\frac{\pi}{2}}^{x_0}$. That means that for fixed $\varepsilon > 0$ there exists a complex number z^* such that $S(z^* e^{-i\varphi}) = 0$ and $|\sigma^* - x_0| < \varepsilon$, where $\sigma^* e^{-i\varphi}$ is the orthogonal projection of $z^* e^{-i\varphi}$ on the straight line $L_{-\varphi}$.

It is worth to notice that an interval of the form $(x_1 e^{-i\varphi}, x_2 e^{-i\varphi}) = \{z \in \mathbb{C} : z = xe^{-i\varphi},\ x_1 < x < x_2\}$, with $x_1 < x_2$, is contained in R_S when every point $xe^{-i\varphi}$ with $x_1 < x < x_2$ is an accumulation point of the set defined by the orthogonal projections of the zeros of $S(z)$ on the straight line $L_{-\varphi}$. That is, if R_S is a perfect set, it is immediate that a point $x_0 e^{-i\varphi}$ is in R_S when $x_0 e^{-i\varphi}$ is an accumulation point of the set defined by the orthogonal projections of the zeros of $S(z)$ on the straight line $L_{-\varphi}$.

Notice that if $\varphi = 0$ then the zeros of $S(z)$ can be seen as those of an exponential polynomial with real frequencies, and the set R_S corresponds with the closure of the set of the real projections of the zeros of $S(z)$. Hence the expression given in (3.2) generalizes that of (1.2) for the case of exponential polynomials with co-linear frequencies.

Remark 3.1. Let $S(z) = m_1 e^{(\gamma_1 + \alpha_1 e^{i\varphi})z} + m_2 e^{(\gamma_1 + \alpha_2 e^{i\varphi})z} + \ldots + m_n e^{(\gamma_1 + \alpha_n e^{i\varphi})z}$ be an exponential polynomial of type (3.1). It is easy to check that

$$R_S = \overline{\{e^{-i\varphi}\operatorname{Re} z :\ S_1(z) = 0\}},$$

where $S_1(z) = 1 + \dfrac{m_2}{m_1} e^{\alpha_2 z} + \ldots + \dfrac{m_n}{m_1} e^{\alpha_n z}$. In fact, $S(z_0) = 0$ if and only if $S_1(z_0 e^{i\varphi}) = 0$. In particular, R_S has the same topological properties as the set $R_{S_1} = \overline{\{\operatorname{Re} z :\ S_1(z) = 0\}}$.

In order to investigate properties of R_S, we shall consider exponential polynomials

$$S(z) = m_1 e^{\gamma_1 z} + m_2 e^{(\gamma_1 + \alpha_2 e^{i\varphi})z} + \ldots + m_n e^{(\gamma_1 + \alpha_n e^{i\varphi})z},\ n \ge 2,$$

of type (3.1) with $0 < \alpha_2 < \ldots < \alpha_n$ linearly independent over the field of the rational numbers. If $S_1(z)$ is the exponential polynomial which was defined in Remark 3.1, consider the points

$$a_{S_1} = \inf\{\operatorname{Re} z :\ S_1(z) = 0\} \quad \text{and} \quad b_{S_1} = \sup\{\operatorname{Re} z :\ S_1(z) = 0\},$$

which are the extreme points of the set R_{S_1} and, by [7, Theorem 3], the unique real solutions of the real equations

$$1 = \sum_{j=2}^{n} \frac{|m_j|}{|m_1|} e^{\alpha_j x}, \quad \frac{|m_n|}{|m_1|} e^{\alpha_n x} = 1 + \sum_{j=2}^{n-1} \frac{|m_j|}{|m_1|} e^{\alpha_j x},$$

respectively. Thus, from Remark 3.1, the extreme points of the set R_S are $a_S = a_{S_1} e^{-i\varphi}$ and $b_S = b_{S_1} e^{-i\varphi}$.

The following proposition provides a pointwise characterization of R_S when the co-linear frequencies of $S(z)$ satisfy the condition above.

Proposition 3.2. *Let* $S(z) = m_1 e^{\gamma_1 z} + m_2 e^{(\gamma_1 + \alpha_2 e^{i\varphi})z} + \ldots + m_n e^{(\gamma_1 + \alpha_n e^{i\varphi})z}$,
$n > 2$, *be an exponential polynomial of type (3.1) with* $0 < \alpha_2 < \ldots < \alpha_n$
linearly independent over the rationals. Then a complex number of the form
$\sigma_0 e^{-i\varphi}$, *where* $\sigma_0 \in \mathbb{R}$, *is in* R_S *if and only if* σ_0 *satisfies*

$$(3.3)$$
$$1 \le \sum_{j=2}^{n} \left| \frac{m_j}{m_1} \right| e^{\alpha_j \sigma_0}; \qquad \left| \frac{m_k}{m_1} \right| e^{\alpha_k \sigma_0} \le 1 + \sum_{j=2, j \ne k}^{n} \left| \frac{m_j}{m_1} \right| e^{\alpha_j \sigma_0}, \ k = 2 \ldots, n.$$

