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Harmonic balance approach to the periodic solutions of the (an)harmonic relativistic oscillator

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ABSTRACT

The first-order harmonic balance method via the first Fourier coefficient is used to construct two approximate frequency-amplitude relations for the relativistic oscillator for which the nonlinearity (anharmonicity) is a relativistic effect due to the time line dilation along the world line. Making a change of variable, a new nonlinear differential equation is obtained and two procedures are used to approximately solve this differential equation. In the first the differential equation is rewritten in a form that does not contain a square-root expression, while in the second the differential equation is solved directly. The approximate frequency obtained using the second procedure is more accurate than the frequency obtained with the first due to the fact that, in the second procedure, application of the harmonic balance method produces an infinite set of harmonics, while in the first procedure only two harmonics are produced. Both approximate frequencies are valid for the complete range of oscillation amplitudes, and excellent agreement of the approximate frequencies with the exact one are demonstrated and discussed. The discrepancy between the first-order approximate frequency obtained by means of the second procedure and the exact frequency never exceeds 1.6%. We also obtained the approximate frequency by applying the second order harmonic balance method and in this case the relative error is as low 0.31% for all the range of values of amplitude of oscillation A.

Keywords: Nonlinear oscillations; Relativistic oscillator; Harmonic balance method; Approximate frequency.

1. Introduction

The study of nonlinear oscillators is of great interest in engineering and physical sciences and many analytical techniques have been developed for solving the second order nonlinear differential equations that govern their motion [1]. It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximate than a numerical one of a given nonlinear oscillatory system [2]. There is a large variety of approximate methods for the determination of solutions of nonlinear second order dynamical systems including perturbation [3], standard and modified Lindstedt-Poincaré [4-6], variational [7], variational iteration [8], homotopy perturbation [9-12], harmonic balance [13-17] methods, etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [2] and [18]. In this paper we apply the first-order harmonic balance method to obtain analytic approximate solutions for the relativistic oscillator. This is a procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

When the energy of a simple harmonic oscillator is such that the velocities become relativistic, the simple harmonic motion (linear oscillations) at low energy becomes anharmonic (nonlinear oscillations) at high energy [19]. Due to this fact we have considered the parentheses around the "an" in the title of this paper. Then, the strength of the nonlinearity increases as the total relativistic energy increases, and at the non-relativistic limit the oscillator becomes linear. Mickens [20] showed that all the solutions to the relativistic (an)harmonic oscillator are periodic and determined a method for calculating analytical approximations to its solutions. Mickens considered the first-order harmonic balance method, but he did not apply the technique correctly and the first analytical approximate frequency he obtained is not the correct one.

2. Nonlinear differential equation for the relativistic oscillator

The governing non-dimensional nonlinear differential equation of motion for the relativistic oscillator is [20]

$$\frac{d^2x}{dt^2}+\left[1-\left(\frac{dx}{dt}\right)^2\right]^{3/2}x=0 \tag{1}$$

where x and t are dimensionless variables. The even power term in Eq. (1), $(dx/dt)^2$, acts like the powers of coordinates in that it does not cause a damping of the amplitude of oscillations with time. Therefore, Eq. (1) is an example of a generalized conservative system [1]. At the limit when $(dx/dt)^2 << 1$, Eq. (1) becomes $(d^2x/dt^2)+x\approx 0$ the oscillator is linear and the proper time τ $(d\tau=\sqrt{1-(dx/dt)^2}\,dt)$ becomes equivalent to the coordinate time t to this order.

Introducing the phase space variable (x,y), Eq. (1) can be written as follows

$$\frac{dx}{dt}=y, \qquad \frac{dy}{dt}=-(1-y^2)^{3/2}x \tag{2}$$

and the trajectories in phase space are given by solutions to the first order, ordinary differential equation

$$\frac{dy}{dx}=-\frac{(1-y^2)^{3/2}x}{y} \tag{3}$$

As Mickens pointed out, since the physical solutions of both Eq. (1) and Eq. (3) are real, the phase space has a "strip" structure [20], i.e.,

