

Primal-dual optimization conditions for the robust sum of functions with applications

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Abstract

This paper associates a dual problem to the minimization of an arbitrary linear perturbation of the robust sum function introduced in [8]. It provides an existence theorem for primal optimal solutions and, under suitable duality assumptions, characterizations of the primal-dual optimal set, the primal optimal set, and the dual optimal set, as well as a formula for the subdifferential of the robust sum function. The mentioned results are applied to get simple formulas for the robust sums of subaffine functions (a class of functions which contains the affine ones) and to obtain conditions guaranteeing the existence of best approximate solutions to inconsistent convex inequality systems.

Keywords Robust sum function · Duality · Optimality conditions · Existence of optimal solutions · Inconsistent convex inequality systems · Best approximation

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1 Introduction

In our previous paper [8] we have introduced the so-called *robust sum* $\sum_{i \in I}^R f_i$ of an infinite family $(f_i)_{i \in I}$ of proper functions from a given locally convex Hausdorff topological vector space X to $\mathbb{R} \cup \{+\infty\}$. To this aim we denoted by $\mathcal{F}(I)$ the collection of all nonempty finite subsets of I and defined the *robust sum* of $(f_i)_{i \in I}$ as

$$\sum_{i \in I}^R f_i(x) := \sup_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i(x), \forall x \in X.$$

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In order to motivate this definition, consider the finite sum $\sum_{i \in J} f_i$ for each $J \in \mathcal{F}(I)$ and interpret $\mathcal{F}(I)$ as an *uncertainty set* for the *uncertain optimization problem*

$$(P_J) \quad f(x) = \inf_{x \in X} \sum_{i \in J} f_i(x).$$

Then, the *robust* (or *pessimistic*) *counterpart* of this parametric problem is (see [1] and references therein) the deterministic problem

$$(RP) \quad \inf_{x \in X} \sup_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i(x), \quad (1.1)$$

whose objective function $\sum_{i \in I} f_i$ cannot be exactly computed at a given x but can be approximated through the finite sums $\sum_{i \in J} f_i(x)$, with $J \in \mathcal{F}(I)$. Observe that the above uncertain problem only makes sense when I is infinite as, otherwise, $\sum_{i \in I} f_i(x)$ is computable at any $x \in \mathbb{R}^n$ and (P_I) is the deterministic problem to be solved. However, this uninteresting case allows to appreciate the pessimistic character of (RP) in comparison with (P_I) . Indeed, defining $I(x) := \{i \in I : f_i(x) \geq 0\}$, the objective function of (RP) reads

$$f(x) = \begin{cases} \max_{i \in I} f_i(x), & \text{if } I(x) = \emptyset, \\ \sum_{i \in I(x)} f_i(x), & \text{else,} \end{cases}$$

with f being an upper estimate of $\sum_{i \in I} f_i$ (the difference $f - \sum_{i \in I} f_i$ may be quite large).

It is worth observing that, in contrast with the well-known *limit sum*

$$\sum_{i \in I} f_i(x) := \lim_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i(x), \forall x \in X$$

(where $\mathcal{F}(I)$ and \lim must be interpreted as a set directed by inclusion and the limit of the corresponding net, respectively), the robust sum $\sum_{i \in I} f_i$ is always well-defined on X .

In [8, Section 1] we gave two examples of optimization problems arising in extended regression and best approximate solution to inconsistent linear system which can be formulated as (RP) , with $(f_i)_{i \in I}$ being families of quadratic functions and maxima of affine functions, respectively.

In this paper we assume that some element \bar{x}^* of the dual space X^* of X is given and introduce a dual problem for the *linearly perturbed robust sum* $\sum_{i \in I} f_i - \langle \bar{x}^*, \cdot \rangle$. More precisely, we are concerned with the non-emptiness and the structure of the optimal sets of the dual pair of optimization problems

$$(RP_{\bar{x}^*}) \quad \inf \{f(x) - \langle \bar{x}^*, x \rangle : x \in X\}$$

and

$$(RD_{\bar{x}^*}) \quad \sup \left\{ - \sum_{j \in J} f_j^*(x_j^*) : (J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*) \right\},$$

where $f := \sum_{i \in I}^R f_i$ represents the robust sum of the family $(f_i)_{i \in I}$, the objective function $-\sum_{j \in J} f_j^*(x_j^*)$ of $(\text{RD}_{\bar{x}^*})$ is well defined thanks to the properness of f_i (guaranteeing that its conjugate function f_i^* does not take the value $-\infty$) for all $i \in I$, and the feasible set of the dual problem, $\mathbb{F}(\bar{x}^*)$, is defined as

$$\mathbb{F}(\bar{x}^*) := \left\{ (J, (x_j^*)_{j \in J}) : J \in \mathcal{F}(I), (x_j^*)_{j \in J} \in (X^*)^J, \sum_{j \in J} x_j^* = \bar{x}^* \right\}.$$

When \bar{x}^* is the null functional, the pair formed by $(\text{RP}_{\bar{x}^*})$ and $(\text{RD}_{\bar{x}^*})$ collapses to the pair of dual problems analyzed in [8], for which we characterized weak duality, zero duality gap, and strong duality, and their corresponding stable versions, but without paying attention to their optimal solution sets.

Many works have been written on the numerical methods for the problem of best least squares solutions of inconsistent finite linear inequality systems (see, e.g., [21] and references therein), for which the existence of optimal solutions has been proved in three different ways in [5]. Unfortunately, as shown in [9], the existence of optimal solution for the best least squares approximation problems relies on the finiteness of the number of constraints and the type of norm used to measure the residual of an approximate solution. The novelties of Section 6, in comparison with its unique antecedent [9], is that, here, we consider convex systems instead of linear ones, describe the structure of the sets of best ℓ_1 and ℓ_∞ approximate solutions (instead of just an existence theorem for best ℓ_∞ approximation problems), and provide strong duality theorems for best ℓ_1 and ℓ_∞ approximation problems.

This paper is organized as follows. Section 2 introduces the necessary notation and some preliminary results. Section 3 provides an existence theorem for primal optimal solutions. Section 4 characterizes the primal-dual optimal solutions with zero duality gap, as well as, under suitable assumptions, primal optimal solutions, dual optimal solutions and also provides a closed formula for the subdifferential of the robust sum function. Section 5 provides formulas for the robust sums of subaffine functions (concept introduced in Section 2). Finally, Section 6 provides existence theorems for best approximate solutions to inconsistent convex inequality systems with respect to the ℓ_∞ and the ℓ_1 pseudo-norms.

2 Preliminaries

We first recall some standard notation regarding locally convex spaces to be used in the sequel. We denote by 0_X and 0_{X^*} the null vectors of X and X^* , respectively. Given a set $A \subset X$, we denote by $\text{co } A$, $\text{cone } A$, $\text{aff } A$, \overline{A} , $\overline{\text{co}}A$, and $\overline{\text{cone}}A$ the convex hull of A , the cone generated by $A \cup \{0_X\}$, the smallest linear manifold containing A , the closure of A , the closed convex hull A , and the closed conic hull of A , respectively. The same notation is used when either $A \subset X^*$ (by default equipped with the w^* -topology) or $A \subset X^* \times \mathbb{R}$ (equipped with the product topology). We represent by

proj_{X^*} the mapping from $X^* \times \mathbb{R}$ to X^* such that $\text{proj}_{X^*}(x^*, r) = x^*$. When $X = \mathbb{R}^n$, we denote by $\text{ri } A$ the relative interior of A .