Proof. The polynomial $S_1(z) = 1 + \dfrac{m_2}{m_1} e^{\alpha_2 z} + \ldots + \dfrac{m_n}{m_1} e^{\alpha_n z}$ has positive
real frequencies $\alpha_2 < \ldots < \alpha_n$ which are linearly independent over the
rationals. Thus, from [14, Theorem 6], we have that $\sigma_0 \in R_{S_1}$ if and only if
(3.3) is satisfied. Now, by Remark 3.1, we have that $\sigma_0 \in R_{S_1}$ if and only if
$\sigma_0 e^{-i\varphi} \in R_S$, and the result follows. $\square$

As a consequence of the result above, given two real numbers $\sigma_0 < \sigma_1$,
it is now immediate that the line segment $\{z \in \mathbb{C} : \ \sigma_0 < z e^{i\varphi} < \sigma_1\}$ is
contained in R_S if and only if

$$(3.4) \quad 1 \le \sum_{j=2}^{n} \left| \frac{m_j}{m_1} \right| e^{\alpha_j x}; \qquad \left| \frac{m_k}{m_1} \right| e^{\alpha_k x} \le 1 + \sum_{j=2, j \ne k}^{n} \left| \frac{m_j}{m_1} \right| e^{\alpha_j x}, \ k = 2 \ldots, n,$$

for every $x \in (\sigma_0, \sigma_1)$. This characterization is analogous to that of [7,
Theorem 1].

Hence the study of inequalities (3.4) is essential to identify the boundary
points of the set R_S. In this respect, under the same conditions as the propo-
sition above, we easily deduce from [7, Lemma 2] that if z_0 is a boundary
point of R_S, then $z_0 e^{i\varphi} \in \mathbb{R}$ satisfies all the inequalities (3.4) and only one
of them is an equality. Likewise, by [7, Theorem 3], we can assure that the
extreme points of the set R_S, denoted above as a_S and b_S, are not isolated
points in the set R_S.

Concerning this subject, let $[x_1 e^{-i\varphi}, x_2 e^{-i\varphi}]$ and $[x_3 e^{-i\varphi}, x_4 e^{-i\varphi}]$, with
$x_1 < x_2 < x_3 < x_4$, be two disjoint line segments contained in R_S. From
now on, we will say that the interval $(x_2 e^{-i\varphi}, x_3 e^{-i\varphi})$ is a *gap* of R_S when
no point of the form $x e^{-i\varphi}$ with $x_2 < x < x_3$ belongs to the set R_S. In
this sense, the fact that R_S does not consist of one segment depends on the
exact number of real solutions of any of the $n - 2$ equations

$$(3.5) \qquad \left| \frac{m_k}{m_1} \right| e^{\alpha_k x} = 1 + \sum_{j=2, j \ne k}^{n} \left| \frac{m_j}{m_1} \right| e^{\alpha_j x}, \ k = 2 \ldots, n - 1.$$

In this respect, by virtue of [7, Lemma 8 and Theorem 9], notice that each
equation (3.5) has at most two real solutions which are in $R_{S_1} \subset [a_{S_1}, b_{S_1}]$

and they are candidates of the boundary points of R_{S_1}. In fact, if we denote as $A_k \subset \mathbb{R}$, $2 \le k \le n-1$, the set of points satisfying the k-th non-strict inequality from (3.5), then there always exist $c_k \le d_k$ such that $A_k = (-\infty, c_k] \cup [d_k, \infty)$, and the boundary points of R_{S_1} appear when $c_k < d_k$ in view of the fact that

$$R_{S_1} = [a_{S_1}, b_{S_1}] \cap \bigcap_{k=2}^{n-1} A_k = [a_{S_1}, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_{l-1}, b_{l-1}] \cup [a_l, b_{S_1}],$$

where $a_{S_1} < b_1 < a_2 < \ldots < b_{l-1} < a_l < b_{S_1}$ and $\{b_1, a_2, b_2, \ldots, a_{l-1}, b_{l-1}, a_l\} \subset \{c_2, d_2, \ldots, c_{n-1}, d_{n-1}\}$ (with $c_k < d_k$). Indeed the following theorem, which is now deduced from [7, theorems 9 and 10], gives a precise description of the set R_S and, in fact, it establishes a bound for the number of gaps of R_S and the reason for which these gaps are produced. Furthermore, we provide a sufficient condition which guarantees that the set R_S consists of one segment.

