Duality for the robust sum of functions

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Abstract

In this paper we associate with an infinite family of real extended functions defined on a locally convex space, a sum, called robust sum, which is always well-defined. We also associate with that family of functions a dual pair of problems formed by the unconstrained minimization of its robust sum and the so-called optimistic dual. For such a dual pair, we characterize weak duality, zero duality gap, and strong duality, and their corresponding stable versions, in terms of multifunctions associated with the given family of functions and a given approximation parameter $\varepsilon \geq 0$ which is related to the ε-subdifferential of the robust sum of the family. We also consider the particular case when all functions of the family are convex, assumption allowing to characterize the duality properties in terms of closedness conditions.

Keywords Robust sum function · Weak duality · Zero duality · Strong duality · Stable duality theorems

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1 Introduction

Given a locally convex Hausdorff topological vector space X and an infinite family $(f_i)_{i\in I}$ of functions $f_i : X \longrightarrow \mathbb{R}_\infty := \mathbb{R}\cup \{+\infty\}$ for all $i \in I$ (situation in the sequel denoted as $(f_i)_{i\in I} \subset (\mathbb{R}_\infty)^X$) of objective proper functions, we are concerned with the uncertain problem of minimizing a finite but unknown sum of the objective functions f_i. Adopting the robust optimization approach under uncertainty (see [4], [7], [8], [13]), and taking the set $\mathcal{F}(I)$ of non-empty finite subsets of I as uncertainty set, the robust counterpart of this uncertain problem is

$$(\mathrm{RP}) \qquad \inf_{x\in X} \sup_{J\in\mathcal{F}(I)} \sum\nolimits_{i\in J} f_i(x). \tag{1.1}$$

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This kind of problem arises in situations where one must minimize the *robust-L_p (pseudo) norm* function $\|h\|_p$ defined on X by

$$\|h\|_p(x) := \left[\sup_{J\in\mathcal{F}(I)} \sum_{i\in J} |h_i(x)|^p\right]^{\frac{1}{p}} = \sup_{J\in\mathcal{F}(I)} \left[\sum_{i\in J} |h_i(x)|^p\right]^{\frac{1}{p}} \in \mathbb{R}_\infty,$$

for any infinite family $h = (h_i)_{i\in I}$ of functions $h_i : X \longrightarrow \mathbb{R}, i \in I$, and $p \geq 1$. Since the exact value of $\|h\|_p$ can hardly be computed, in practice it should be replaced by the maximum of $\left[\sum_{i\in J} |h_i|^p\right]^{\frac{1}{p}}$ for a sample of non-empty finite sets J picked at random from I. This way, $\left[\sum_{i\in J} |h_i|^p\right]^{\frac{1}{p}}$ can be interpreted as an uncertain function with uncertain parameter J ranging on the uncertainty set $\mathcal{F}(I)$.

As a first example, in the extension of the classical least squares linear regression model (see, e.g., [1, Subsection 3.2.1]) to the case of infinite point clouds $\{(t_i, s_i),\ i \in I\} \subset \mathbb{R}^2$, when the shape of the latter set suggests a linear dependence of magnitude s with respect to magnitude t, the problem consists in computing the ordinate at the origin, x_1, and the slope, x_2, of the line $s = x_1 + x_2 t$ better fitted to that set. To find it, one should minimize $\|h\|_2^2(x_1, x_2)$ on $\mathbb{R}^2$, where the $i-$th component of the residual function h, is $h_i(x_1, x_2) := x_1 + x_2 t_i - s_i,\ i \in I$. In the terminology of robust optimization, the uncertain objective function $\sum_{i\in J} |h_i(x)|^2$ at x represents the sum of squares error for the line $s = x_1 + x_2 t$ relative to the finite point cloud $\{(t_i, s_i),\ i \in J\}$, the worst-case objective function $\sup_{J\in\mathcal{F}(I)} \sum_{i\in J} |h_i(x)|^2$ is the least upper bound, for $J \in \mathcal{F}(I)$, of the errors corresponding to that line, and a robust optimal solution of (RP) is a best infinite regression line for the point cloud $\{(t_i, s_i),\ i \in I\}$.

A second example comes from the search of a best approximate solution to an inconsistent system $\{\langle a_i, x\rangle \leq b_i,\ i \in I\}$ in $\mathbb{R}^n$. Denote by $h(x)$ the residual of $x \in \mathbb{R}^n$, i.e., $h_i(x) := \max\{\langle a_i, x\rangle - b_i.0\},\ i \in I$. When I is finite, the minimization of the L_p norm of the residual vector $h(x)$ on $\mathbb{R}^n$ is a convex optimization problem [1, page 120] that can be reformulated as a linear programming problem for $p \in \{1, \infty\}$ by using the linearization technique described in [1, Subsubsection 1.1.5.4] or directly solved by means of ad hoc numerical methods for $p = 2$ (see [16] and references therein). Assuming that I is the union of a discrete set with a finite union of pairwise disjoint boxes as well as the continuity on these boxes of the function $i \longmapsto (a_i, b_i)$, [10] analyzes the minimization of the components of the residual function h, involving integrals whose existence is guaranteed by the continuity assumption. One can get rid of any assumption on I and the function $i \longmapsto (a_i, b_i)$ by considering the minimization of the robust pseudonorm function $\|h(x)\|_p$ for an arbitrary infinite set I, in which case an optimal solution of (RP) provides a best robust-L_p approximate solution of $\{\langle a_i, x\rangle \leq b_i,\ i \in I\}$.

The third example is an extension of the classical Markowitz [15] portfolio model, which is based on historical return data allowing to estimate the expected return and the expected variance (identified with the risk) of each portfolio. We now briefly describe Markowitz's model. The decision maker (DM) tries to invest, in an optimal way, a unit of capital into a finite set I of assets with expected return r_i, $i \in I$, and estimated covariance v_{ij} of the returns of assets $i, j \in I$. In the absence of other constraints, the

DM has to decide the amount x_i to be invested at asset $i \in I$, so that the expected return $\sum\limits_{i\in I} r_i x_i$ is maximized while the expected variance $\sum\limits_{i,j\in I} v_{ij} x_i x_j$ is minimized. In other words, she must solve the ordinary bi-objective problem

$$
\begin{array}{lll}
(\mathrm{P}_1) & \max & \sum\limits_{i\in I} r_i x_i \\
 & \min & \sum\limits_{i,j\in I} v_{ij} x_i x_j \\
 & \text{s.t.} & \sum\limits_{i\in I} x_i \leq 1, \\
 & & x_i \geq 0, i \in I.
\end{array}
$$

This model implicitly assumes a quadratic utility (or risk) function, but there exists a wide literature on utility functions in portfolio models [12]. Taking into account the almost unlimited number of existing assets in the global economy, it is natural to replace (P_1) by the following bi-objective infinite dimensional optimization problem involving robust sums of linear functions and quadratic forms:

$$
\begin{array}{lll}
(\mathrm{P}_2) & \min & \sup_{J\in\mathcal{F}(\mathbb{N})} \sum_{i\in J} (-r_i x_i) \\
 & \min & \sup_{K\in\mathcal{F}(\mathbb{N}^2)} \sum_{(i,j)\in K} v_{ij} x_i x_j \\
 & \text{s.t.} & \sup_{J\in\mathcal{F}(\mathbb{N})} \sum_{i\in J} x_i \leq 1, \\
 & & x_i \geq 0, i \in \mathbb{N}.
\end{array}
$$

It is worth observing that this third example involves a problem of minimizing a vector-valued robust sum function under some constraints. In this paper, however, we consider only unconstrained scalar problems. The class of vector robust sum problems with constraints will be considered in another work.

The aim of this paper is to establish some duality principles for the problem (RP) and to characterize in various ways the zero duality gap property. We call *robust sum* of the family $(f_i)_{i\in I} \subset (\mathbb{R}_\infty)^X$, represented by $\sum_{i\in I}^R f_i : X \longrightarrow \mathbb{R}_\infty$, the objective function of (RP), namely,

$$
\sum\nolimits_{i\in I}^R f_i(x) := \sup_{J\in\mathcal{F}(I)} \sum_{i\in J} f_i(x), \forall x \in X.
$$

The term "robust sum" is not new in the literature, but it has been only used in the framework of the uncertain optimization of finite sums (see, e.g., [2], [5]).

In the case where all functions f_i are non-negative,

$$
\sum\nolimits_{i\in I}^R f_i(x) = \sum_{i\in I} f_i(x) := \lim_{J\in\mathcal{F}(I)} \sum_{i\in J} f_i(x), \forall x \in X, \tag{1.2}
$$

where the limit is taken respect to the directed set $\mathcal{F}(I)$ ordered by the inclusion relation. The advantage of the robust sum $\sum_{i\in I}^R f_i$ in comparison with the *infinite sum* $\sum_{i\in I} f_i$ is that $\sum_{i\in I}^R f_i(x)$ is well defined for each $x \in X$ while $\sum_{i\in I} f_i(x)$ may not exist (see Remark 2.1 and Lemma 2.6 below). Formulas for the subdifferential of $\sum_{i\in I} f_i$ in the case that all functions f_i are continuous have been given in [18] and

[19, Proposition 2.3], while duality theorems on infinite sums of proper, convex and lower semicontinuous (lsc in short) functions can be found in [14, Section 3]. The mentioned subdifferential formulas and duality theorems for $\sum_{i\in I} f_i$ have been used in [19, Proposition 2.3] and [14, Section 5] to obtain error bounds for convex infinite systems and optimality conditions for convex infinite programs, respectively.