Given $A, B \subset X$, A is said [2] to be *closed regarding to* B if $B \cap \bar{A} = B \cap A$. Clearly, A is closed regarding B if and only if A is closed regarding each subset of B .

We denote by $\bar{\mathbb{R}}$ the extended real line with $\pm\infty$ and by $\bar{\mathbb{R}}^X$ the linear space of functions from X to $\bar{\mathbb{R}}$. Given $h \in \bar{\mathbb{R}}^X$, its *lower level sets* are $[h \leq r] := \{x \in X : h(x) \leq r\}$, with $r \in \mathbb{R}$, its *domain* is the set $\text{dom } h := \{x \in X : h(x) < +\infty\}$, its *epigraph* is $\text{epi } h := \{(x, r) \in X \times \mathbb{R} : h(x) \leq r\}$, its *strict epigraph* is $\text{epi}_s h := \{(x, r) \in X \times \mathbb{R} : h(x) < r\}$, and its *Fenchel conjugate* the function $h^* \in \bar{\mathbb{R}}^{X^*}$ such that

$$h^*(x^*) := \sup\{\langle x^*, x \rangle - h(x) : x \in X\}, \forall x^* \in X^*.$$

Moreover, the *closed hull* of h is the function $\bar{h} \in \bar{\mathbb{R}}^X$ whose epigraph $\text{epi } \bar{h}$ is the closure of $\text{epi } h$ in $X \times \mathbb{R}$. The definitions are similar if $h \in \bar{\mathbb{R}}^{X^*}$; in particular, \bar{h} is the w^* -closed hull of h . The *subdifferential* of h at $a \in X$ is

$$\partial h(a) := \begin{cases} \{x^* \in X^* : h(x) \geq h(a) + \langle x^*, x - a \rangle, \forall x \in X\}, & \text{if } h(a) \in \mathbb{R}, \\ \emptyset, & \text{else.} \end{cases}$$

The *indicator function* of $A \subset X$ is represented by δ_A (i.e. $\delta_A(x) = 0$ if $x \in A$, and $\delta_A(x) = +\infty$ if $x \notin A$). The *support function* of $A \neq \emptyset$, $\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle$, is the conjugate of its indicator, i.e., $\sigma_A = \delta_A^*$. The support functions are *sublinear*, i.e., they are subadditive and positively homogeneous.

We denote by $\Gamma(X)$ the cone of $\bar{\mathbb{R}}^X$ formed by the proper closed convex functions on X . For instance, $\delta_A \in \Gamma(X)$ if and only if A is a nonempty closed convex set while $\sigma_A \in \Gamma(X^*)$ for all nonempty $A \subset X$. The sublinear elements of $\Gamma(X)$ are the support functions of the nonempty w^* -closed convex subsets of X^* .

The continuous affine functions on X are the sums of continuous linear functionals with constants, i.e., functions of the form $\langle a^*, \cdot \rangle + r = \sigma_{\{a^*\}} + r$, with $a^* \in X^*$ and $r \in \mathbb{R}$. In the same vein, we define the *subaffine functions* on X as those functions which can be expressed as $\sigma_A + r$, with A being a nonempty w^* -closed convex subset of X^* and $r \in \mathbb{R}$. For instance, the polar A° of such a set A is the lower level set of some subaffine function. Indeed,

$$A^\circ := \{x \in X : \langle a^*, x \rangle \leq 1, \forall a^* \in A\} = [\sigma_A - 1 \leq 0].$$

Obviously, any continuous affine function is subaffine.

Remark 2.1 *The above class of subaffine functions is not related with others types of functions introduced under the same name in different settings:*

1. *Generalized convexity (see, e.g., [20], [16],[19], [22]): a function $f \in \bar{\mathbb{R}}^X$ is called subaffine (or truncated affine) if it can be written as $f = \min\{x^* + r, s\}$, for $x^* \in X^*$ and $r, s \in \mathbb{R}$.*
2. *Elliptic PDEs (see, e.g., [11], [18]): a function $f \in \bar{\mathbb{R}}^{\mathbb{R}^n}$ is called subaffine if it is*

upper semicontinuous and there exists a ball B such that for each affine function h , $f \leq h$ on $\text{bd} B$ implies that $f \leq h$ on B . A \mathcal{C}^2 function is subaffine in this sense iff its Hessian matrix has at least one nonnegative eigenvalue at each point.

We now come back to the pair of problems $(\text{RP}_{\bar{x}^*})$ and $(\text{RD}_{\bar{x}^*})$, whose optimal sets are respectively denoted

$$\text{sol}(\text{RP}_{\bar{x}^*}) = \{x \in X : f(x) - \langle \bar{x}^*, x \rangle = \inf(\text{RP}_{\bar{x}^*})\}$$

and

$$\text{sol}(\text{RD}_{\bar{x}^*}) = \left\{ (J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*) : - \sum_{j \in J} f_j^*(x_j^*) = \sup(\text{RD}_{\bar{x}^*}) \right\}.$$

When $\text{sol}(\text{RP}_{\bar{x}^*}) \neq \emptyset$ we write $\min(\text{RP}_{\bar{x}^*})$ instead of $\inf(\text{RP}_{\bar{x}^*})$. Similarly, we write $\max(\text{RD}_{\bar{x}^*})$ instead of $\sup(\text{RD}_{\bar{x}^*})$ if $\text{sol}(\text{RD}_{\bar{x}^*}) \neq \emptyset$.

Adopting the robust optimization approach under uncertainty (as in [4], [6], [7], [15], etc.) we have shown in [8] that $(\text{RP}_{\bar{x}^*})$ may be interpreted as the robust optimization counterpart of some uncertain optimization problem and $(\text{RD}_{\bar{x}^*})$ as its optimistic dual. In particular, the relation

$$\sup(\text{RD}_{\bar{x}^*}) \leq \inf(\text{RP}_{\bar{x}^*}) \tag{2.1}$$

always holds [8, Proposition 3.1]. The characterization of the *strong duality*, namely $\inf(\text{RP}_{\bar{x}^*}) = \max(\text{RD}_{\bar{x}^*})$, involves the set

$$\mathcal{A} := \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^*. \tag{2.2}$$

As shown below, the set \mathcal{A} may be convex in favorable circumstances.

Lemma 2.1 *Let $(A_i)_{i \in I}$ be a family of convex subsets of a linear space Z such that $0_Z \in \bigcap_{i \in I} A_i$. Then $A := \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} A_j$ is a convex subset of Z .*

Proof. Notice that $\left(\sum_{j \in J} A_j \right)_{J \in \mathcal{F}(I)}$ is a family of convex subsets of Z which is directed with respect to the inclusion. It follows that A is convex. \square

Example 2.1 *The set $\mathcal{A} = \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^*$ is convex if the functions f_j , $j \in J$, are non-negative.*

Example 2.2 *The set $A := \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^*$ is convex if each function f_j , $j \in J$, is bounded below.*

We have the following characterization of strong duality under convexity.

Theorem 2.1 (Strong zero duality gap under convexity) [8, Theorem 6.1] *Assume the $f_i \in \Gamma(X)$, $i \in I$, and $\text{dom } f \neq \emptyset$. The next statements are equivalent:*

(i) $\inf(\text{RP}_{\bar{x}^*}) = \max(\text{RD}_{\bar{x}^*})$.