Theorem 3.3. *Let* $S(z) = m_1 e^{\gamma_1 z} + m_2 e^{(\gamma_1 + \alpha_2 e^{i\varphi})z} + \ldots + m_n e^{(\gamma_1 + \alpha_n e^{i\varphi})z}$, $n \ge 2$, *be an exponential polynomial of type (3.1) with* $0 < \alpha_2 < \ldots < \alpha_n$ *linearly independent over the rationals.*

    i) R_S is either $[a_S, b_S]$ or the union of at most $n-1$ disjoint nondegenerate line segments. In the latter case, the gaps of R_S are associated with those equations (3.5) which have two real solutions, i.e. $(xe^{-i\varphi}, ye^{-i\varphi})$ is a gap of R_S if and only if $x = b_j$ and $y = a_{j+1}$, where $1 \le j \le l-1$ and the a_j's and b_j's are considered above.

    ii) If $|m_1| = |m_2| = \ldots = |m_n|$, then $R_S = [a_S, b_S]$, i.e. R_S consists of one segment.

Finally, in the more general case of an exponential polynomial $P(z)$ with distinct complex frequencies (not necessarily co-linear), according to Theorem 2.1, we must consider the union of q sets of type (3.2), where q is the number of sides of the convex polygon C_P. In this manner, let R_P denote the union of those sets, and let us introduce the following notation.

Notation 3.4. *Consider $P(z) = S_1(z) + \ldots + S_q(z)$ of type (1.1), with $S_r(z)$ an exponential polynomial such that the complex conjugates of its frequencies lie on the side l_r of the polygon C_P, where $r = 1, \ldots, q$. If that is the case, let $\mathcal{C}_I$ denote the set of those $P(z)$ for which each $S_r(z)$ can be written as an exponential polynomial of type (3.1) with $0 < \alpha_2 < \ldots < \alpha_n$ linearly independent over the rationals.*

Now, we deduce from Theorem 3.3 that R_P is the union of a finite amount of disjoint nondegenerate line segments for any $P(z)$ in the class $\mathcal{C}_I$. Also, as a clear consequence of Theorem 2.1, the following result for this particular class of exponential polynomials holds.

Corollary 3.5. *Let $P(z) = S_1(z) + \ldots + S_q(z)$ be an exponential polynomial in the class $\mathcal{C}_I$ where q is the number of sides of the convex polygon C_P. If the coefficients of any $S_r(z)$ have the same absolute value, then each one of the q sets of R_P consists of one segment.*

It is clear that the results above can be applied to any exponential polynomial with the same set of zeros. In this sense, in the paper [12] the authors identify the families of exponential polynomials with the same set of zeros.

Example 3.6. Consider the exponential polynomial

$$S(z) = e^{(-1+2i)z} + 5e^{(-1+2i+\log(2)e^{\frac{3\pi i}{4}})z} + 7ie^{(-1+2i+\log(11)e^{\frac{3\pi i}{4}})z} +$$

$$+ 17e^{(-1+2i+\log(83)e^{\frac{3\pi i}{4}})z} + 25e^{(-1+2i+\log(179)e^{\frac{3\pi i}{4}})z} +$$

$$+ 11ie^{(-1+2i+\log(601)e^{\frac{3\pi i}{4}})z} + ie^{(-1+2i+\log(1193)e^{\frac{3\pi i}{4}})z}.$$

Observe that $S(z)$ is of type (3.1) where $\gamma_1 = -1 + 2i$, $\varphi = \dfrac{3\pi}{4}$, $\alpha_1 = 0$, $\alpha_2 = \log 2$, $\alpha_3 = \log(11)$, $\alpha_4 = \log(83)$, $\alpha_5 = \log(179)$, $\alpha_6 = \log(601)$ and $\alpha_7 = \log(1193)$. Note also that $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ is linearly independent over the rationals. Moreover, $S(z)$ is associated (see Remark 3.1) with the exponential polynomial

$$S_1(z) = 1 + 5e^{\log(2)z} + 7ie^{\log(11)z} + 17e^{\log(83)z} +$$

$$+ 25e^{\log(179)z} + 11ie^{\log(601)z} + ie^{\log(1193)z},$$

which satisfies $R_S = \overline{\{e^{-i\varphi}\operatorname{Re} z : \ S_1(z) = 0\}}$.