Throughout this paper we assume that all functions f_i, $i \in I$, are proper, as well as their robust sum $f := \sum_{i\in I}^{R} f_i$. The paper is organized as follows. Section 2 introduces the robust sum of an infinite family in $\mathbb{R}_\infty$ and analyzes its relationship with the infinite sum of the family. Sections 3, 4 and 5 provide results characterizing weak duality, zero duality gap, and strong duality, for the robust sum of a family of arbitrary functions, respectively, in terms of multifunctions associated with $(f_i)_{i\in I}$. Section 6 analyzes the robust sum under the assumption that $(f_i)_{i\in I}$ is a family of proper, lsc and convex functions; the main result of this section is Theorem 6.1, which characterizes the strong zero duality gap of f under a closedness assumption instead of ε-subdifferentials and epigraphs of the family of corresponding conjugate functions, as in [14, Theorem 3.2] for $\sum_{i\in I} f_i$. Example 6.1 shows that the properness of a family of continuous affine functions $(f_i)_{i\in I}$ does not imply the properness of its robust sum and illustrates the checkability of the conditions involved in the duality theorems. Finally, Section 7 provides a stable zero duality theorem for the infinite sum of proper, lsc, and non-negative convex functions (as in the above infinite regression problem).

2 Some rules for the robust sum

We associate with a given infinite family of extended real numbers $(a_i)_{i\in I}$ (situation in the sequel denoted as $(a_i)_{i\in I} \in (\mathbb{R}_\infty)^I$) its *robust sum*

$$\sum_{i\in I}^{R} a_i := \sup_{J\in\mathcal{F}(I)} \sum_{i\in J} a_i, \tag{2.1}$$

together with its *inferior and superior limits*,

$$\liminf_{J\in\mathcal{F}(I)} \sum_{i\in J} a_i := \sup_{J\in\mathcal{F}(I)} \inf_{J\subset K\in\mathcal{F}(I)} \sum_{i\in K} a_i,$$

and

$$\limsup_{J\in\mathcal{F}(I)} \sum_{i\in J} a_i := \inf_{J\in\mathcal{F}(I)} \sup_{J\subset K\in\mathcal{F}(I)} \sum_{i\in K} a_i,$$

respectively.

Lemma 2.1 *One has*

$$-\infty < \sup_{i\in I} a_i \leq \sum_{i\in I}^{R} a_i \leq +\infty, \tag{2.2}$$

and

$$-\infty < \liminf_{J\in\mathcal{F}(I)} \sum_{i\in J} a_i \leq \sum_{i\in I}^{R} a_i \leq +\infty. \tag{2.3}$$

Proof. Let $j \in I$. Setting $J = \{j\}$ in (2.1) we get

$$-\infty < a_j \leq \sum\nolimits_{i \in I}^{R} a_i \leq +\infty.$$

Taking the supremum over $j \in I$, (2.2) holds true.

Let $J \in \mathcal{F}(I)$. We have

$$-\infty \leq \inf_{K \in \mathcal{F}(I)} \sum_{i \in K} a_i \leq \sum_{i \in J} a_i \leq \sum\nolimits_{i \in I}^{R} a_i \leq +\infty.$$

Taking the supremum over $J \in \mathcal{F}(I)$, (2.3) holds true. ∎

We also define the *infinite sum* of the family $(a_i)_{i \in I}$ as

$$\sum_{i \in I} a_i := \lim_{J \in \mathcal{F}(I)} \sum_{i \in J} a_i,$$

provided that the unconditional limit $\lim\limits_{J \in \mathcal{F}(I)} \sum_{i \in J} a_i$ exists as a member of $\overline{\mathbb{R}}$, i.e.,

$$-\infty \leq \liminf_{J \in \mathcal{F}(I)} \sum_{i \in J} a_i = \limsup_{J \in \mathcal{F}(I)} \sum_{i \in J} a_i \leq +\infty.$$

In the case when $(a_i)_{i \in I} \in [0, +\infty]^I$, we have

$$0 \leq \sum\nolimits_{i \in I}^{R} a_i = \sum_{i \in I} a_i \leq +\infty.$$

For each $\theta \in \overline{\mathbb{R}}$ we consider $\theta^+ := \max\{\theta, 0\}$ and $\theta^- := \max\{-\theta, 0\}$.

Lemma 2.2 $\left(\sum_{i \in I}^{R} a_i\right)^+ = \sum_{i \in I}^{R} a_i^+ = \sum\limits_{i \in I} a_i^+$.

Proof. Since $a_i \leq a_i^+$ and $a_i^+ \geq 0$ for all $i \in I$, we have

$$\sum\nolimits_{i \in I}^{R} a_i \leq \sum\nolimits_{i \in I}^{R} a_i^+ = \sum_{i \in I} a_i^+.$$

Since $\sum_{i \in I} a_i^+ \geq 0$ we obtain $\left(\sum_{i \in I}^{R} a_i\right)^+ \leq \sum_{i \in I} a_i^+$. Let us prove the reverse inequality. Let $J \in \mathcal{F}(I)$ and $K_J := \{i \in J : a_i > 0\}$. If $K_J = \emptyset$ then $\sum_{i \in J} a_i^+ = 0 \leq \left(\sum_{i \in I}^{R} a_i\right)^+$. If, alternatively, $K_J \neq \emptyset$ then

$$0 \leq \sum_{i \in J} a_i^+ = \sum_{i \in K_J} a_i \leq \sum\nolimits_{i \in I}^{R} a_i \leq \left(\sum\nolimits_{i \in I}^{R} a_i\right)^+.$$

In both cases we have $\sum_{i\in J} a_i^+ \leq \left(\sum_{i\in I}^R a_i\right)^+$, and, since $J \in \mathcal{F}(I)$ is arbitrary, we obtain

$$\sum_{i\in I} a_i^+ = \sum_{i\in I}^R a_i^+ \leq \left(\sum_{i\in I}^R a_i\right)^+,$$

and the proof is complete. ∎

As an immediate consequence of Lemma 2.2 we have:

Lemma 2.3 *One has*

$$\sum_{i\in I}^R a_i \in \mathbb{R} \Longleftrightarrow \sum_{i\in I} a_i^+ < +\infty.$$

Lemma 2.4 *Next statements are equivalent:*
(i) $\sup_{i\in I} a_i \geq 0$.
(ii) $\sum_{i\in I}^R a_i \geq 0$.
(iii) $\sum_{i\in I}^R a_i = \sum_{i\in I} a_i^+$.

Proof. One has $[(i) \Longrightarrow (ii)]$ by Lemma 2.1 and $[(ii) \Longrightarrow (i)]$ by Lemma 2.2. Assume now that (i) does not hold, i.e., there exists $\varepsilon > 0$ such that $a_i \leq -\varepsilon$ for all $i \in I$. For each $J \in \mathcal{F}(I)$ we have $\sum_{i\in J} a_i \leq -\varepsilon \times \operatorname{card} J \leq -\varepsilon$. Since J is arbitrary we get $\sum_{i\in I}^R a_i \leq -\varepsilon$ and (iii) does not hold. So, $[(iii) \Longrightarrow (i)]$ and the proof is complete. ∎

Lemma 2.5 *One has*

$$\sum_{i\in I}^R a_i = \begin{cases} \sum_{i\in I} a_i^+, & \textit{if} \quad \sup_{i\in I} a_i \geq 0, \\ \sup_{i\in I} a_i, & \textit{if} \quad \sup_{i\in I} a_i \leq 0. \end{cases} \tag{2.4}$$

Proof. If $\sup_{i\in I} a_i \geq 0$, (2.4) holds by Lemma 2.4. Assume now that $\sup_{i\in I} a_i \leq 0$. By Lemma 2.1 we have just to check that $\sum_{i\in I}^R a_i \leq \sup_{i\in I} a_i$. Let $J \in \mathcal{F}(I)$. Picking $j \in J$, we have

$$\sum_{i\in J} a_i \leq a_j \leq \sup_{i\in I} a_i,$$

and, since J is arbitrary, we are done. ∎

Remark 2.1 *We note that $\sum_{i\in I}^R a_i$ always exists in $\mathbb{R}_\infty$ while $\sum_{i\in I} a_i$ may not exist in $\overline{\mathbb{R}}$. This is for instance the case when $I = \mathbb{N}$ and $a_i = (-1)^i$ or $a_i = \frac{(-1)^i}{i}$. In both cases we have $\sum_{i\in I} a_i^+ = +\infty$ and, by Lemma 2.3, $\sum_{i\in I}^R a_i = +\infty$, while there are subnets of $\left\{\sum_{i\in J} a_i\right\}_{J\in\mathcal{F}(I)}$ converging towards distinct limits, so that $\sum_{i\in I} a_i$ does not exist in $\overline{\mathbb{R}}$. However, in the case when $\sum_{i\in I}^R a_i \in \mathbb{R}$, $\sum_{i\in I} a_i$ does exist in $\mathbb{R}\cup\{-\infty\}$ as the next lemma shows.*

Lemma 2.6 *Assume that* $\sum_{i\in I}^{R} a_i \in \mathbb{R}$*. Then either* $\sum_{i\in I} a_i^- < +\infty$ *or* $\sum_{i\in I} a_i^- = +\infty$*, and in the first case it holds* $\sum_{i\in I} a_i \in \mathbb{R}$ *while in the latter one,* $\sum_{i\in I} a_i = -\infty$*.*