(ii) \mathcal{A} is w^* -closed convex regarding $\{\bar{x}^*\} \times \mathbb{R}$.

In particular, (i) holds for any $\bar{x}^* \in X^*$ if and only if \mathcal{A} is w^* -closed convex.

3 Minimizing the robust sum: existence of primal optimal solutions

In this section we assume that $(f_i)_{i \in I} \subset \Gamma(X)$ and, unless specified otherwise, that $f = \sum_{i \in I}^R f_i$ is proper. We thus have $f \in \Gamma(X)$. Additionally, we suppose that

$$f \text{ is weakly inf-locally compact} \tag{3.1}$$

in the sense that the lower level set $[f \leq r]$ is weakly locally compact for each $r \in \mathbb{R}$. Let us note that this condition is always satisfied if X is finitely dimensional. It is also satisfied if $\sup_{i \in I} f_i$ is weakly inf-locally compact or, a fortiori, if there exists $i \in I$ such that f_i is weakly inf-locally compact.

By [12, Chapter 1, Proposition 5.4] or by [14, Theorem 7.7.6], (3.1) is equivalent to:

f^* is quasicontinuous with respect to the Mackey topology $\tau(X^*, X)$ on X^* .

Let us recall that a convex function $\xi : X^* \rightarrow \overline{\mathbb{R}}$ is said to be $\tau(X^*, X)$ -quasicontinuous if the following four properties are satisfied ([12], [13], [14]):

- $\text{aff dom } \xi$ is $\tau(X^*, X)$ -closed (or w^* -closed).
- $\text{aff dom } \xi$ is of finite codimension.
- The $\tau(X^*, X)$ -relative interior of $\text{dom } \xi$, say $\text{ri dom } \xi$, is nonempty.
- The restriction of ξ to $\text{aff dom } \xi$ is $\tau(X^*, X)$ -continuous on $\text{ri dom } \xi$.

Remark 3.1 *A convex function majorized by a $\tau(X^*, X)$ -quasicontinuous one is $\tau(X^*, X)$ -quasicontinuous, too (see [17, Theorem 2.4], [23, Proposition 2.2.15]). If $X = X^* = \mathbb{R}^n$, any extended real-valued convex function with nonempty domain is quasicontinuous.*

Let us consider the subdifferential of f^* at $\bar{x}^* \in X^*$, namely,

$$\partial f^*(\bar{x}^*) = \begin{cases} \{x \in X : f^*(x^*) \geq f^*(\bar{x}^*) + \langle x^* - \bar{x}^*, x \rangle, \forall x^* \in X^*\}, & \text{if } f^*(\bar{x}^*) \in \mathbb{R}, \\ \emptyset, & \text{else.} \end{cases}$$

For $\bar{x}^* \in \text{dom } f^*$, since $f \in \Gamma(X)$ entails $f^{**} = f$, one has

$$\partial f^*(\bar{x}^*) = \text{argmin}(f - \langle \bar{x}^*, \cdot \rangle) = \text{sol}(\text{RP}_{\bar{x}^*}). \quad (3.2)$$

We are faced with the subdifferentiability of f^* at \bar{x}^* , for which the dual version [17, Theorem III.3] gives a very useful criterion:

Lemma 3.1 *Assume that $g \in \Gamma(X)$ is weakly inf-locally compact and*

$$\overline{\text{cone}}(\text{dom } g^* - \bar{x}^*) \text{ is a linear subspace of } X^*. \quad (3.3)$$

Then $\partial g^(\bar{x}^*)$ is the sum of a nonempty weakly compact convex set and a finitely dimensional linear subspace of X .*

Remark 3.2 *Condition (3.3) means that the sets $\text{dom } g^*$ and $\{x^*\}$ are united in the sense that they cannot be properly separated (all weak*-closed hyperplanes which separate them contain both of them). A sufficient (in general not necessary) condition for this is that x^* belongs to the relative algebraic interior of $\text{dom } g^*$ (see [23, Proposition 1.2.8] for more details).*

To exploit Lemma 3.1 in the case that $g = f = \sum_{i \in I}^R f_i$, we need an explicit formulation of the criterion (3.3) in terms of the functions f_j^* . To this end, let us consider the function φ defined on X^* by

$$\varphi(x^*) := \inf \left\{ \sum_{j \in J} f_j^*(x_j^*) : (J, (x_j^*)_{j \in J}) \in \mathbb{F}(x^* + \bar{x}^*) \right\}, \forall x^* \in X^*. \quad (3.4)$$

One has straightforwardly

$$\varphi^*(x) = f(x) - \langle \bar{x}^*, x \rangle, \forall x \in X,$$

$$\varphi^{**}(x^*) = f^*(x^* + \bar{x}^*), \forall x^* \in X^*,$$

and

$$\text{dom } f^* - \bar{x}^* = \text{dom } \varphi^{**}. \quad (3.5)$$

Since $\text{dom } \varphi^* = \text{dom } f \neq \emptyset$, the biconjugate function φ^{**} coincides with the w^* -closed convex hull $\overline{\text{co}}\varphi$ of φ , which satisfies

$$\text{epi } \overline{\text{co}}\varphi = \overline{\text{co}} \text{epi } \varphi. \quad (3.6)$$

Let us observe that

$$\text{proj}_{X^*}(\text{co epi } \varphi) = \text{co dom } \varphi. \quad (3.7)$$

Now, by (3.6) and (3.7), one has

$$\text{dom } \overline{\text{co}}\varphi = \text{proj}_{X^*}(\overline{\text{co}} \text{epi } \varphi) \subset \overline{\text{proj}_{X^*}(\text{co epi } \varphi)} = \overline{\text{co}} \text{dom } \varphi,$$

and, since $\overline{\text{co}} \text{dom } \varphi$ is w^* -closed,

$$\overline{\text{dom } \overline{\text{co}} \varphi} \subset \overline{\text{co}} \text{dom } \varphi.$$

Conversely, since $\overline{\text{co}} \varphi \leq \varphi$, we have $\text{dom } \varphi \subset \text{dom } \overline{\text{co}} \varphi$ and, since $\text{dom } \overline{\text{co}} \varphi$ is convex, $\text{co } \text{dom } \varphi \subset \text{dom } \overline{\text{co}} \varphi$. So, $\overline{\text{co}} \text{dom } \varphi = \overline{\text{co } \text{dom } \varphi} \subset \overline{\text{dom } \overline{\text{co}} \varphi}$. Consequently,

$$\overline{\text{co}} \text{dom } \varphi = \overline{\text{dom } \overline{\text{co}} \varphi}, \quad (3.8)$$

and hence, it follows from (3.5) that

$$\begin{aligned} \overline{\text{cone}}(\text{dom } f^* - \bar{x}^*) &= \overline{\text{cone}} \text{dom } \varphi^{**} = \overline{\text{cone}} \text{dom } \overline{\text{co}} \varphi \\ &= \overline{\text{cone}}(\overline{\text{dom } \overline{\text{co}} \varphi}) = \overline{\text{cone}}(\overline{\text{co}} \text{dom } \varphi) \\ &= \overline{\text{cone}}(\text{co } \text{dom } \varphi). \end{aligned}$$

Now, from the very definition of φ , one has

$$\text{dom } \varphi = \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right) - \bar{x}^*,$$

and the criterion (3.3) writes, for $g = f$,

$$\overline{\text{cone}} \text{co} \left\{ \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right) - \bar{x}^* \right\} \text{ is a linear subspace of } X^*. \quad (3.9)$$