$$-\infty < x < +\infty \qquad \text{and} \qquad -1 < y < +1 \tag{4}$$

Then unlike the usual non-relativistic harmonic oscillator, the relativistic oscillator is bounded in the y variable. This is due to the fact that the nondimensional variable y is related with the relativistic relation $\beta = v/c$, where v is the velocity of the particle and c the velocity of light. In the relativistic case, the condition $-c < v(t) < +c$ must be met, and so we obtain $-1 < y(t) < +1$. Mickens proved that all the trajectories to Eq. (3) are closed in the open region of phase space given by Eq. (4) and then all the physical solutions to Eq. (1) are periodic. However, unlike the usual (non-relativistic) harmonic oscillator, the relativistic (an)harmonic oscillator contains higher-order multiples of the fundamental frequency.

In order to apply the harmonic balance method, we make a change of variable, $y \to u$, such that $-\infty < u < +\infty$, as follows

$$y = \frac{u}{\sqrt{1+u^2}} \tag{5}$$

and the corresponding second order nonlinear differential equation for u is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{u}{\sqrt{1+u^2}} = 0 \tag{6}$$

We consider the following initial conditions in Eq. (7)

$$u(0) = B \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0 \tag{7}$$

Eq. (6) is an example of conservative nonlinear oscillatory system having irrational form for the restoring force. This is a conservative nonlinear oscillatory system and all the motions corresponding to Eq. (6) are periodic [20], the system will oscillate symmetric bounds [-B, B], and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude B.

The main objective of this paper is to solve Eq. (6) by applying the first-order harmonic balance method and to compare the approximate frequency obtained with the exact one and with another approximate frequency obtained by applying the method of harmonic balance to the same oscillatory system but rewriting Eq. (6) in a way suggested previously by Mickens [20]. Comparing with the approximate solution obtained by this last procedure, the approximate frequency derived here is more accurate with respect to exact solution. The errors of the resulting frequency are reduced and the maximum relative error is less than 1.6% for the complete range of oscillation amplitudes, including the limiting cases of amplitude approaching zero and infinity.

3. Solution method

3.1.- First adaptation of harmonic balance method

Eq. (6) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. Eq. (6) can be rewritten in a form that does not contain the square-root expression

$$(1+u^2)\left(\frac{\mathrm{d}^2u}{\mathrm{d}t^2}\right)^2 = u^2 \tag{8}$$

It is possible to solve Eq. (8) by applying the method of harmonic balance. Following the lowest order harmonic balance method, a reasonable and simple initial approximation satisfying conditions in Eq. (7) can be taken as

$$u(t) = B\cos\omega t \tag{9}$$

The angular frequency of the oscillator is ω, which is unknown to be further determined. The corresponding period of the nonlinear oscillation is given by $T = 2\pi/\omega$. Both the periodic solution $u(t)$ and frequency ω (thus period T) depends on B. Substitution of Eq.

(9) into Eq. (8), and expanding and simplifying the resulting expression gives

$$\frac{1}{2}\omega^4 + \frac{3}{8}\omega^4 B^2 - \frac{1}{2} + \left(\frac{1}{6}\omega^4 B^2 - \frac{1}{2}\right) + (\text{higher}-\text{order harmonics}) = 0 \qquad (10)$$

and setting the coefficient of the resulting term $\cos(0\omega t)$ (the lowest harmonic) equal to zero gives the first analytical approximate frequency ω_{a1} as a function of B

$$\omega_{a1}(B) = \left(1 + \frac{3}{4}B^2\right)^{-1/4} \qquad (11)$$

which is valid for the whole range of values of B. By applying the first order homotopy perturbation method to Eq. (8) the same approximate frequency is obtained [21] and this is due to the fact that in both methods the first Fourier coefficient is obtained. We can conclude that this is a general result for conservative oscillators. The approximate frequency in Eq. (11) is the correct one when the harmonic balance method is applied to Eq. (8), and not the frequency obtained in Ref. [20].