Proof. Since $\sum_{i\in I}^{R} a_i \in \mathbb{R}$, Lemma 2.3 says that $\sum_{i\in I} a_i^+ < +\infty$.
Assume that $\sum_{i\in I} a_i^- < +\infty$. Then,

$$\sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^- = \sum_{i\in I} \left(a_i^+ - a_i^-\right) = \sum_{i\in I} a_i \in \mathbb{R}.$$

Assume that $\sum_{i\in I} a_i^- = +\infty$ and let us prove that $\sum_{i\in I} a_i = -\infty$.
Let $r \in \mathbb{R}$ and $s := \sum_{i\in I} a_i^+ \in \mathbb{R}$. There exist $J_1, J_2 \in \mathcal{F}(I)$ such that,

$$\forall J \in \mathcal{F}(I),\ J_1 \subset J \implies \sum_{i\in J} a_i^+ \leq s+1,$$

$$\forall J \in \mathcal{F}(I),\ J_2 \subset J \Longrightarrow \sum_{i\in J} a_i^- \geq s+1-r.$$

Thus, for each $J \in \mathcal{F}(I)$ such that $J_1 \cup J_2 \subset J$, we have

$$\sum_{i\in J} a_i = \sum_{i\in J} a_i^+ - \sum_{i\in J} a_i^- \leq r,$$

which means that $\sum_{i\in I} a_i = -\infty$. ∎

Example 2.1 *Let* $I = \mathbb{N}$ *and, for each* $i \in I$*,*

$$a_i = \begin{cases} \frac{1}{i^2}, & \text{if } i \text{ is even,} \\ -\frac{1}{i}, & \text{if } i \text{ is odd.} \end{cases}$$

By Lemma 2.5 we have $\sum_{i\in I}^{R} a_i = \sum_{i\in I} a_i^+ = \frac{\pi^2}{24}$ *and, since* $\sum_{i\in I} a_i^- = +\infty$*, by Lemma 2.6, we have* $\sum_{i\in I} a_i = -\infty$*.*

3 Weak duality

We now introduce the notation that will be used in the rest of the paper. The topological dual space of X is denoted by X^*. We denote by 0_X and 0_X^* the null vector of X and X^*, respectively. The closure of a subset $A \subset X$ will be denoted by $\overline{A}$ and the same symbol will be used for the closure of a subset of the dual space X^*.

Given a function $h \in \overline{\mathbb{R}}^X$, its domain is the set $\operatorname{dom} h := \{x \in X : h(x) < +\infty\}$, its epigraph is $\operatorname{epi} h := \{(x,r) \in X \times \mathbb{R} : h(x) \leq r\}$, its strict epigraph is $\operatorname{epi}_s h := \{(x,r) \in X \times \mathbb{R} : h(x) < r\}$, and its Fenchel conjugate is the function $h^* \in \overline{\mathbb{R}}^{X^*}$ such that $h^*(x^*) := \sup\{\langle x^*, x\rangle - h(x) : x \in X\}$ for any $x^* \in X^*$. Moreover, the lsc hull of h is the function $\overline{h} \in \overline{\mathbb{R}}^X$ whose epigraph $\operatorname{epi} \overline{h}$ is the closure of $\operatorname{epi} h$ in $X \times \mathbb{R}$.

Given $\varepsilon \in \mathbb{R}$, we denote by $[h \leq \varepsilon] := \{x \in X : h(x) \leq \varepsilon\}$ the lower level set of h at level ε. The definition of the strict lower level set $[h < \varepsilon]$ is similar.

Given $\varepsilon \geq 0$, we define the $\varepsilon-$minimizers of h as

$$\varepsilon - \operatorname{argmin} h := \begin{cases} \left\{x \in X : h(x) \leq \inf\limits_X h + \varepsilon\right\}, & \text{if } \inf\limits_X h \in \mathbb{R}, \\ \emptyset, & \text{else.} \end{cases}$$

Given $a \in X$ and $\varepsilon \geq 0$, we denote by

$$\partial^\varepsilon h(a) := \begin{cases} \{x^* \in X^* \ : \ h(x) \geq h(a) + \langle x^*, x - a\rangle - \varepsilon, \forall x \in X\}, & \text{if } h(a) \in \mathbb{R}, \\ \emptyset, & \text{else,} \end{cases}$$

the $\varepsilon-$subdifferential of h at a. For $\varepsilon = 0$ one sets $\partial h(a)$ instead of $\partial^0 h(a)$. By definition, $\partial^\varepsilon h : X \rightrightarrows X^*$ is a multifunction whose inverse multifunction we denote by $M^\varepsilon h : X^* \rightrightarrows X$. For each $x^* \in X^*$ one has

$$M^\varepsilon h(x^*) = \begin{cases} \varepsilon - \operatorname{argmin}(h - x^*), & \text{if } h^*(x^*) \in \mathbb{R}, \\ \emptyset, & \text{else.} \end{cases}$$

The multifunction $M^\varepsilon h(\cdot)$ will be of a crucial importance in the paper. Notice that, with the rule $(+\infty) - (-\infty) = (-\infty) + (+\infty) = +\infty$, one has

$$x \in M^\varepsilon h(x^*) \Longleftrightarrow x^* \in \partial^\varepsilon h(x) \Longleftrightarrow h(x) + h^*(x^*) \leq \langle x^*, x\rangle + \varepsilon. \tag{3.1}$$

We are now turning back to the problem (RP) defined in (1.1) by an infinite family $(f_i)_{i\in I} \subset (\mathbb{R}_\infty)^X$ of proper functions with $f = \sum_{i\in I}^R f_i$, which is assumed to be proper as well. Note that as the functions f, f_i are proper, the conjugate functions f^*, f_i^*, $i \in I$, never take the value $-\infty$.

For each $x^* \in X^*$ consider the dual pair of problems

$$\begin{array}{ll} (\mathrm{RP}_{x^*}) & \inf\limits_{x\in X} [f(x) - \langle x^*, x\rangle], \\ (\mathrm{RD}_{x^*}) & \sup\limits_{\substack{J\in\mathcal{F}(I) \\ (x_i^*)_{i\in J}\in(X^*)^J \\ \sum_{i\in J} x_i^* = x^*}} -\sum\limits_{i\in J} f_i^*(x_i^*). \end{array}$$

It is clear that (RP) is nothing else but $(\mathrm{RP}_{0_{X^*}})$ and from now on, we will write (RP) and (RD) instead of $(\mathrm{RP}_{0_{X^*}})$ and $(\mathrm{RD}_{0_{X^*}})$, respectively. Note that (RD) is nothing but the optimistic dual problem of (RP).

Let us now introduce the function $\varphi : X^* \longrightarrow \overline{\mathbb{R}}$ defined as

$$\varphi(x^*) := \inf_{J\in\mathcal{F}(I)} \left\{ \sum_{i\in J} f_i^*(x_i^*) : (x_i^*)_{i\in J} \in (X^*)^J, \sum_{i\in J} x_i^* = x^* \right\}, \forall x^* \in X^*.$$

Then it is clear that for each $x^* \in X^*$,

$$\inf(\mathrm{RP}_{x^*}) = -f^*(x^*) \ \text{ and } \ \sup(\mathrm{RD}_{x^*}) = -\varphi(x^*).$$

Proposition 3.1 (Weak duality) *For each $x^* \in X^*$ we have*

$$-\infty \leq \sup(\mathrm{RD}_{x^*}) \leq \inf(\mathrm{RP}_{x^*}) < +\infty,$$

or, equivalently,

$$-\infty < f^*(x^*) \leq \varphi(x^*) \leq +\infty. \tag{3.2}$$

Proof. Since f is proper, its conjugate does not take the value $-\infty$. Let $x^* \in X^*$, $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$, $\sum_{i\in J} x_i^* = x^*$, and $x \in X$. One has to check that $\langle x^*, x\rangle - f(x) \leq \sum_{i\in J} f_i^*(x_i^*)$. By definition of f_i^* we have

$$\sum_{i\in J} f_i^*(x_i^*) \geq \sum_{i\in J} (\langle x_i^*, x\rangle - f_i(x)) = \langle x^*, x\rangle - \sum_{i\in J} f_i(x) \geq \langle x^*, x\rangle - f(x),$$

as by the definition of f, $\sum_{i\in J} f_i(x) \leq f(x)$ for all $x \in X$. ∎

4 Zero duality gap

Definition 4.1 *We say that the robust sum problem* (RP_{x^*}) *has zero duality gap at a given $x^* \in X^*$ if*

$$\inf(\mathrm{RP}_{x^*}) = \sup(\mathrm{RD}_{x^*}), \tag{4.1}$$

(or equivalently, $f^*(x^*) = \varphi(x^*)$). If (4.1) holds at each $x^* \in X^*$, we will say that (RP_{x^*})*, seen as a parametric problem, has stable zero duality gap.*

The characterization of the zero duality gap involves two mutually inverse multifunctions associated with the given family of functions $(f_i)_{i\in I}$ with robust sum f.

For each $\alpha \geq 0$ let us define $S_f^\alpha : X \rightrightarrows \mathcal{F}(I)$ such that

$$S_f^\alpha(x) := \begin{cases} \left\{J \in \mathcal{F}(I) : f(x) \leq \sum_{i\in J} f_i(x) + \alpha\right\}, & \text{if } x \in \operatorname{dom} f, \\ \emptyset, & \text{else,} \end{cases}$$

and $T_f^\alpha : \mathcal{F}(I) \rightrightarrows X$ such that

$$T_f^\alpha(J) := \left\{x \in \operatorname{dom} f : f(x) \leq \sum_{i\in J} f_i(x) + \alpha\right\}.$$

So, for any $(x, J) \in X \times \mathcal{F}(I)$ we have

$$J \in S_f^\alpha(x) \Longleftrightarrow x \in T_f^\alpha(J).$$

We first characterize the zero duality gap at a fixed $x^* \in X^*$ and then the same property on the whole of X^*, i.e., the so-called stable zero duality gap.