Together with (3.2) and Lemma 3.1, we have thus proved the following result:

Theorem 3.1 (Existence of optimal solution) *Assume that $(f_i)_{i \in I} \subset \Gamma(X)$, $f = \sum_{i \in I}^R f_i$ is proper weakly inf-locally compact and (3.9) holds. Then $(\text{RP}_{\bar{x}^*})$ admits at least an optimal solution. More precisely, $\text{sol}(\text{RP}_{\bar{x}^*})$ is the sum of a nonempty convex weakly compact set and a finitely dimensional linear subspace of X .*

For nonnegative functions we obtain:

Corollary 3.1 *Let $(f_i)_{i \in I}$ be a family of nonnegative $\Gamma(X)$ -functions such that the infinite sum $\sum_{i \in I} f_i$ is proper weakly inf-locally compact. Assume that*

$$\overline{\text{cone}} \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \text{ is a linear subspace of } X^*. \quad (3.10)$$

Then the optimal solution set of the problem

$$\inf_{x \in X} \sum_{i \in I} f_i(x)$$

is the sum of a nonempty convex weakly compact set and a finitely dimensional linear subspace of X .

Proof. Since the functions f_i , $i \in I$ are nonnegative, their robust sum coincides with the infinite sum $\sum_{i \in I} f_i$. Moreover, one has $0_{X^*} \in \text{dom } f_i^*$ for each $i \in I$, and the set $\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^*$ is convex (see Example 2.2). We conclude the proof with Theorem

3.1. □

Remark 3.3 *If I is finite and all functions f_i , $i \in I$, are nonnegative, then*

$$\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* = \sum_{i \in I} \text{dom } f_i^*,$$

and condition (3.10) becomes

$$\overline{\text{cone}} \sum_{i \in I} \text{dom } f_i^* \text{ is a linear subspace of } X^*.$$

Observe that, under the assumptions of Theorem 3.1, one has in particular $\inf(\text{RP}_{\bar{x}^*}) \in \mathbb{R}$. Observe also that when $X = X^* = \mathbb{R}^n$, (3.3) writes $\bar{x}^* \in \text{ri}(\text{dom } g^*)$, and in such a case, one has the next corollary.

Corollary 3.2 *Assume that $(f_i)_{i \in I} \subset \Gamma(\mathbb{R}^n)$, $\text{dom } f \neq \emptyset$, and*

$$\bar{x}^* \in \text{ri co} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right). \quad (3.11)$$

Then, $\text{sol}(\text{RP}_{\bar{x}^})$ is the sum of a nonempty convex compact set and a linear subspace of \mathbb{R}^n .*

Remark 3.4 *If each function f_i , $i \in I$, is bounded below, then (see Example 2.2) the criteria (3.9) and (3.11) collapse respectively to*

$$\overline{\text{cone}} \left\{ \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right) - \bar{x}^* \right\} \text{ is a linear subspace of } X^*$$

and

$$\bar{x}^* \in \text{ri} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right).$$

Note that the conclusion of Theorem 3.1 does not entail that

$$\min(\text{RP}_{\bar{x}^*}) = \sup(\text{RD}_{\bar{x}^*}). \quad (3.12)$$

One has in fact, with φ defined as in (3.4), the following lemma.

Lemma 3.2 *Assume that either $\sup(\text{RD}_{\bar{x}^*}) = +\infty$ or φ is subdifferentiable at 0_{X^*} . Then (3.12) holds.*

Proof. Since $\inf(\text{RP}_{\bar{x}^*}) \geq \sup(\text{RD}_{\bar{x}^*})$, (3.12) is obvious if $\sup(\text{RD}_{\bar{x}^*}) = +\infty$. Assume now that $\bar{x} \in \partial\varphi(0_{X^*})$. Then $\varphi(0_{X^*}) + \varphi^*(\bar{x}) = \langle 0_{X^*}, \bar{x} \rangle = 0$ and we thus have

$$\inf(\text{RP}_{\bar{x}^*}) \leq f(\bar{x}) - \langle \bar{x}^*, \bar{x} \rangle = \varphi^*(\bar{x}) = -\varphi(0_{X^*}) = \sup(\text{RD}_{\bar{x}^*}) \leq \inf(\text{RP}_{\bar{x}^*}),$$

and (3.12) follows. \square

Remark 3.5 *Recall that $\mathcal{A} = \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^*$ and $\text{dom } \varphi = \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right) - \bar{x}^*$.*

From (3.4) one has

$$\text{epi}_s \varphi \subset \mathcal{A} - (\bar{x}^*, 0) \subset \text{epi } \varphi$$

and, consequently,

$$\varphi(\bar{x}^*) = \inf \{t \in \mathbb{R} : (x^*, t) \in \mathcal{A} - (\bar{x}^*, 0)\}.$$

It follows that, if \mathcal{A} is convex, then φ is convex too.

Theorem 3.2 (Primal attainment) *Assume that $(f_i)_{i \in I} \subset \Gamma(X)$, φ defined by (3.4) is convex and Mackey-quasicontinuous, and that*

$$\overline{\text{cone}} \left\{ \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right) - \bar{x}^* \right\} \text{ is a linear subspace of } X^*. \quad (3.13)$$

Then,

$$\min(\text{RP}_{\bar{x}^*}) = \sup(\text{RD}_{\bar{x}^*}).$$

Proof. By Lemma 3.2 one may assume that $\varphi(0_{X^*}) \neq -\infty$. By [17, Theorem 3.3] we have $\partial\varphi(0_{X^*}) \neq \emptyset$ and by Lemma 3.2 again we are done. \square

Remark 3.6 *Since for each $(i, x^*) \in I \times X^*$ one has $\varphi(x^*) \leq f_i^*(x^* + \bar{x}^*)$, the function φ (assumed to be convex) is Mackey-quasicontinuous whenever there exists $i_0 \in I$ such that f_{i_0} is weakly inf-locally compact (see Remark 3.1).*

Corollary 3.3 *Let $(f_i)_{i \in I} \subset \Gamma(\mathbb{R}^n)$ be such that $\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^*$ is convex and*

$$\bar{x}^* \in \text{ri} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right). \quad (3.14)$$

Then $\min(\text{RP}_{\bar{x}^}) = \sup(\text{RD}_{\bar{x}^*})$.*

Proof. As $\mathcal{A} = \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^*$ is convex, φ is convex, too (Remark 3.5). Moreover,

as $X = \mathbb{R}^n$ and $\text{dom } \varphi \neq \emptyset$, φ is Mackey-quasicontinuous. Now, again, as $X = \mathbb{R}^n$, (3.14) \Leftrightarrow (3.13), and the conclusion follows from Theorem 3.2. \square

4 Primal-dual optimality relations

We need to introduce some additional notations. Given $g : X \rightarrow \overline{\mathbb{R}}$, we denote by $M_g : X^* \rightrightarrows X$ the set-valued mapping defined, for each $x^* \in X^*$, as

$$(M_g)(x^*) = \begin{cases} \text{argmin}(g - \langle x^*, \cdot \rangle), & \text{if } g^*(x^*) \in \mathbb{R}, \\ \emptyset, & \text{else.} \end{cases}$$

In fact, M_g is nothing else than the inverse of the subdifferential mapping $\partial g : X \rightrightarrows X^*$, i.e.,

$$x \in (M_g)(x^*) \iff x^* \in \partial g(x).$$

One has $(M_g)(x^*) \subset \partial g^*(x^*)$ and equality holds whenever $g = g^{**}$ (e.g., when $g \in \Gamma(X)$).