3.2.- Second adaptation of harmonic balance method

As we pointed out previously, the main objective of this paper is to solve Eq. (6) instead of Eq. (8) by applying the harmonic balance method. Substitution of Eq. (9) into Eq. (6) gives

$$-B\omega^2 \cos\omega t + \frac{B\cos\omega t}{\sqrt{1+\cos^2 \omega t}} = 0 \qquad (12)$$

The power-series expansion of $u/\sqrt{1+u^2}$ is

$$\frac{u}{\sqrt{1+u^2}} = u + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!}{2^{2n-1}n!(n-1)!}u^{2n+1} \qquad (13)$$

Substituting Eq. (13) into Eq. (12) and taking into account Eq. (9) gives

$$(-\omega^2+1)\cos\omega t+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!}{2^{2n-1}n!(n-1)!}B^{2n}\cos^{2n+1}\omega t=0 \tag{14}$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2n+1}\omega t=\frac{1}{2^{2n}}\left\{\binom{2n+1}{n}\cos\omega t+\binom{2n+1}{n-1}\cos 3\omega t+\ldots+\binom{2n+1}{0}\cos[(2n+1)\omega t]\right\} \tag{15}$$

Substituting Eq. (15) into Eq. (16) gives

$$\left[-\omega^2+\sum_{n=0}^{\infty}c_{2n+1}B^{2n}\right]\cos\omega t+(\text{higher}-\text{order harmonics})=0 \tag{16}$$

where the coefficients c_{2n+1} are given by

$$c_1=1 \tag{17}$$

and

$$c_{2n+1}=(-1)^n\frac{(2n-1)!(2n+1)!}{2^{4n-1}(n!)^2(n-1)!(n+1)!}\qquad \text{for } n\geq 1 \tag{18}$$

For the lowest order harmonic to be equal to zero, it is necessary to set the coefficient of $\cos\omega t$ equal to zero in Eq. (16), then

$$\omega(B)=\left(\sum_{n=0}^{\infty}c_{2n+1}B^{2n}\right)^{1/2} \tag{19}$$

In order to obtain the value of $\sum_{n=0}^{\infty}c_{2n+1}B^{2n}$ in Eq. (19) we consider the following relations

$$\frac{(2n-1)!}{2^{2n-1}(n-1)!} = \left(1/2\right)_n , \qquad \frac{(2n+1)!}{2^{2n}n!} = \left(3/2\right)_n , \qquad (n+1)! = (2)_n \tag{20}$$

where $(a)_n$ is the Pochhammer symbol [22]

$$(a)_n = a(a+1)\ldots(a+n-1) \tag{21}$$

Taking into account Eqs. (17), (20) and (21) it is possible to write $\sum_{n=0}^{\infty} c_{2n+1} B^{2n}$ as follows

$$\sum_{n=0}^{\infty} c_{2n+1} B^{2n} = \sum_{n=0}^{\infty} \frac{\left(1/2\right)_n \left(3/2\right)_n}{(2)_n} \frac{(-B^2)^n}{n!} = {}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-B^2\right) \tag{22}$$

where ${}_2F_1\left(a,b;c;z\right)$ is the hypergeometric function [22]

$${}_2F_1\left(a,b;c;z\right) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{23}$$

Substituting Eq. (22) into Eq. (19) gives

$$\omega_{b1}(B) = \left[{}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-B^2\right)\right]^{1/2} \tag{24}$$

which is the angular frequency obtained applying the low-order harmonic balance method directly to Eq. (6). By the software MATHEMATICA, we can readily obtain the following relation

$$\omega_{b1}(B) = \left[{}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-B^2\right)\right]^{1/2} = \frac{2}{\sqrt{\pi}B}\left[E(-B^2) - K(-B^2)\right]^{1/2} \tag{25}$$

To obtain Eq. (25), the command "FunctionExpand" has been applied to

${}_2F_1\left(1/2,3/2;2;-B^2\right)$. $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, defined as follows [23]

$$K(m)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-m\cos^2\theta}} \tag{26}$$

$$E(m)=\int_0^{\pi/2}\sqrt{1-m\cos^2\theta}\,\mathrm{d}\theta \tag{27}$$

By applying the first order homotopy perturbation method to Eq. (6), the same approximate frequency is obtained [24].