4.1 Zero duality at a given linear functional

Let us now put in light a necessary condition for the robust sum problem to have zero duality gap at a given $x^* \in X^*$. So, assume that

$$f^*(x^*) \geq \varphi(x^*) \in \mathbb{R} \tag{4.2}$$

(see the weak duality (3.2)) and let $\varepsilon \geq 0$. For any $\eta > 0$ and $x \in M^\varepsilon f(x^*)$, one has

$$f(x) - \langle x^*, x\rangle \leq -f^*(x^*) + \varepsilon < -\varphi(x^*) + \varepsilon + \eta.$$

Then, by definition of φ, there exist $J \in \mathcal{F}(I)$ and $(x_i^*)_{i\in J} \in (X^*)^J$ such that $\sum_{i\in J} x_i^* = x^*$ and

$$f(x) - \langle x^*, x\rangle \leq -\sum_{i\in J} f_i^*(x_i^*) + \varepsilon + \eta.$$

This last inequality can be rewritten as

$$\left[f(x) - \sum_{i\in J} f_i(x)\right] + \sum_{i\in J} \left[f_i(x) + f_i^*(x_i^*) - \langle x_i^*, x\rangle\right] \leq \varepsilon + \eta.$$

All the above brackets being non-negative, there exists $\left(\alpha, (\varepsilon_i)_{i\in J}\right) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon + \eta$ and

$$f(x) - \sum_{i\in J} f_i(x) \leq \alpha \text{ and } f_i(x) + f_i^*(x_i^*) - \langle x_i^*, x\rangle \leq \varepsilon_i, i \in J.$$

In other words,

$$x \in T_f^\alpha(J) \text{ and } x \in M^{\varepsilon_i} f(x_i^*), \forall i \in J. \tag{4.3}$$

Hence we have quoted that for any $x \in M^\varepsilon f(x^*)$ and any $\eta > 0$, there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$ and $\left(\alpha, (\varepsilon_i)_{i\in J}\right) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $\sum_{i\in J} x_i^* = x^*$, $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon + \eta$ and (4.3) holds.

Thus, if (4.2) holds, then, for any $\varepsilon \geq 0$ we have $M^\varepsilon f(x^*) \subset N^\varepsilon f(x^*)$, where the multifunction $N^\varepsilon f : X^* \rightrightarrows X$ is defined, for each $x^* \in X^*$, by

$$N^\varepsilon f(x^*) = \bigcap_{\eta>0} \bigcup_{J\in\mathcal{F}(I)} \bigcup_{\substack{(x_i^*)_{i\in J} \in (X^*)^J \\ \sum_{i\in J} x_i^* = x^*}} \bigcup_{\substack{(\alpha,(\varepsilon_i)_{i\in J}) \in \mathbb{R}_+\times\mathbb{R}_+^J \\ \alpha + \sum_{i\in J}\varepsilon_i = \varepsilon+\eta}} \left(T_f^\alpha(J) \bigcap \left(\bigcap_{i\in J} M^{\varepsilon_i} f(x_i^*)\right)\right). \tag{4.4}$$

Since $M^\varepsilon f(x^*) = \emptyset$ when $f^*(x^*) \notin \mathbb{R}$, we can state:

Lemma 4.1 *If* (RP_{x^*}) *has zero duality gap at* $x^* \in X^*$*, then*

$$M^\varepsilon f(x^*) \subset N^\varepsilon f(x^*), \forall \varepsilon \geq 0.$$

It turns out that the reverse inclusion always holds:

Lemma 4.2 *For any $x^* \in X^*$, it holds*

$$M^\varepsilon f(x^*) \supset N^\varepsilon f(x^*), \forall \varepsilon \geq 0.$$

If, moreover, $f^*(x^*) = \varphi(x^*)$, then

$$M^\varepsilon f(x^*) = N^\varepsilon f(x^*), \forall \varepsilon \geq 0.$$

Proof. Let $\varepsilon \geq 0$ and $x \notin M^\varepsilon f(x^*)$. If $f(x) = +\infty$, then, by definition of $T_f^\alpha(J)$, we have $T_f^\alpha(J) = \emptyset$ for any $(\alpha, J) \in \mathbb{R}_+ \times \mathcal{F}(I)$. Consequently, $x \notin N^\varepsilon f(x^*) = \emptyset$.

Assume now that $f(x) \in \mathbb{R}$. Since $x \notin M^\varepsilon f(x^*)$, there exists $\eta > 0$ such that

$$f(x) + f^*(x^*) - \langle x^*, x \rangle > \varepsilon + \eta. \tag{4.5}$$

Let us suppose now that $x \in N^\varepsilon f(x^*)$. Then, there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i \in J} \in (X^*)^J$, and $\left(\alpha, (\varepsilon_i)_{i \in J}\right) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $\sum_{i \in J} x_i^* = x^*$, $\alpha + \sum_{i \in J} \varepsilon_i = \varepsilon + \eta$, and $x \in T_f^\alpha(J) \cap \left(\bigcap_{i \in J} M^{\varepsilon_i} f(x_i^*)\right)$. By definition of φ we thus have $f^*(x^*) \leq \varphi(x^*) \leq \sum_{i \in J} f_i^*(x_i^*)$ and, so,

$$\begin{aligned} f(x) + f^*(x^*) - \langle x^*, x \rangle &\leq \left[f(x) - \sum_{i \in J} f_i(x) \right] + \sum_{i \in J} \left[f_i(x) + f_i^*(x_i^*) - \langle x_i^*, x \rangle \right] \\ &\leq \alpha + \textstyle\sum_{i \in J} \varepsilon_i = \varepsilon + \eta, \end{aligned}$$

which contradicts (4.5). So, $x \notin N^\varepsilon f(x^*)$ and we are done. ∎

We now observe that the converse statement in Lemma 4.1 always holds. In fact, we can prove a little more:

Lemma 4.3 *Let $x^* \in X^*$ and assume that there exists $\overline{\varepsilon} > 0$ such that*

$$M^\varepsilon f(x^*) \subset N^\varepsilon f(x^*), \forall \varepsilon \in \left]0, \overline{\varepsilon}\right[.$$

Then $f^*(x^*) = \varphi(x^*)$.

Proof. We have just to check that $\varphi(x^*) \leq f^*(x^*)$. This is obvious if $f^*(x^*) = +\infty$. Assume now that $f^*(x^*) \in \mathbb{R}$. Let us assume that $\varphi(x^*) > f^*(x^*)$.

There exists $\varepsilon \in \left]0, \overline{\varepsilon}\right[$ such that

$$\varphi(x^*) > f^*(x^*) + 3\varepsilon. \tag{4.6}$$

Let us pick $x \in M^\varepsilon f(x^*)$, which is non-empty since $f^*(x^*) \in \mathbb{R}$. By hypothesis $x \in N^\varepsilon f(x^*)$ and, by (4.4), with $\eta = \varepsilon$, there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i \in J} \in (X^*)^J$, and

$\left(\alpha,(\varepsilon_i)_{i\in J}\right)\in\mathbb{R}_+\times\mathbb{R}_+^J$ such that $\sum_{i\in J}x_i^*=x^*$, $\alpha+\sum_{i\in J}\varepsilon_i=2\varepsilon$, $f(x)-\sum_{i\in J}f_i(x)\leq\alpha$, and $f_i(x)+f_i^*(x_i^*)\leq\langle x_i^*,x\rangle+\varepsilon_i$, for all $i\in J$. We thus have

$$\begin{aligned}-f^*(x^*)&\leq f(x)-\langle x^*,x\rangle\\&=\left[f(x)-\sum_{i\in J}f_i(x)\right]+\sum_{i\in J}\left[f_i(x)+f_i^*(x_i^*)-\langle x_i^*,x\rangle\right]-\sum_{i\in J}f_i^*(x_i^*)\\&\leq\alpha+\sum_{i\in J}\varepsilon_i-\sum_{i\in J}f_i^*(x_i^*)\\&\leq 2\varepsilon-\varphi(x^*),\end{aligned}$$

which contradicts (4.6). So, $\varphi(x^*)\leq f^*(x^*)$, which together with the weak duality shows that $\varphi(x^*)=f^*(x^*)$ and we are done. ∎

We now state the main result of this section.