Given $x \in X$, we denote by $S_f(x)$ the (possibly empty) set of those $J \in \mathcal{F}(I)$ that realize the supremum in the definition of the robust sum when $f(x)$ is finite:

$$S_f(x) = \begin{cases} \left\{ J \in \mathcal{F}(I) : \sum_{j \in J} f_j(x) = f(x) \right\}, & \text{if } x \in \text{dom } f, \\ \emptyset, & \text{else.} \end{cases}$$

The inverse of the set-valued mapping $S_f : X \rightrightarrows \mathcal{F}(I)$ is denoted by T_f . One has $T_f : \mathcal{F}(I) \rightrightarrows X$ and

$$x \in T_f(J) \iff J \in S_f(x).$$

If I is finite one has of course $S_f(x) \neq \emptyset$ for each $x \in \text{dom } f$. We now make explicit $S_f(x)$ in different situations. To this aim, we introduce the supremum function $f_0 := \sup_{i \in I} f_i$.

- If $f_0(x) \leq 0$ we have $f(x) = f_0(x)$ [8, Lemma 2.5]. Then

$$S_f(x) = \begin{cases} \{\{j\} : j \in I, f_j(x) = f_0(x)\}, & \text{if } f_0(x) < 0, \\ \{J \in \mathcal{F}(I) : f_j(x) = 0, \forall j \in J\}, & \text{if } f_0(x) = 0. \end{cases}$$

- If $f_0(x) \in]0, +\infty[$ we have $f(x) = \sum_{i \in I} f_i^+(x) := \sum_{i \in I} \max\{f_i(x), 0\}$ [8, Lemma 2.5] and

$$S_f(x) = \begin{cases} \{i \in I : f_i(x) > 0\}, & \text{if this set is finite,} \\ \emptyset, & \text{else.} \end{cases}$$

Theorem 4.1 (Primal-dual optimality with zero duality gap) *Assume that all functions f_i are proper and let $x \in \text{dom } f$ and $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$. Next statements are equivalent:*

(i) $x \in \text{sol}(\text{RP}_{\bar{x}^*})$, $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$, and $\inf(\text{RP}_{\bar{x}^*}) = \sup(\text{RD}_{\bar{x}^*})$.
(ii) $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$.

(iii) $x \in T_f(J) \cap \left(\bigcap_{j \in J} M_{f_j}(x_j^*) \right)$.

If $(f_i)_{i \in I} \subset \Gamma(X)$ we can add

(iv) $x \in T_f(J) \cap \left(\bigcap_{j \in J} \partial f_j^*(x_j^*) \right)$.

Proof. From the definitions of the set-valued mappings S_f , T_f , and M_{f_j} it is clear that (ii) \iff (iii) and (iii) \iff (iv) under the assumption that $(f_i)_{i \in I} \subset \Gamma(X)$.

[(i) \implies (ii)] Since $x \in \text{dom } f$ we have $\sum_{j \in J} f_j(x) \in \mathbb{R}$ and

$$\sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle \leq f(x) - \langle \bar{x}^*, x \rangle = \inf(\text{RP}_{\bar{x}^*}) = \sup(\text{RD}_{\bar{x}^*}) = - \sum_{j \in J} f_j^*(x_j^*). \quad (4.1)$$

By Fenchel and Young inequality we have

$$- \sum_{j \in J} f_j^*(x_j^*) \leq \sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle. \quad (4.2)$$

Since $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$ we have

$$\sum_{j \in J} (f_j(x) - \langle x_j^*, x \rangle) = \sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle. \quad (4.3)$$

Combining (4.1), (4.2), and (4.3), we obtain $\sum_{j \in J} f_j(x) = f(x)$, that means $J \in S_f(x)$ and

$$\sum_{j \in J} (f_j(x) + f_j^*(x_j^*) - \langle x_j^*, x \rangle) = 0.$$

By Fenchel and Young inequality all terms of the above sum are nonnegative, hence equal to zero, that means $x_j^* \in \partial f_j(x)$ for all $j \in J$.

[(ii) \implies (i)] Since $J \in S_f(x)$, $\sum_{j \in J} x_j^* = \bar{x}^*$, $x_j^* \in \partial f_j(x)$ for all $j \in J$, and $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$, we have

$$\begin{aligned} f(x) - \langle \bar{x}^*, x \rangle &= \sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle \\ &= \sum_{j \in J} (f_j(x) - \langle x_j^*, x \rangle) \\ &= - \sum_{j \in J} f_j^*(x_j^*) \\ &\leq \sup(\text{RD}_{\bar{x}^*}) \\ &\leq \inf(\text{RP}_{\bar{x}^*}) \\ &\leq f(x) - \langle \bar{x}^*, x \rangle. \end{aligned}$$

All terms of the above chain of inequalities are thus equal and this proves that (i) holds. \square

Next corollary assumes that $\inf(\text{RP}_{\bar{x}^*}) = \max(\text{RD}_{\bar{x}^*})$ (i.e., strong duality), which is characterized (in the convex case) in Theorem 2.1.

Corollary 4.1 *Assume that all functions f_i are proper and let $x \in \text{dom } f$ and $\inf(\text{RP}_{\bar{x}^*}) = \max(\text{RD}_{\bar{x}^*})$. Next statements are equivalent:*

- (i) $x \in \text{sol}(\text{RP}_{\bar{x}^*})$.
- (ii) For all $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$ one has $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$.
- (iii) There exists $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$ such that $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$.
- (iv) There exists $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$ such that $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$.

Moreover, for any $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$ one has

$$\text{sol}(\text{RP}_{\bar{x}^*}) = T_f(J) \cap \left(\bigcap_{j \in J} M_{f_j}(x_j^*) \right). \quad (4.4)$$

Proof. [(i) \implies (ii)] It follows from the statement [(i) \implies (ii)] in Theorem 4.1.

[(ii) \implies (iii)] It is obvious as $\text{sol}(\text{RD}_{\bar{x}^*}) \neq \emptyset$.

[(iii) \implies (iv)] It is obvious.

[(iv) \implies (i)] Since $J \in S_f(x)$, $\sum_{j \in J} x_j^* = \bar{x}^*$, and $x \in M_{f_j}(x_j^*)$ for each $j \in J$,

$$\begin{aligned} \inf(\text{RP}_{\bar{x}^*}) &\leq f(x) - \langle \bar{x}^*, x \rangle = \sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle \\ &= \sum_{j \in J} (f_j(x) - \langle x_j^*, x \rangle) \\ &= - \sum_{j \in J} f_j^*(x_j^*) \\ &\leq \sup(\text{RD}_{\bar{x}^*}) \leq \inf(\text{RP}_{\bar{x}^*}). \end{aligned}$$

This ensures that $f(x) - \langle \bar{x}^*, x \rangle = \inf(\text{RP}_{\bar{x}^*})$ and (i) holds.