The corresponding approximation to y is obtained from Eqs. (5) and (9)

$$y(t)=\frac{u(t)}{\sqrt{1+u^2(t)}}=\frac{B\cos\omega t}{\sqrt{1+B^2\cos^2\omega t}} \tag{28}$$

Likewise, $x(t)$ can be calculated by integrating equation $y=\mathrm{d}x/\mathrm{d}t$ subject to the restrictions

$$x(0)=0, \qquad y(0)=\frac{B}{\sqrt{1+B^2}} \tag{29}$$

which can be easily obtained from Eqs. (5) and (7). This integration gives

$$x_j(t)=\frac{1}{\omega_j(B)}\sin^{-1}\left[\frac{B}{\sqrt{1+B^2}}\sin[\omega_j(B)t]\right], \quad j=1,2 \tag{30}$$

However, we should not forget what we are really looking for is an approximate analytical solution to Eq. (1), that is, $x(t)$. Moreover, it is convenient to express the approximate angular frequency and the solution in terms of oscillation amplitude A rather than as a function of B. It is now necessary to find a relation between oscillation amplitude A and parameter B used to solve Eqs. (6) and (8) approximately. From Eq. (3) we get

$$\frac{1}{(1-y^2)^{1/2}}+\frac{1}{2}x^2=C \tag{31}$$

where C is a constant to be determined as a function of initial conditions. From Eq. (28) we can easily obtain $C=(1+B^2)^{1/2}$ and Eq. (31) can be written as follows

$$\frac{1}{(1-y^2)^{1/2}}+\frac{1}{2}x^2=(1+B^2)^{1/2} \tag{32}$$

In addition, when $x = A$, the velocity $y=\mathrm{d}x/\mathrm{d}t$ is zero. Taking this into account in Eq. (32), we obtain the following relation between amplitude A and parameter B

$$1+\frac{1}{2}A^2=(1+B^2)^{1/2} \tag{33}$$

From the above equation we can easily find that the solution for B is

$$B=A\left(1+\frac{1}{4}A^2\right)^{1/2} \tag{34}$$

Substituting Eq. (34) into Eqs. (11), (26) and (29), the first analytical approximate periodic solutions for the relativistic oscillator as a function of the oscillation amplitude A are given by

$$\omega_{a1}(A)=\left(1+\frac{3}{4}A^2+\frac{3}{16}A^2\right)^{-1/4} \tag{35}$$

$$\omega_{b1}(A) = \frac{4}{A\sqrt{\pi(4+A^2)}}\left[E\left(-\frac{1}{4}A^2(4+A^2)\right) - K\left(-\frac{1}{4}A^2(4+A^2)\right)\right]^{1/2} \quad (36)$$

$$x_j(t) = \frac{1}{\omega_j(A)}\sin^{-1}\left[\sqrt{\frac{4A^2+A^4}{4+4A^2+A^4}}\sin[\omega_j(A)t]\right], \quad j = a1, b1 \quad (37)$$

3.3.- Results and discussion

In this section we illustrate the accuracy of the proposed approach by comparing the approximate frequencies $\omega_{a1}(B)$ and $\omega_{b1}(B)$ obtained in this paper with the exact frequency $\omega_{ex}(A)$. The exact angular frequency is calculated as follows. Substituting Eq. (34) into Eq. (32), we obtain

$$\frac{1}{(1-y^2)^{1/2}} = \frac{1}{2}(A^2 - x^2) \quad (38)$$

The exact frequency can then be derived as follows

$$\omega_{ex}(A) = \frac{\pi}{2}\left[\int_0^A \frac{1+\frac{1}{2}(A^2 - x^2)}{\sqrt{A^2 - x^2 + \frac{1}{4}(A^2 - x^2)^2}}\mathrm{d}x\right]^{-1} \quad (39)$$

which can be written in terms of elliptical integrals as follows

$$\omega_{ex}(A) = 2\pi\left[4\sqrt{4+A^2}E\left(\frac{A^2}{4+A^2}\right) - \frac{8}{\sqrt{4+A^2}}K\left(\frac{A^2}{4+A^2}\right)\right]^{-1} \quad (40)$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, defined in Eqs. (26) and (27).