Theorem 4.1 (Zero duality gap) *Let $(f_i)_{i\in I}$ be a family of proper functions with $f=\sum_{i\in I}^R f_i$ proper, and let $x^*\in X^*$. The next statements are equivalent:*
(i) (RP_{x^*}) *has zero duality gap,*
(ii) $M^\varepsilon f(x^*)=N^\varepsilon f(x^*),\forall\varepsilon\geq 0$,
(iii) There exists $\overline{\varepsilon}>0$ such that

$$M^\varepsilon f(x^*)=N^\varepsilon f(x^*),\forall\varepsilon\in\left]0,\overline{\varepsilon}\right[,$$

(iv) There exists $\overline{\varepsilon}>0$ such that

$$M^\varepsilon f(x^*)\subset N^\varepsilon f(x^*),\forall\varepsilon\in\left]0,\overline{\varepsilon}\right[.$$

Proof. Lemma 4.2 says that $[(i)\Longrightarrow(ii)]$, while $[(ii)\Longrightarrow(iii)]$ and $[(iii)\Longrightarrow(iv)]$ are obvious. Finally, $[(iv)\Longrightarrow(i)]$ is Lemma 4.3. ∎

4.2 Stable zero duality gap

We now characterize stable zero duality gap for the robust sum problem. To this end, let us introduce $\Pi^\varepsilon f:=(N^\varepsilon f)^{-1}$, i.e., the inverse multifunction of $N^\varepsilon f$. One has $\Pi^\varepsilon f:X\rightrightarrows X^*$ and, for any $(x^*,x)\in X^*\times X$,

$$x^*\in\Pi^\varepsilon f(x)\Longleftrightarrow x\in N^\varepsilon f(x^*).$$

The next explicit formula holds:

Lemma 4.4 *For any $(x,\varepsilon)\in X\times\mathbb{R}_+$ we have*

$$\Pi^\varepsilon f(x)=\bigcap_{\eta>0}\ \bigcup_{0\leq\alpha\leq\varepsilon+\eta}\ \bigcup_{J\in S_f^\alpha(x)}\ \bigcup_{\substack{(\varepsilon_i)_{i\in J}\in\mathbb{R}_+^J\\ \sum_{i\in J}\varepsilon_i=\varepsilon+\eta-\alpha}}\ \sum_{i\in J}\partial^{\varepsilon_i}f_i(x).\tag{4.7}$$

Proof. By (4.4) we have $x^* \in \Pi^\varepsilon f(x)$ if and only if for any $\eta > 0$ there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$, and $\left(\alpha, (\varepsilon_i)_{i\in J}\right) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $\sum_{i\in J} x_i^* = x^*$, $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon + \eta$, $x \in T_f^\alpha(J)$ (i.e., $J \in S_f^\alpha(x)$), and $x \in \bigcap_{i\in J} M^{\varepsilon_i} f_i(x_i^*)$ (i.e., $x_i^* \in \partial^{\varepsilon_i} f_i(x)$ for all $i \in J$). This exactly means that x^* belongs to the set in the right hand side of (4.7). ∎

Lemma 4.5 *For any* $(x, \varepsilon) \in X \times \mathbb{R}_+$ *one has*

$$\Pi^\varepsilon f(x) \subset \partial^\varepsilon f(x).$$

Proof. Let $x^* \in \Pi^\varepsilon f(x)$. We have $x \in N^\varepsilon f(x^*)$ and, by Lemma 4.2, $x \in M^\varepsilon f(x^*)$, that means $x^* \in \partial^\varepsilon f(x)$. ∎

We now characterize the stable zero duality gap for the robust sum problem.

Theorem 4.2 (Stable zero duality gap) *Let* $(f_i)_{i\in I}$ *be a family of proper functions with* $f = \sum_{i\in I}^R f_i(x)$ *proper. The next statements are equivalent:*
(i) $f^*(x^*) = \varphi(x^*), \forall x^* \in X^*$,
(ii) $\partial^\varepsilon f(x) = \Pi^\varepsilon f(x), \forall (x, \varepsilon) \in X \times \mathbb{R}_+$,
(iii) *There exists* $\overline{\varepsilon} > 0$ *such that*

$$\partial^\varepsilon f(x) = \Pi^\varepsilon f(x), \forall (x, \varepsilon) \in X \times]0, \overline{\varepsilon}[,$$

(iv) *There exists* $\overline{\varepsilon} > 0$ *such that*

$$\partial^\varepsilon f(x) \subset \Pi^\varepsilon f(x), \forall (x, \varepsilon) \in X \times]0, \overline{\varepsilon}[.$$

Proof. $[(i) \Longrightarrow (ii)]$ Let $(x, \varepsilon) \in X \times \mathbb{R}_+$. We know that $x^* \in \partial^\varepsilon f(x)$ if and only if $x \in M^\varepsilon f(x^*)$. By Theorem 4.1, $M^\varepsilon f(x^*) = N^\varepsilon f(x^*)$. So,

$$x^* \in \partial^\varepsilon f(x) \Longleftrightarrow x \in N^\varepsilon f(x^*) \Longleftrightarrow x^* \in \Pi^\varepsilon f(x)$$

and (ii) holds.

$[(ii) \Longrightarrow (iii)]$ and $[(iii) \Longrightarrow (iv)]$ are obvious.

$[(vi) \Longrightarrow (i)]$ Let $(x^*, \varepsilon) \in X^* \times]0, \overline{\varepsilon}[$ and $x \in M^\varepsilon f(x^*)$. We have $x^* \in \partial^\varepsilon f(x)$ and, by Theorem 4.1, $x^* \in \Pi^\varepsilon f(x)$, that means $x \in N^\varepsilon f(x^*)$. So, $M^\varepsilon f(x^*) \subset N^\varepsilon f(x^*)$ for any $\varepsilon \in]0, \overline{\varepsilon}[$, and, again by Theorem 4.1, $f^*(x^*) = \varphi(x^*)$. ∎

5 Strong duality

Definition 5.1 *We say that the robust sum problem* (RP_{x^*}) *has a strong zero duality gap at a given* $x^* \in X^*$ *if there exist* $J \in \mathcal{F}(I)$ *and* $(x_i^*)_{i\in J} \in (X^*)^J$ *such that* $x^* = \sum_{i\in J} x_i^*$ *and*

$$\inf(\mathrm{RP}_{x^*}) = -f^*(x^*) = -\sum_{i\in J} f_i^*(x_i^*) = \sup(\mathrm{RD}_{x^*}). \tag{5.1}$$

If the above condition holds at each $x^* \in X^*$ we will say that (RP_{x^*}) has a stable strong zero duality gap.

To characterize the strong zero duality gap of the robust sum problem (RP_{x^*}), let us fix some notation first. Given $x^* \in X^*$, $\varepsilon \geq 0$, $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$, define

$$B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*) := \begin{cases} \bigcup\limits_{\substack{(\alpha,(\varepsilon_i)_{i\in J})\in\mathbb{R}_+\times\mathbb{R}_+^J \\ \alpha+\sum_{i\in J}\varepsilon_i=\varepsilon}} T_f^{\alpha}(J) \bigcap \left(\bigcap_{i\in J} M^{\varepsilon_i} f(x_i^*)\right), & \text{if} \sum\limits_{i\in J} x_i^* = x^*, \\ \emptyset, & \text{else.} \end{cases}$$

Theorem 5.1 (Strong zero duality gap) *Let $(f_i)_{i\in I}$ be a family of proper functions with $f = \sum_{i\in I}^R f_i$ proper, and let $x^* \in X^*$. The next statements are equivalent:*
(i) The robust sum problem (RP_{x^*}) has a strong zero duality gap,
(ii) $\exists J \in \mathcal{F}(I)$, $\exists (x_i^*)_{i\in J} \in (X^*)^J$: $M^{\varepsilon} f(x^*) = B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*)$, $\forall \varepsilon \geq 0$,
(iii) There exist $\overline{\varepsilon} > 0$, $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$ such that

$$M^{\varepsilon} f(x^*) = B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*), \quad \forall \varepsilon \in \left]0, \overline{\varepsilon}\right[, \tag{5.2}$$

Proof. $[(i) \Longrightarrow (ii)]$ By the very definition of $B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*)$, (4.4), and Lemma 4.2 we have

$$B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*) \subset N^{\varepsilon} f(x^*) \subset M^{\varepsilon} f(x^*).$$

Let $x \in M^{\varepsilon} f(x^*)$. By (i) there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$ such that $\sum_{i\in J} x_i^* = x^*$ and

$$\sum_{i\in J} f_i^*(x_i^*) = f^*(x^*) \leq \langle x^*, x\rangle - f(x) + \varepsilon.$$

Consequently,

$$\sum_{i\in J} \Big[f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle\Big] + \Big[f(x) - \sum_{i\in J} f_i(x)\Big] \leq \varepsilon.$$

Since all the above brackets are non negative, there exist $(\alpha, (\varepsilon_i)_i) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $f(x) - \sum_{i\in J} f_i(x) \leq \alpha$, that means $x \in T_f^{\alpha}(J)$, $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon$, and for each $i \in J$,

$$f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle \leq \varepsilon_i,$$

that means $x \in \cap_{i\in J} M^{\varepsilon_i} f_i(x_i^*)$. So $x \in B^{\varepsilon}_{(J,(x_i^*)_{i\in J})} f(x^*)$ and (ii) holds.

$[(ii) \Longrightarrow (iii)]$ is obvious.