Let us prove the last assertion of Corollary 4.1. Let $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$. From [(i) \iff (ii)] one has $x \in \text{sol}(\text{RP}_{\bar{x}^*})$ if and only if $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$ or, equivalently,

$$x \in T_f(J) \cap \left(\bigcap_{j \in J} M_{f_j}(x_j^*) \right). \quad \square$$

Notice that, if $(f_i)_{i \in I} \subset \Gamma(X)$, then $M_{f_j}(x_j^*) = \partial f_j^*(x_j^*)$ for each $j \in J$ and the equation (4.4) writes

$$\text{sol}(\text{RP}_{\bar{x}^*}) = T_f(J) \cap \left(\bigcap_{j \in J} \partial f_j^*(x_j^*) \right).$$

Corollary 4.2 Assume that all functions f_i and f are proper and let $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$ and $\min(\text{RP}_{\bar{x}^*}) = \sup(\text{RD}_{\bar{x}^*})$. Next statements are equivalent:

- (i) $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$,
- (ii) For all $x \in \text{sol}(\text{RP}_{\bar{x}^*})$ one has $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$,
- (iii) There exists $x \in \text{sol}(\text{RP}_{\bar{x}^*})$ such that $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$,
- (iv) There exists $x \in X$ such that $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$.

Moreover, for any $(J, (x_j^*)_{j \in J}) \in \text{sol}(\text{RD}_{\bar{x}^*})$ one has

$$\text{sol}(\text{RD}_{\bar{x}^*}) = \{(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*) : J \in S_f(x) \text{ and } x_j^* \in \partial f_j(x), \forall j \in J\}$$

for either some (all) $x \in \text{sol}(\text{RP}_{\bar{x}^*})$ or for some $x \in X$.

Proof. [(i) \implies (ii)] It comes from the statement [(i) \implies (ii)] in Theorem 4.1.

[(ii) \implies (iii)] It is obvious as $\text{sol}(\text{RP}_{\bar{x}^*}) \neq \emptyset$.

[(iii) \implies (iv)] It is obvious.

[(iv) \implies (i)] Since $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$, $x_j^* \in \partial f_j(x)$ for all $j \in J$, $\sum_{j \in J} x_j^* = \bar{x}^*$, and $J \in S_f(x)$, one has

$$\begin{aligned} \sup(\text{RD}_{\bar{x}^*}) &\geq - \sum_{j \in J} f_j^*(x_j^*) \\ &= \sum_{j \in J} (f_j(x) - \langle x_j^*, x \rangle) \\ &= \sum_{j \in J} f_j(x) - \langle \bar{x}^*, x \rangle \\ &= f(x) - \langle \bar{x}^*, x \rangle \\ &\geq \inf(\text{RP}_{\bar{x}^*}) \\ &\geq \sup(\text{RD}_{\bar{x}^*}). \end{aligned}$$

Consequently, $\sup(\text{RD}_{\bar{x}^*}) = - \sum_{j \in J} f_j^*(x_j^*)$ and (i) holds.

The last assertion of Corollary 4.2 comes directly from the equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv). \square

For the last result of this section we still assume $(f_i)_{i \in I} \subset (\mathbb{R} \cup \{+\infty\})^X$ is an infinite family of proper functions, but we do not consider a fixed element $\bar{x}^* \in X^*$. The equation (4.5) is called *stable strong duality* in [3].

Corollary 4.3 Assume that

$$\inf(\text{RP}_{x^*}) = \max(\text{RD}_{x^*}), \forall x^* \in \bigcup_{x \in X} \partial f(x). \quad (4.5)$$

Then one has

$$\partial f(x) = \bigcup_{J \in S_f(x)} \sum_{j \in J} \partial f_j(x), \forall x \in X. \quad (4.6)$$

Proof. Let us show that the inclusion \supset always holds in (4.6).

Let $x^* := \sum_{j \in J} x_j^*$ with $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$. We thus have,

$$\begin{aligned} f(x) - \langle x^*, x \rangle &= \sum_{j \in J} (f_j(x) - \langle x_j^*, x \rangle) \\ &= - \sum_{j \in J} f_j^*(x_j^*) \\ &\leq \sup(\text{RD}_{x^*}) \\ &\leq \inf(\text{RP}_{x^*}) \\ &= -f^*(x^*) \\ &\leq f(x) - \langle x^*, x \rangle. \end{aligned}$$

Finally, $f(x) - \langle x^*, x \rangle = -f^*(x^*)$, that means $x^* \in \partial f(x)$.

We now prove the reverse inclusion \subset in (4.6).

Let $x^* \in \partial f(x)$. Then $x \in \partial f^*(x^*)$ and, by (3.2), $x \in \text{sol}(\text{RP}_{\bar{x}^*})$. By (4.5) and Corollary 4.1, there exists $(J, (x_j^*)_{j \in J}) \in \mathbb{F}(\bar{x}^*)$ such that $J \in S_f(x)$ and $x_j^* \in \partial f_j(x)$ for all $j \in J$. We thus have $x^* = \sum_{j \in J} x_j^* \in \sum_{j \in J} \partial f_j(x)$. \square

5 Robust sum of subaffine functions

Let $(A_i)_{i \in I}$ be a family of nonempty, w^* -closed convex subsets of X^* , $t_i \in \mathbb{R}$ for all $i \in I$ and the subaffine functions $f_i := \sigma_{A_i} - t_i$, $i \in I$. Then $(f_i)_{i \in I} \subset \Gamma(X)$ and we have $f_i^* := \delta_{A_i} + t_i$ and $\text{epi } f_i^* = A_i \times [t_i, +\infty[= A_i \times \{t_i\} + \{0_{X^*}\} \times \mathbb{R}_+$ for each $i \in I$. The robust sum f of this family is

$$f(x) = \sum_{i \in I}^R f_i(x) = \sup_{J \in \mathcal{F}(I)} \sum_{j \in J} [\sigma_{A_j}(x) - t_j], \forall x \in X$$

and the set \mathcal{A} defined by (2.2) now becomes

$$\mathcal{A} := \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} [A_j \times \{t_j\}] \right) + \{0_{X^*}\} \times \mathbb{R}_+. \quad (5.1)$$

Let us introduce the set-valued mapping

$$\mathbb{A} : \mathcal{F}(I) \rightrightarrows X^* \quad \text{such that} \quad \mathbb{A}(J) = \sum_{j \in J} A_j.$$

Then the problem $(\text{RP}_{\bar{x}^*})$ and its dual $(\text{RD}_{\bar{x}^*})$ write as

$$\inf(\text{RP}_{\bar{x}^*}) = \inf\{f(x) - \langle \bar{x}^*, x \rangle : x \in X\} = -f^*(\bar{x}^*)$$

and

$$\sup(\text{RD}_{\bar{x}^*}) = \sup \left\{ - \sum_{j \in J} f_j^*(x_j^*) : J \in \mathbb{A}^{-1}(\bar{x}^*) \right\} = - \inf \left\{ \sum_{j \in J} t_j : J \in \mathbb{A}^{-1}(\bar{x}^*) \right\},$$

and hence, the zero duality gap relation amounts to

$$f^*(\bar{x}^*) = \inf \left\{ \sum_{j \in J} t_j : J \in \mathbb{A}^{-1}(\bar{x}^*) \right\}.$$

We now briefly quote some remarkable properties on the duality and the convexity and closedness of the qualifying set \mathcal{A} :

- It is worth observing firstly that if $\mathbb{A}^{-1}(\bar{x}^*) = \emptyset$ (i.e., $\bar{x}^* \notin \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} A_j$), one has $\bar{x}^* \notin \text{dom } f^*$ and $\text{sup}(\text{RD}_{\bar{x}^*}) = -\infty$.