For small values of the amplitude A it is possible to take into account the following approximation,

$$\omega_{ex}(A) \approx 1 - \frac{3}{16}A^2 + \frac{51}{1024}A^4 - \frac{233}{16384}A^6 + \ldots \qquad (41)$$

For small values of A it is also possible to do the power-series expansion of the approximate angular frequencies ω_{a1} (Eq. (35)) and ω_{b1} (Eq. (36)). In this way the following equations can be obtained

$$\omega_{a1}(A) \approx 1 - \frac{3}{16}A^2 + \frac{42}{1024}A^4 - \frac{90}{16384}A^6 + \ldots \qquad (42)$$

$$\omega_{b1}(A) \approx 1 - \frac{3}{16}A^2 + \frac{54}{1024}A^4 - \frac{278}{16384}A^6 + \ldots \qquad (43)$$

These series expansions were carried out using MATHEMATICA. As can be seen, in the expansions of the angular frequencies, ω_{a1} (Eq. (42)) and ω_{b1} (Eq. (43)), the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact angular frequency, ω_{ex} (Eq. (41)). By comparing the third terms in Eqs. (42) and (43) with the third term in the series expansion of the exact frequency ω_{ex} (Eq. (41)), we can see that the third term in the series expansions of ω_{b1} (Eq. (43)) is more accurate than the third term in the expansion of ω_{a1} (Eq. (42)).

Now we are going to obtain an asymptotic representation for large amplitudes. For very large values of the amplitude A it is possible to take into account the following power series expansions

$$\omega_{ex}(A) \approx \frac{\pi}{2A} + \ldots = \frac{1.57080}{A} + \ldots \qquad (44)$$

$$\omega_{a1}(A) \approx \frac{2}{3^{1/4}A} + \ldots = \frac{1.51967}{A} + \ldots \qquad (45)$$

$$\omega_{b1}(A) \approx \sqrt{\frac{8}{\pi}}\frac{1}{A} + \ldots = \frac{1.59577}{A} + \ldots \tag{46}$$

Once again we can also see than the frequency $\omega_{b1}(A)$ is more accurate than $\omega_{a1}(A)$ and can provide excellent approximations to the exact frequency $\omega_{ex}(A)$ for very large values of oscillation amplitude. Now, the relative errors of the first term of series expansions of $\omega_{a1}(A)$ and $\omega_{b1}(A)$ are 3.3% and 1.6%, respectively. These results confirm the fact that ω_{b1} is a better approximation to the exact frequency ω_{ex} than the approximate frequency ω_{a1}, not only for small amplitudes but also for large values of the amplitude of oscillation.

The exact periodic solutions $x(t)$ achieved by numerically integrating Eq. (1), and the proposed normalized first-order approximate periodic solutions $x_1(t)$ and $x_2(t)$ in Eq. (37), respectively, for one complete cycle are plotted in Figures 1 and 2 for A = 1 and 10, respectively. In these figures parameter h is defined as follows

$$h = T_{ex}(A)t = \frac{2\pi t}{\omega_{ex}(A)} \tag{48}$$

As the dimensionless variable y is equal to v/c, where v and c are the velocity of the particle and the velocity of light, respectively,

$$\beta_0 = \frac{v_0}{c} = y(0) = \frac{B}{\sqrt{1+B^2}} = \sqrt{\frac{4A^2 + A^4}{4 + 4A^2 + A^4}} \tag{49}$$

Figures 1 and 2 show that Eq. (37) provides a good approximation to the exact periodic solution and it is adequate to obtain the approximate analytical expression of $x(t)$. As we can see, for small values of A (Figure 1) $x(t)$ is very close to the sine function form of nonrelativistic simple harmonic motion. For higher values of A (Figure 2) the curvature becomes more concentrated at the turning points $(x = \pm A)$. For these values of A, $x(t)$ becomes markedly anharmonic and is almost straight between the turning points. Only in the vicinity of the turning points, where the magnitude of the Hooke's law force is maximum and the velocity becomes relativistic, is the force effective in changing the velocity [19]. Figure 2 is typical example of the motion in the ultrarelativistic region where $\beta_0 \to 1$.