$[(iii) \Longrightarrow (i)]$ Assume that (iii) holds. So, there exist $\overline{\varepsilon} > 0$, $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$ such that (5.2) holds. Let us first prove that $\sum_{i\in J} f_i^*(x_i^*) \leq f^*(x^*)$. Assume the contrary, i.e., there exists $\varepsilon > 0$, that we can choose $\varepsilon < \overline{\varepsilon}$, such that

$$f^*(x^*) + \varepsilon < \sum_{i\in J} f_i^*(x_i^*). \tag{5.3}$$

We have $f^*(x^*) \in \mathbb{R}$. Picking $x \in M^\varepsilon f(x^*)$ which is non-empty, we have $x \in B^\varepsilon_{(J,(x_i^*)_{i\in J})} f(x^*)$ and hence, there exist $(\alpha, (\varepsilon_i)_i) \in \mathbb{R}_+ \times \mathbb{R}_+^J$ such that $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon$, $\sum_{i\in J} x_i^* = x^*$ and $x \in T_f^\alpha(J) \cap (\cap_{i\in J} M^{\varepsilon_i} f_i(x_i^*))$. Then

$$\begin{aligned}
\sum_{i\in J} f_i^*(x_i^*) &\le \sum_{i\in J} [\langle x_i^*, x\rangle - f_i(x) + \varepsilon_i] = \langle x^*, x\rangle - \sum_{i\in J} f_i(x) + \sum_{i\in J} \varepsilon_i \\
&\le \langle x^*, x\rangle - f(x) + \alpha + \sum_{i\in J} \varepsilon_i = \langle x^*, x\rangle - f(x) + \varepsilon \\
&\le f^*(x^*) + \varepsilon,
\end{aligned}$$

which contradicts (5.3). We then have

$$\varphi(x^*) \le \sum_{i\in J} f_i^*(x_i^*) \le f^*(x^*) \le \varphi(x^*).$$

So, $\varphi(x^*) = \sum_{i\in J} f_i^*(x_i^*) = f^*(x^*)$ with $\sum_{i\in J} x_i^* = x^*$, that means that (i) holds. ∎

In order to characterize the stable strong zero duality gap for the robust sum problem (RP_{x^*}), let us introduce, for each $\varepsilon \ge 0$, the set-valued mapping $N_s^\varepsilon f : X^* \rightrightarrows X$ defined by

$$N_s^\varepsilon f(x^*) := \bigcup_{\substack{J\in\mathcal{F}(I), \left(x_i^*\right)_{i\in J}\in (X^*)^J \\ \sum_{i\in J} x_i^* = x^*}} B^\varepsilon_{(J,(x_i^*)_{i\in J})} f(x^*), \ \forall x^* \in X^*,$$

and its inverse $\Pi_s^\varepsilon f : X \rightrightarrows X^*$. For each $(x, x^*) \in X \times X^*$ one has

$$x^* \in \Pi_s^\varepsilon f\,(x) \Longleftrightarrow x \in N_s^\varepsilon f\,(x^*).$$

More explicitly one has straightforwardly, for each $x \in X$,

$$\Pi_s^\varepsilon f\,(x) = \bigcup_{0\le\alpha\le\varepsilon} \ \bigcup_{J\in S_f^\alpha(x)} \ \bigcup_{\substack{(\varepsilon_i)_{i\in J}\in\mathbb{R}_+^J \\ \sum_{i\in J}\varepsilon_i = \varepsilon-\alpha}} \sum_{i\in J} \partial^{\varepsilon_i} f_i\,(x),$$

where $S_f^\alpha\,(x) = \{J \in \mathcal{F}(I) : \sum_{i\in J} f_i(x) + \alpha \ge f(x) \in \mathbb{R}\}$ as in Section 4. We have

$$N_s^\varepsilon f(x^*) \subset (N^\varepsilon f)(x^*) \subset (M^\varepsilon f)(x^*),$$

and, passing to the inverse multivalued mappings,

$$(\Pi_s^\varepsilon f)(x) \subset (\Pi^\varepsilon f)(x) \subset \partial^\varepsilon f(x), \forall x \in X, \forall \varepsilon \ge 0. \tag{5.4}$$

Theorem 5.2 (Stable strong zero duality gap) *Let $(f_i)_{i\in I}$ be a family of proper functions with $f = \sum_{i\in I}^R f_i$ proper. The next statements are equivalent:*
(i) The robust sum problem (RP_{x^*}) has stable strong zero duality gap,
(ii) $\partial^\varepsilon f\,(x) = \Pi_s^\varepsilon f\,(x)\,, \forall\,(x,\varepsilon) \in X \times \mathbb{R}_+$,
(iii) $\exists \bar\varepsilon > 0$: $\partial^\varepsilon f\,(x) = \Pi_s^\varepsilon f\,(x)\,, \forall\,(x,\varepsilon) \in X \times [0, \bar\varepsilon]\,.$

Proof. $[(i) \Longrightarrow (ii)]$ We only have to prove the inclusion "$\subset$" in (ii). So, let $x^* \in \partial^\varepsilon f(x)$. By (i), there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$ such that $\sum_{i\in J} x_i^* = x^*$ and

$$\sum_{i\in J} f_i^*(x_i^*) = f^*(x^*) \leq \langle x^*, x\rangle - f(x) + \varepsilon.$$

Consequently,

$$\sum_{i\in J} \Big[f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle \Big] + \Big[f(x) - \sum_{i\in J} f_i^*(x_i^*) \Big] \leq \varepsilon,$$

and there exist $((\varepsilon_i)_{i\in J}, \alpha) \in \mathbb{R}^J \times \mathbb{R}$ such that $f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle \leq \varepsilon_i$ for each $i \in J$, $f(x) - \sum_{i\in J} f_i(x) \leq \alpha$, and $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon$. We thus have $x^* = \sum_{i\in J} x_i^* \in \sum_{i\in J} \partial^\varepsilon f_i(x)$ with $0 \leq \alpha \leq \varepsilon$, $J \in S_f^\alpha(x)$, and $\sum_{i\in J} \varepsilon_i = \varepsilon - \alpha$, that means $x^* \in (\Pi_s^\varepsilon f)(x)$, and (ii) holds.

$[(ii) \Longrightarrow (iii)]$ is obvious.

$[(iii) \Longrightarrow (i)]$ Let $x^* \in X^*$. If $f^*(x^*) = +\infty$ then $\varphi(x^*) = +\infty$ and f has obviously a strong zero duality gap at x^*. Since $\operatorname{dom} f \neq \emptyset$ we have $f^*(x^*) \neq -\infty$ and it remains to consider the case $f^*(x^*) \in \mathbb{R}$. Pick $x \in M^{\bar\varepsilon} f(x^*)$ which is non-empty, and set $\varepsilon := f^*(x^*) + f(x) - \langle x^*, x\rangle$. One has $\varepsilon \in [0, \bar\varepsilon]$, $x^* \in \partial^\varepsilon f(x)$ and, by (iii), there exist $\alpha \in [0, \varepsilon]$, $J \in S_f^\alpha(x)$, $(x_i^*)_{i\in J} \in (X^*)^J$, $(\varepsilon_i)_{i\in J} \in \mathbb{R}_+^J$ such that $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon$, $\sum_{i\in J} x_i^* = x^*$, and $x_i^* \in \partial^{\varepsilon_i} f_i(x)$ for each $i \in J$. We thus have

$$\begin{aligned}
\varphi(x^*) &\leq \sum_{i\in J} f_i^*(x_i^*) \leq \sum_{i\in J} \Big[\langle x_i^*, x\rangle - f_i(x) + \varepsilon_i \Big] \\
&= \langle x^*, x\rangle - \sum_{i\in J} f_i(x) + \sum_{i\in J} \varepsilon_i \\
&\leq \langle x^*, x\rangle - f(x) + \alpha + \sum_{i\in J} \varepsilon_i \\
&= \langle x^*, x\rangle - f(x) + \varepsilon = f^*(x^*) \leq \varphi(x^*).
\end{aligned}$$

Consequently, $f^*(x^*) = \sum_{i\in J} f_i^*(x_i^*)$ with $J \in \mathcal{F}(I)$ and $\sum_{i\in J} x_i^* = x^*$, that means f^* has strong zero duality gap at x^* and we are done. ∎

6 Duality for the robust sum of closed convex functions

Denote by $\operatorname{co} A$ the convex hull of $A \subset X^* \times \mathbb{R}$, by $\overline{A}$ its closure w.r.t. the w^*-topology and by $\overline{\operatorname{co}} A$ its w^*-closed convex hull. We also denote by $\Gamma(X)$ the set of all proper convex lsc functions on X. In this section we assume that

$$(f_i)_{i\in I} \subset \Gamma(X) \text{ and } \operatorname{dom} f \neq \emptyset \tag{6.1}$$

(recall that $f = \sum_{i\in I}^R f_i$). We thus have $f \in \Gamma(X)$.

Let us introduce the set

$$\mathcal{A} := \bigcup_{J \in \mathcal{F}(I)} \sum\nolimits_{i \in J} \operatorname{epi} f_i^*,$$

which is related with the function

$$\varphi(x^*) := \inf_{J \in \mathcal{F}(I)} \left\{ \sum_{i \in J} f_i^*(x^*) : (x_i^*)_{i \in J} \in (X^*)^J, \sum_{i \in J} x_i^* = x^* \right\}, \forall x^* \in X^*,$$

by the (easily checkable) double inclusion

$$\operatorname{epi}_s \varphi \subset \mathcal{A} \subset \operatorname{epi} \varphi. \tag{6.2}$$

Thus,

$$\overline{\operatorname{co}} \mathcal{A} = \overline{\operatorname{co}} \operatorname{epi} \varphi. \tag{6.3}$$

Lemma 6.1 *Assume that (6.1) holds. Then* $\varphi^* = f$ *and* $\operatorname{epi} f^* = \overline{\operatorname{co}} \mathcal{A}$.