- In the case when $\text{dom } f \neq \emptyset$ (for instance, if $\sum_{i \in I} t_i \in \mathbb{R}$), Theorem 2.1 says that the *stable strong duality of the pair* (RP_{x^*}) - (RD_{x^*}) *holds*, i.e.,

$$f^*(x^*) = \min \left\{ \sum_{j \in J} t_j : J \in \mathbb{A}^{-1}(x^*) \right\}, \forall x^* \in \text{dom } f^* \quad (5.2)$$

if and only if the set

$$\mathcal{A} = \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} [A_j \times \{t_j\}] \right) + \{0_{X^*}\} \times \mathbb{R}_+ \text{ is } w^* \text{-closed and convex.} \quad (5.3)$$

- According to Lemma 2.1 and Example 2.1, we know that the set \mathcal{A} in (5.1) is convex if $0_{X^*} \in \bigcap_{i \in I} A_i \neq \emptyset$ and $\text{sup}_{i \in I} t_i \leq 0$. Moreover, the set \mathcal{A} is w^* -closed if

$$\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} (A_j \times \{t_j\}) \text{ is } w^* \text{-compact.}$$

On the primal attainment and the strong duality of the robust sum for subaffine functions (RP_{x^*}) , one has the following consequence of Theorem 3.1 and Lemma 2.1.

Proposition 5.1 *Assume that $0_{X^*} \in \bigcap_{i \in I} A_i$ and the robust sum $\sum_{i \in I}^R (\sigma_{A_i} - t_i)$ is proper and weakly inf-locally compact. Let $\bar{x}^* \in X^*$ be such that*

$$\overline{\text{cone}} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} A_j - \bar{x}^* \right) \text{ is a linear subspace of } X^*. \quad (5.4)$$

Then the optimal solution set of the problem

$$(\text{RP}_{\bar{x}^*}) \quad \inf_{x \in X} \left(\sum_{i \in I}^R (\sigma_{A_i}(x) - t_i) - \langle \bar{x}^*, x \rangle \right)$$

is the sum of a nonempty weakly compact set and a finitely dimensional linear subspace of X .

Applying Theorem 3.2 we get

Proposition 5.2 *Assume that $0_{X^*} \in \bigcap_{i \in I} A_i \neq \emptyset$ and $\sup_{i \in I} t_i \leq 0$, and there exists $i_0 \in I$ such that $\delta_{A_{i_0}}$ is Mackey quasicontinuous. Then for each $\bar{x}^* \in X^*$ satisfying (5.4) we have*

$$\min_{x \in X} \left(\sum_{i \in I}^R (\sigma_{A_i}(x) - t_i) - \langle \bar{x}^*, x \rangle \right) = \sup \left\{ - \sum_{j \in J} t_j : J \in \mathbb{A}^{-1}(\bar{x}^*) \right\}.$$

Proof. By Lemma 2.1 (Example 2.1) the set \mathcal{A} is convex and the function φ is convex, too (Remark 3.5). On the other hand, by Remark 3.6, the function φ is Mackey quasicontinuous. The conclusion follows from Theorem 3.2. \square

In finite dimension we have (as an immediate consequence of Proposition 5.2):

Proposition 5.3 *Let $(A_i)_{i \in I}$ be a family of closed convex subsets of \mathbb{R}^n such that $0_{X^*} \in \bigcap_{i \in I} A_i$. Assume that $\sup_{i \in I} t_i \leq 0$. Then for any $\bar{x}^* \in \text{ri} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} A_j \right)$ one has*

$$\min_{x \in X} \left(\sum_{i \in I}^R (\sigma_{A_i}(x) - t_i) - \langle \bar{x}^*, x \rangle \right) = \sup \left\{ - \sum_{j \in J} t_j : J \in \mathbb{A}^{-1}(\bar{x}^*) \right\}.$$

We end this section with a formula on the subdifferential of the robust sum $f = \sum_{i \in I}^R (\sigma_{A_i} - t_i)$. Let us recall that for each $x \in X$ one has, by definition,

$$S_f(x) = \left\{ J \in \mathcal{F}(I) : \sum_{j \in J} (\sigma_{A_j}(x) - t_j) = f(x) \right\}.$$

We observe also that

$$\partial \sigma_{A_i}(x) = \{ x^* \in A_i : \langle x^*, x \rangle = \sigma_{A_i}(x) \}$$

or, in other words,

$$\partial \sigma_{A_i}(x) = \text{argmax}_{A_i} \langle \cdot, x \rangle. \quad (5.5)$$

We then have:

Proposition 5.4 *Assume that $0_{X^*} \in \bigcap_{i \in I} A_i$, $\sup_{i \in I} t_i \leq 0$, f is proper, and the set*

$$\left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} [A_j \times \{t_j\}] \right) + \{0_{X^*}\} \times \mathbb{R}_+, \quad (5.6)$$

is w^ -closed regarding the set $\bigcup_{u \in X} \partial f(u)$. Then one has*

$$\partial f(x) = \bigcup_{J \in S_f(x)} \sum_{j \in J} \text{argmax}_{A_j} \langle \cdot, x \rangle, \forall x \in X.$$

Proof. Noting that the set in (5.6) is nothing but

$$\mathcal{A} = \bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi}(\sigma_{A_j} - t_j)^*,$$

which is convex. The conclusion now follows from Theorem 2.1, Corollary 4.3, and (5.5). \square

6 Approximate solutions to inconsistent convex inequality systems

In this section $(f_i)_{i \in I} \subset \Gamma(X)$. We consider the system

$$(S) \quad \{f_i(x) \leq 0, i \in I\},$$

that we assume to be inconsistent. Defining

$$f_0(x) := \sup_{i \in I} f_i(x), \tag{6.1}$$

we have $f_0(x) > 0$ for all $x \in X$.

The i -th residual of x is given by $f_i^+(x)$ and, in some sense, the infeasibility of x is measured by $\sup_{i \in I} f_i^+(x)$, that is, $f_0(x)$ too. We may also consider the cumulative infeasibility of x , namely the infinite sum $\sum_{i \in I} f_i^+(x)$ (see [9]). Since $f_0(x) > 0$ we know that $\sum_{i \in I} f_i^+$ coincides with the robust sum $\sum_{i \in I}^R f_i$ of the family $(f_i)_{i \in I}$ (see [8, Lemma 2.5]).

In formal terms, let us define a best ℓ_∞ -approximate solution of the inconsistent system (S) as an optimal solution to the problem

$$\inf_{x \in X} f_0(x) = \sup_{i \in I} f_i(x) = \sup_{i \in I} f_i^+(x)$$

and, similarly, a best ℓ_1 -approximate solution of (S) as an optimal solution to the problem

$$\inf_{x \in X} \sum_{i \in I} f_i^+(x) = \sum_{i \in I}^R f_i(x).$$

We denote by ℓ_∞ -sol(S) (resp., ℓ_1 -sol(S)) the set of best ℓ_∞ (resp., ℓ_1) approximate solutions of the inconsistent system (S).