The extreme points of the set R_{S_1}, denoted as a_{S_1} and b_{S_1}, are respectively determined by the equations

$$5 \cdot 2^\sigma + 7 \cdot 11^\sigma + 17 \cdot 83^\sigma + 25 \cdot 179^\sigma + 11 \cdot 601^\sigma + 1193^\sigma = 1,$$

and

$$1 + 5 \cdot 2^\sigma + 7 \cdot 11^\sigma + 17 \cdot 83^\sigma + 25 \cdot 179^\sigma + 11 \cdot 601^\sigma = 1193^\sigma,$$

whose solutions are $a_{S_1} \approx -2.358$ and $b_{S_1} \approx 3.544$. The distribution of zeros of $S_1(z)$ with imaginary part between -200 and 200 can be observed
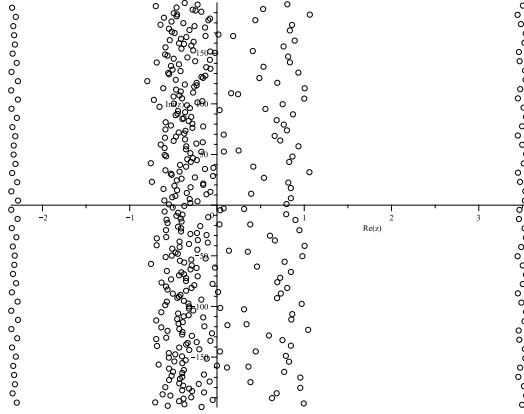
Figure 2: Distribution of zeros of the polynomial $S_1(z)$ defined in Example 3.6

on Figure 2. Hence the extreme points of R_S are $a_S \approx -2.358e^{-\frac{3\pi}{4}i}$ and $b_S \approx 3.544e^{-\frac{3\pi}{4}i}$. Furthermore, equalities (3.5) are given in this case by

$$(3.6) \qquad 5 \cdot 2^\sigma = 1 + 7 \cdot 11^\sigma + 17 \cdot 83^\sigma + 25 \cdot 179^\sigma + 11 \cdot 601^\sigma + 1193^\sigma,$$

$$(3.7) \qquad 7 \cdot 11^\sigma = 1 + 5 \cdot 2^\sigma + 17 \cdot 83^\sigma + 25 \cdot 179^\sigma + 11 \cdot 601^\sigma + 1193^\sigma,$$

$$(3.8) \qquad 17 \cdot 83^\sigma = 1 + 5 \cdot 2^\sigma + 7 \cdot 11^\sigma + 25 \cdot 179^\sigma + 11 \cdot 601^\sigma + 1193^\sigma,$$

$$(3.9) \qquad 25 \cdot 179^\sigma = 1 + 5 \cdot 2^\sigma + 7 \cdot 11^\sigma + 17 \cdot 83^\sigma + 11 \cdot 601^\sigma + 1193^\sigma,$$

$$(3.10) \qquad 11 \cdot 601^\sigma = 1 + 5 \cdot 2^\sigma + 7 \cdot 11^\sigma + 17 \cdot 83^\sigma + 25 \cdot 179^\sigma + 1193^\sigma.$$
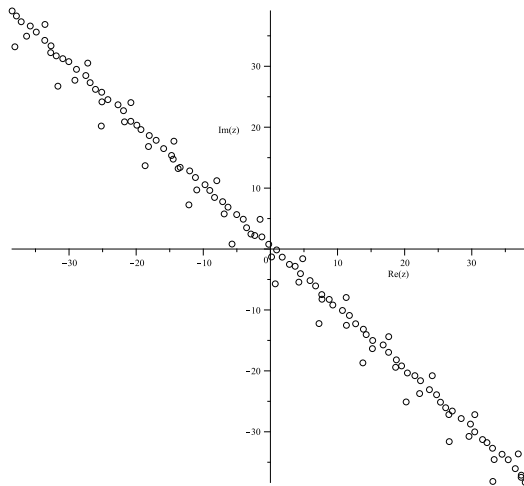

Figure 3: Distribution of zeros of the polynomial $S(z)$ defined in Example 3.6

We next use Theorem 3.3 in order to calculate the disjoint line segments of R_S. The equality in (3.6) is reached at $d_1 \approx -2.278$ and $d_2 \approx -0.824$. The equality in (3.10) is reached at $d_3 \approx 1.078$ and $d_4 \approx 3.443$. Finally,

equalities (3.7), (3.8) and (3.9) have no real solutions. Therefore, the gaps in R_{S_1} are (d_1, d_2) and (d_3, d_4), as we observe in Figure 2. Consequently, the gaps in R_S are determined by the line segments $(d_1 e^{-\frac{3\pi}{4}i}, d_2 e^{-\frac{3\pi}{4}i})$ and $(d_3 e^{-\frac{3\pi}{4}i}, d_4 e^{-\frac{3\pi}{4}i})$. Figure 3 presents the distribution of zeros of $S(z)$ with the imaginary and real part between -40 and 40.

Acknowledgements

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