Proof. We have $\varphi = \inf\limits_{J \in \mathcal{F}(I)} (\square_{i \in J} f_i^*)$, where

$$\square_{i \in J} f_i^*(x^*) := \inf \left\{ \sum_{i \in J} f_i^*(x_i^*) : \sum_{i \in J} x_i^* = x^* \right\}, \forall x^* \in X^*,$$

is the infimal convolution of the finite family of functions $\{f_i^*, i \in J\}$. So,

$$\varphi^* = \sup_{J \in \mathcal{F}(I)} (\square_{i \in J} f_i^*)^* = \sup_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i^{**} = \sup_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i = f.$$

For the second statement, one has $f^* = \varphi^{**}$ and, since f^* is proper, $\operatorname{epi} f^* = \operatorname{epi} \varphi^{**} = \overline{\operatorname{co}} \operatorname{epi} \varphi = \overline{\operatorname{co}} \mathcal{A}$ (the last equality follows from (6.3)). ∎

To go further let us recall the following notions (see, e.g., [3], [6], and [9]).

Definition 6.1 *A subset $A \subset X^* \times \mathbb{R}$ is said to be closed (respectively, closed convex) regarding another subset $B \subset X^* \times \mathbb{R}$ if $B \cap \overline{A} = B \cap A$ (respectively, $B \cap \overline{\operatorname{co}} A = B \cap A$).*

Theorem 6.1 (Strong zero duality gap under convexity) *Assume that (6.1) holds and let $x^* \in X^*$. The next statements are equivalent:*
(i) The robust sum problem (RP_{x^*}) *has a strong zero duality gap,*
(ii) $\mathcal{A}$ is closed convex regarding $\{x^*\} \times \mathbb{R}$.

Proof. Assume that $f^*(x^*) = +\infty$. By Lemma 6.1, we have $(\{x^*\} \times \mathbb{R}) \cap \overline{\operatorname{co}} \mathcal{A} = \emptyset$ and (ii) holds. By Proposition 3.1, $\varphi(x^*) = +\infty$ and (i) holds too. So, in this case, both statements (i) and (ii) hold. Thus, we can assume that $f^*(x^*) < +\infty$.

$[(i) \implies (ii)]$ Let $r \in \mathbb{R}$ be such that $(x^*, r) \in \overline{\operatorname{co}} \mathcal{A}$. By Lemma 6.1 we have $f^*(x^*) \leq r$ and, by (i), there exist $J \in \mathcal{F}(I)$ and $(x_i^*)_{i \in J} \in (X^*)^J$ such that $\sum_{i \in J} x_i^* = x^*$ and

$f^*(x^*) = \sum_{i\in J} f_i^*(x_i^*) \le r$. From this last inequality, there exists $(r_i)_{i\in J} \in \mathbb{R}^J$ such that $f_i^*(x_i^*) \le r_i$, for all $i \in J$, and $\sum_{i\in J} r_i = r$. It follows that

$$(x^*, r) = \sum_{i\in J} (x_i^*, r_i) \in \sum_{i\in J} \operatorname{epi} f_i^* \in \mathcal{A}.$$

$[(ii) \Longrightarrow (i)]$ Since f is proper we have $r := f^*(x^*) \in \mathbb{R}$ and, by Lemma 6.1, $(x^*, r) \in \overline{\operatorname{co}}\mathcal{A}$. Then, by (ii), $(x^*, r) \in \mathcal{A}$ and there exist $J \in \mathcal{F}(I)$, $(x_i^*)_{i\in J} \in (X^*)^J$, and $(r_i)_{i\in J} \in \mathbb{R}^J$ such that $\sum_{i\in J} x_i^* = x^*$, $\sum_{i\in J} r_i = r$, and $f_i^*(x_i^*) \le r_i$ for all $i \in J$. Then, again by Proposition 3.1, we have

$$\varphi(x^*) \le \sum_{i\in J} f_i^*(x_i^*) \le \sum_{i\in J} r_i = r = f^*(x^*) \le \varphi(x^*),$$

that means that (i) holds. ∎

Since $\mathcal{A}$ is closed convex if and only if it is closed convex regarding $\{x^*\} \times \mathbb{R}$ for all $x^* \in X^*$, we have:

Corollary 6.1 (Stable strong zero duality gap) *Assume that (6.1) holds. The next statements are equivalent:*
(i) *The robust sum problem* (RP_{x^*}) *has stable strong zero duality gap,*
(ii) $\mathcal{A}$ *is closed and convex.*

We now consider the simple, but non-trivial case that $(f_i)_{i\in I}$ is a family of affine functions with a proper robust sum f.

Example 6.1 *Let*

$$f_i = \langle a_i^*, \cdot\rangle - t_i,\ (a_i^*,\ t_i) \in X^* \times \mathbb{R},\ \forall i \in I,$$

be a family of continuous affine functions. According to Remark 2.1, if $a_i^* = 0_X^*$ for all $i \in I$, we may have $f = \sum_{i\in I}^R (-t_i)_{i\in I} = +\infty$ or not depending on the given family $(t_i)_{i\in I}$ of real numbers; in particular, by Lemma 2.5, f is finite whenever $\inf_{i\in I} t_i \ge 0$. So, to ensure the properness of $f = \sum_{i\in I}^R f_i$, we suppose the existence of $\overline{x} \in X$ and $M \in \mathbb{R}$ such that

$$\sum_{i\in J} (\langle a_i^*, \overline{x}\rangle - t_i) \le M,\ \forall J \in \mathcal{F}(I). \tag{6.4}$$

For each $i \in I$, we have $f_i^* = \delta_{a_i^*} + t_i$, where $\delta_{a_i^*} : X^* \longrightarrow \mathbb{R}\cup\{+\infty\}$ represents the indicator function of a_i^*, i.e., $\delta_{a_i^*}(x^*) = 0$, if $x^* = a_i^*$, and $\delta_{a_i^*}(x^*) = +\infty$, otherwise. Defining $A : \mathcal{F}(I) \longrightarrow X^*$ such that $A(J) = \sum_{i\in J} a_i^*$, the function φ writes

$$\varphi(x^*) := \inf_{J\in A^{-1}(x^*)} \sum_{i\in J} t_i,\ \forall x^* \in X^*.$$

The robust sum problem (RP_{x^*}) *has a zero duality gap means that*

$$\inf_{x\in X}\sup_{J\in\mathcal{F}(I)}\sum_{i\in J}\left(\langle a_i^*-x^*,x\rangle-t_i\right)=\sup_{J\in A^{-1}(\overline{x}^*)}-\sum_{i\in J}t_i. \tag{6.5}$$

We note that, given $\varepsilon\geq 0$,

$$M^\varepsilon f_i(x^*)=\begin{cases}X, & \text{if } x^*=a_i^*,\\ \emptyset, & \textit{else.}\end{cases}$$

Consequently, from (4.4),

$$N^\varepsilon f(x^*)=\bigcap_{\eta>0}\bigcup_{J\in A^{-1}(\overline{x}^*)}T_f^{\varepsilon+\eta}(J). \tag{6.6}$$

By Theorem 4.1, (6.5) holds if and only if

$$M^\varepsilon f(\overline{x}^*)=\bigcap_{\eta>0}\bigcup_{J\in A^{-1}(\overline{x}^*)}T_f^{\varepsilon+\eta}(J),\ \forall\varepsilon\geq 0.$$

By (6.6) one has $x^*\in\left(N^\varepsilon f\right)^{-1}(x)$ *if and only if for each* $\eta>0$ *there exists* $J\in\mathcal{F}(I)$ *such that* $x^*\in A(J)$ *and* $J\in S_f^{\varepsilon+\eta}(x)$, *that means*

$$x^*\in\bigcap_{\eta>0}\bigcup_{J\in S_f^{\varepsilon+\eta}(x)}\sum_{i\in J}a_i^*.$$

Consequently, from Theorem 4.2, (6.5) holds for each $x^*\in X^*$ *if and only if*

$$\partial^\varepsilon f(x)=\bigcap_{\eta>0}\bigcup_{J\in S_f^{\varepsilon+\eta}(x)}\sum_{i\in J}a_i^*,\ \forall(x,\varepsilon)\in X\times\mathbb{R}_+.$$

Regarding the closedness criteria in Theorem 6.1 and Corollary 6.1, observe that

$$\begin{aligned}\mathcal{A}&=\bigcup_{J\in\mathcal{F}(I)}\sum_{i\in J}\operatorname{epi}f_i^*\\&=\bigcup_{J\in\mathcal{F}(I)}\left[\left\{\sum_{i\in J}(a_i^*,t_i)\right\}+\{0_{X^*}\}\times\mathbb{R}_+\right]\end{aligned}$$

is the union of infinitely many vertical closed half-lines.

It is worth observing that in case all functions are linear (i.e., $t_i=0$ *for all* $i\in I$*),* $\mathcal{A}$ *is closed (convex, respectively) if and only if* $\left\{\sum_{i\in J}a_i^*: J\in\mathcal{F}(I)\right\}$ *is closed (convex). When* I *is countable (as in the robust sums of linear functions in the third example* *of the introduction),* $\left\{\sum\limits_{i\in J} a_i^* : J \in \mathcal{F}(I)\right\}$ *is countable too, so that it cannot be convex. Finally, in the simplest case that all functions are constants (i.e.,* $a_i^* = 0_{X^*}$ *for all* $i \in I$ *and, according to (6.4),* $\theta := \sum_{i\in I}^R -t_i \in \mathbb{R}$*), we have* $\{0_{X^*}\}\times]-\theta, +\infty[\subset \mathcal{A} \subset \{0_{X^*}\} \times [-\theta, +\infty[$ *and either* $\mathcal{A} = \{0_{X^*}\}\times]-\theta, +\infty[$ *or* $\mathcal{A} = \{0_{X^*}\} \times [-\theta, +\infty[$*. So,* $\mathcal{A}$ *is convex. However,* $\mathcal{A}$ *is closed if and only if there exists* $J \in \mathcal{F}(I)$ *such that* $\theta = \sum\limits_{i\in J} -t_i$*.*

7 Duality for the infinite sum of non-negative convex functions and related situations

The case that the functions f_i, $i \in I$, are non-negative presents many specificities. For example, in such a case the robust sum coincides with the infinite sum, i.e.,

$$f(x) = \sum\nolimits_{i\in I}^R f_i(x) = \lim_{J\in\mathcal{F}(I)} \sum_{i\in J} f_i(x) = \sum_{i\in I} f_i(x), \forall x \in X.$$

As in Section 2, the limit is taken with respect to the directed set $\mathcal{F}(I)$ ordered by the inclusion relation. We have the next important convexity properties.