In order to associate a suitable dual problem with $\inf_{x \in X} f_0(x)$ we define, as in [10], the *unit simplex* in the linear space $\mathbb{R}^{(I)}$ of real-valued functions $\lambda \in \mathbb{R}^I$ with finite *support set* $\text{supp } \lambda := \{i \in I : \lambda_i \neq 0\}$ as

$$S_I := \left\{ \lambda \in \mathbb{R}^{(I)} : \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I \right\}$$

and the *modified Lagrangian function* as $L : X \times S_I$ such that

$$L(x, \lambda) := \sum_{i \in \text{supp } \lambda} \lambda_i f_i(x), \forall (x, \lambda) \in X \times S_I.$$

Proposition 6.1 (Structure of ℓ_∞ -sol(S) and strong duality) *Assume that f_0 is proper and weakly inf-locally compact, and that $\overline{\text{co}} \bigcup_{i \in I} \text{dom } f_i^*$ is a linear subspace of X^* . Then $\ell_\infty - \text{sol}(S)$ is the sum of a nonempty convex compact set and a finitely dimensional linear subspace of X . Moreover, one has*

$$\inf_{x \in X} \sup_{i \in I} f_i(x) = \max \left\{ \inf_{x \in X} \sum_{i \in I} \lambda_i f_i(x) : \lambda \in S_I \right\}$$

if and only if

$$\bigcup_{\lambda \in S_I} \text{epi} \left(\sum_{i \in I} \lambda_i f_i \right)^* \text{ is } w^* - \text{closed regarding } \{0_{X^*}\} \times \mathbb{R}.$$

Proof. Since $f_0 \in \Gamma(X)$ one has $\ell_\infty - \text{sol}(S) = \partial f_0^*(0_{X^*})$. We intend to apply Lemma 3.1 for $g = f_0$ and $x^* = 0_{X^*}$. We have to make explicit the criterion (3.3) in terms of the conjugate of the data functions f_i . To this end consider the function $\Psi := \inf_{i \in I} f_i^*$. One has $\text{dom } \Psi = \bigcup_{i \in I} \text{dom } f_i^*$, $\Psi^* = f_0$ and, since $\text{dom } f_0 \neq \emptyset$, $f_0^* = \overline{\text{co}} \Psi$. Now, as in (3.8), we have $\overline{\text{co}} \text{dom } \Psi = \overline{\text{dom } \overline{\text{co}} \Psi}$ and, consequently,

$$\overline{\text{co}} \text{dom } f_0^* = \overline{\text{co}}(\text{dom}(\overline{\text{co}} \Psi)) = \overline{\text{co}} \text{co}(\text{dom } \Psi) = \overline{\text{co}} \left(\text{co} \bigcup_{i \in I} \text{dom } f_i^* \right).$$

The strong duality theorem is consequence of [10, Corollary 3.4]. \square

Observe that, if at least one of the functions f_i is weakly inf-locally compact, then f_0 is weakly inf-locally compact, too. The next corollary is an immediate consequence of Proposition 6.1.

Corollary 6.1 *Assume that $(f_i)_{i \in I} \subset \Gamma(\mathbb{R}^n)$, $\text{dom } f_0 \neq \emptyset$, and $0_{\mathbb{R}^n} \in \text{ri co} \left(\bigcup_{i \in I} \text{dom } f_i^* \right)$.*

Then $\ell_\infty - \text{sol}(S)$ is the sum of a nonempty convex compact set and a linear subspace of \mathbb{R}^n .

Example 6.1 *Let $\{\langle a_i, x \rangle \leq b_i, i \in I\}$ be an inconsistent linear system posed in \mathbb{R}^n . This is a particular case of system (S) above, with $f_i = \langle a_i, \cdot \rangle - b_i$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all $i \in I$. Denoting by 0_n the null vector in \mathbb{R}^n , by Corollary 6.1, if $\text{dom } f_0 \neq \emptyset$ and $0_n \in \text{ri co} \{a_i, i \in I\}$, then $\ell_\infty - \text{sol}(S)$ is the sum of a nonempty convex compact set and a linear subspace of \mathbb{R}^n ([9, Proposition 1(S)] only asserts that, under these assumptions, $\ell_\infty - \text{sol}(S) \neq \emptyset$). Moreover, since*

$$\bigcup_{\lambda \in S_I} \text{epi} \left(\sum_{i \in I} \lambda_i f_i \right)^* = \left\{ \sum_{i \in I} \lambda_i (a_i, b_i) : \lambda \in S_I \right\} + \{0_n\} \times \mathbb{R}_+,$$

the strong duality theorem becomes here

$$\inf_{x \in \mathbb{R}^n} \sup_{i \in I} (\langle a_i, x \rangle - b_i) = \max \left\{ \inf_{x \in \mathbb{R}^n} \sum_{i \in I} \lambda_i (\langle a_i, x \rangle - b_i) : \lambda \in S_I \right\},$$

if and only if

$$\left\{ \sum_{i \in I} \lambda_i (a_i, b_i) : \lambda \in S_I \right\} + \{0_n\} \times \mathbb{R}_+ \text{ is closed regarding } \{0_n\} \times \mathbb{R}_+.$$

Proposition 6.2 (Structure of ℓ_1 -sol (S) and strong duality) *Assume that the robust sum $\sum_{i \in I}^R f_i$ is proper, weakly inf-locally compact, and $\overline{\text{con}} \text{co} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{dom } f_j^* \right)$ is a linear subspace of X^* . Then, ℓ_1 -sol (S) is the sum of a nonempty convex weakly compact set and a finitely dimensional linear subspace of X . Moreover, one has*

$$\inf_{x \in X} \sum_{i \in I} f_i^+(x) = \max \left\{ - \sum_{j \in J} f_j^*(x_j^*) : J \in \mathcal{F}(I), (x_j^*)_{j \in J} \in (X^*)^J, \sum_{j \in J} x_j^* = 0_{X^*} \right\}$$

if and only if

$$\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} \text{epi } f_j^* \text{ is } w^* \text{-closed convex regarding } \{0_{X^*}\} \times \mathbb{R}.$$

Proof. It is direct consequence of Theorem 3.1 and Theorem 2.1 for $x^* = 0_{X^*}$, due to the relation $\sum_{i \in I}^R f_i = \sum_{i \in I} f_i^+$. \square

Example 6.2 *Consider again the linear system (S) in Example 6.1. By Proposition 6.2, if $\sum_{i \in I}^R (\langle a_i, \cdot \rangle - b_i)$ is proper and $0_n \in \text{ri} \left(\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} a_j \right)$, then ℓ_1 -sol (S) is the sum of a nonempty convex compact set and a finitely dimensional linear subspace of \mathbb{R}^n . Observe that, for each $(x_j)_{j \in J} \in (\mathbb{R}^n)^J$, one has*

$$\sum_{j \in J} f_j^*(x_j) = \sum_{j \in J} \left(\delta_{\{a_j\}}^*(x_j) + b_j \right) = \begin{cases} \sum_{j \in J} b_j, & \text{if } x_j = a_j, \forall j \in J, \\ +\infty, & \text{else.} \end{cases}$$

So, again by Proposition 6.2,

$$\inf_{x \in \mathbb{R}^n} \sum_{i \in I} (\langle a_i, x \rangle - b_i)^+ = \max \left\{ - \sum_{J \in \mathcal{F}(I)} b_j : J \in \mathcal{F}(I), \sum_{j \in J} a_j = 0_n \right\}$$

if and only if

$$\bigcup_{J \in \mathcal{F}(I)} \sum_{j \in J} (\{a_j\} \times [b_j, +\infty[) \text{ is closed convex regarding } \{0_n\} \times \mathbb{R}.$$

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