Lemma 7.1 *Assume that* $f_i \geq 0$ *for each* $i \in I$*. Then the set* $\mathcal{A} = \bigcup\limits_{J\in\mathcal{F}(I)} \sum\limits_{i\in J} \operatorname{epi} f_i^*$ *and the function* φ *are convex.*

Proof. Let $(x^*, r), (y^*, s) \in \mathcal{A}$ and $t \in [0,1]$. There exist $J, K \in \mathcal{F}(I)$ such that $(x^*, r) \in \sum_{j\in J} \operatorname{epi} f_j^*$ and $(y^*, s) \in \sum_{k\in K} \operatorname{epi} f_k^*$.

Let $l \in I$. As $f_l \geq 0$, we have $f_l^*(0_{X^*}) \leq 0$, that is $(0_{X^*}, 0) \in \operatorname{epi} f_l^*$. Let $L := J \cup K \in \mathcal{F}(I)$. Since $(0_{X^*}, 0) \in \operatorname{epi} f_l^*$ for all $l \in L$, $(x^*, r), (y^*, s) \in \sum_{l\in L} \operatorname{epi} f_l^*$ which is a convex subset of $\mathcal{A}$. Thus, $(1-t)(x^*, r) + t(y^*, s) \in \mathcal{A}$ and $\mathcal{A}$ is convex.

The convexity of φ is a consequence of (6.2). In fact, we have

$$\varphi(x^*) = \inf\{r \in \mathbb{R} : (x^*, r) \in \mathcal{A}\}, \forall x^* \in X^*,$$

which is a convex functions thanks to the convexity of $\mathcal{A}$. ∎

In what follows we assume that

$$(f_i)_{i\in I} \subset \Gamma(X),\ f = \sum\nolimits_{i\in I}^R f_i \text{ is proper, and } \mathcal{A} = \bigcup_{J\in\mathcal{F}(I)} \sum\nolimits_{i\in J} \operatorname{epi} f_i^* \text{ is convex.} \quad (7.1)$$

Lemma 7.2 *Assume that (7.1) holds. Then* $f^* = \overline{\varphi}$ *(the* w^**-lsc hull of* φ*).*

Proof. By Lemma 6.1, we have $\varphi^{**} = f^*$. As shown in the proof of Lemma 7.1, φ is convex due to the convexity of $\mathcal{A}$. Since f is proper, one has $\operatorname{dom} \varphi^* \neq \emptyset$ and, consequently, $\overline{\varphi} = \varphi^{**} = f^*$. ∎

Lemma 7.3 *Assume that (7.1) holds. Then for any $x \in X$ and any $\varepsilon > 0$, we have*

$$\partial^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)}.$$

Proof. If $f(x) = +\infty$, then $\partial^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)} = \emptyset$. Assume now $f(x) \in \mathbb{R}$. By Lemma 7.2, $f^* = \overline{\varphi}$ and it now follows from (3.1) that

$$\partial^{\varepsilon} f(x) = [\overline{\varphi} - \langle \cdot, x \rangle + f(x) \leq \varepsilon] = \left[\overline{\varphi - \langle \cdot, x \rangle + f(x)} \leq \varepsilon\right]. \tag{7.2}$$

As $\varphi^*(x) = f(x)$ (by Lemma 6.1), we have $0 = f(x) - \varphi^*(x) = \inf_{X^*} \{\varphi - \langle \cdot, x \rangle + f(x)\} < \varepsilon$. By [11, Lemma 1.1] (applies to the function $\varphi - \langle \cdot, x \rangle + f(x)$) we have $\left[\overline{\varphi - \langle \cdot, x \rangle + f(x)} \leq \varepsilon\right] = \overline{[\varphi - \langle \cdot, x \rangle + f(x) < \varepsilon]}$. Taking (7.2) into account, we have

$$\partial^{\varepsilon} f(x) = \overline{[\varphi - \langle \cdot, x \rangle + f(x) < \varepsilon]}.$$

Now it is straightforward to check that $[\varphi - \langle \cdot, x \rangle + f(x) < \varepsilon] \subset \Pi_s^{\varepsilon} f(x)$, and hence,

$$\partial^{\varepsilon} f(x) = \overline{[\varphi - \langle \cdot, x \rangle + f(x) < \varepsilon]} \subset \overline{\Pi_s^{\varepsilon} f(x)}.$$

It now follows from (5.4) and Lemma 4.5,

$$\Pi_s^{\varepsilon} f(x) \subset \Pi^{\varepsilon} f(x) \subset \partial^{\varepsilon} f(x) \subset \overline{\Pi_s^{\varepsilon} f(x)}.$$

Since $\partial^{\varepsilon} f(x)$ is w^*-closed, we get $\partial^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)}$. ∎

Theorem 7.1 (Stable zero duality gap under convexity) *Assume that (7.1) holds. The next statements are equivalent:*
(i) The robust sum problem (RP_{x^*}) *has stable zero duality gap,*
(ii) $\Pi^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)}, \ \forall x \in X, \ \forall \varepsilon > 0,$
(iii) There exists $\overline{\varepsilon} > 0$ such that

$$\Pi^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)}, \ \forall (x, \varepsilon) \in X \times]0, \overline{\varepsilon}[,$$

(iv) There exists $\delta > 0$ such that

$$\partial^{\varepsilon} f(x) \subset \Pi_s^{\varepsilon\delta} f(x), \forall (x, \varepsilon) \in X \times]0, +\infty[.$$

Proof. The equivalence of (i), (ii) and (iii) follows from Theorem 4.2 and Lemma 7.3.

$[(i) \Longrightarrow (iv)]$ By Theorem 4.2 we have $\partial^{\varepsilon} f(x) = \Pi^{\varepsilon} f(x)$. Now

$$\Pi^{\varepsilon} f(x) = \bigcap_{\eta > 0} \Pi_s^{\varepsilon + \eta} f(x) \subset \Pi_s^{2\varepsilon} f(x)$$

and (iv) holds with $\delta = 2$.

$[(iv) \Longrightarrow (i)]$ Assume that (i) does not hold and let $\delta > 0$. We will show that there exist $x \in X$ and $\varepsilon > 0$ such that

$$\overline{\Pi_s^{\varepsilon} f(x)} \nsubseteq \Pi_s^{\varepsilon\delta} f(x). \tag{7.3}$$

Since (i) does not hold, there exist $x^* \in X^*$ and $\varepsilon > 0$ such that $f^*(x^*) + \varepsilon\delta < \underline{\varphi(x^*)}$. Pick $\overline{x} \in \partial^{\varepsilon} f^*(x^*)$ (which is non-empty since $\varepsilon > 0$). We have $x^* \in \partial^{\varepsilon} f(\overline{x}) = \overline{\Pi_s^{\varepsilon} f(\overline{x})}$. Assume that $x^* \in \Pi_s^{\varepsilon\delta} f(\overline{x})$. Then, exist $\alpha \in [0, \varepsilon\delta]$, $J \in S_f^{\alpha}(\overline{x})$, $(\varepsilon_i)_{i\in J} \in \mathbb{R}_+^J$, and $x_i^* \in \partial^{\varepsilon_i} f_i(\overline{x})$ for all $i \in J$, such that $\alpha + \sum_{i\in J} \varepsilon_i = \varepsilon\delta$ and $\sum_{i\in J} x_i^* = x^*$. Then

$$\begin{aligned}
\varphi(x^*) &\leq \sum_{i\in J} f_i^*(x_i^*) \\
&\leq \sum_{i\in J} (\langle x_i^*, \overline{x}\rangle - f_i(\overline{x}) + \varepsilon_i) \\
&= \langle x^*, \overline{x}\rangle - \sum_{i\in J} f_i(\overline{x}) + \varepsilon\delta - \alpha \\
&\leq \langle x^*, \overline{x}\rangle - f(\overline{x}) + \alpha + \varepsilon\delta - \alpha \\
&\leq f^*(x^*) + \varepsilon\delta < \varphi(x^*),
\end{aligned}$$

which contradicts $f^*(x^*) + \varepsilon\delta < \varphi(x^*)$. So $x^* \notin \Pi_s^{\varepsilon\delta} f(\overline{x})$, (7.3) is proved and the proof is complete. ∎

Corollary 7.1 *Assume that (7.1) holds and $\Pi_s^{\varepsilon} f(x)$ is w^*-closed for each $x \in X$ and $\varepsilon > 0$. Then the robust sum problem* (RP_{x^*}) *has stable zero duality gap.*

Proof. Under the assumption, it follows from Lemma 7.3 that $\partial^{\varepsilon} f(x) = \overline{\Pi_s^{\varepsilon} f(x)} = \Pi_s^{\varepsilon} f(x)$, which means that statement (iv) of Theorem 7.1 holds with $\delta = 1$. ∎

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