At this point it is necessary to answer the following questions: (a) why does substitution of Eq. (9) into Eq. (6) not give the same result as substitution of Eq. (9) into Eq. (8)?, and (b) why does application of the first-order harmonic balance method to Eq. (6) give a more accurate frequency than application of the method to Eq. (8)? These questions have been analyzed in detail in reference [24] and here we are going only to summarize the results. If we substitute Eq. (9) into Eq. (8) and we divide the resulting equation by $\cos\omega t$ we obtain an equation that includes only two odd powers of $\cos\omega t$, $\cos\omega t$ and $\cos^3\omega t$, and then there are only two contributions to the coefficient of the first harmonic $\cos\omega t$, which are 1 from $\cos\omega t$ and 3/4 from $\cos^3\omega t$. Therefore, substituting Eq. (9) into Eq. (8) produces only the first harmonic, $\cos\omega t$, and the third harmonic, $\cos 3\omega t$,

$$\left[\omega^4\left(1+\frac{3}{4}B^2\right)-1\right]\cos\omega t+\frac{1}{4}\omega^4 B^2\cos 3\omega t=0 \qquad (50)$$

However, Eq. (14) includes all odd powers of $\cos\omega t$, which are $\cos^{2n+1}\omega t$ with n = 0, 1, 2, …, ∞, and then there are infinite contributions to the coefficient of the first harmonic $\cos\omega t$, that is, 1 from $\cos\omega t$, 3/4 from $\cos^3\omega t$, 5/8 from $\cos^5\omega\tau$, …, $2^{-2n}\binom{2n+1}{n}$ from $\cos^{2n+1}\omega t$, and so on. Therefore, substituting Eq. (9) into Eq. (6) produces the infinite set of higher harmonics, $\cos\omega t$, $\cos 3\omega t$, …, $\cos[(2n+1)\omega t]$, and so on. Similar phenomenon occurred in [10] and [15] for the Duffing-harmonic oscillator.

It can be seen that Eqs. (11) and (25) have the form

$$\omega_j(B)=\left[f_j(B)\right]^{-1/4}, \quad j=a1, b1 \qquad (51)$$

which allows the approximate frequency ω to be determined in terms of the oscillation amplitude B. From this equation we can conclude that application of the first-order harmonic balance method to Eqs. (6) and (8) gives the same functional form for the approximate frequency ω. The difference between these approximate frequencies is the function $f(B)$

$$\omega_{a1}(B) = \left(1 + \frac{3}{4}B^2\right)^{-1/4} \quad \text{and} \quad f_{a1}(B) = 1 + \frac{3}{4}B^2 \tag{52}$$

$$\omega_{b1}(B) = \left({}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-B^2\right)^{-2}\right)^{-1/4} \quad \text{and} \quad f_{b1}(B) = {}_2F_1\left(\frac{1}{2},\frac{3}{2};2;-B^2\right)^{-2} \tag{53}$$

If we do the power-series expansion of ${}_2F_1\left(1/2, 3/2; 2; -B^2\right)^{-2}$ we have

$$\omega_{b1}(B) = \left(1 + \frac{3}{4}B^2 - \frac{3}{64}B^4 + \frac{13}{512}B^6 + \ldots\right)^{-1/4} \tag{54}$$

As can be seen, in this equation the first two terms in the brackets are identical to the two terms in brackets in Eq. (52); whereas powers B^4, B^6,… are due to the infinite set of higher harmonics in Eq. (14). Applying the harmonic balance method to Eqs. (6) and (9) with higher harmonics, the two procedures will give more accurate results. In the limit in which we include all the harmonics, they must give us exactly the same solution, since Eq. (9) is equivalent to Eq. (6).

4. Higher order approximation

As the application of the first-order harmonic balance method to Eq. (6) give a more accurate frequency than application of the method to Eq. (8), we have obtain the second order approximation for Eq. (6). The next level of harmonic balance uses the form [16]

$$u_2(t) = u_1(t) + \Delta u_1(t) \tag{55}$$

where $u_1(t) = B\cos\omega t$ (Eq. (9)) and $\Delta u_1(t)$ is the correction part. Substituting Eq. (55) into Eq. (6) and linearizing the resulting equation with respect to the correction $\Delta u_1(t)$ at $u(t) = u_1(t)$ leads to

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 \Delta u_1}{\mathrm{d}t^2} + \frac{u_1}{\sqrt{1+u_1^2}} + \frac{\Delta u_1}{(1+u_1^2)^{3/2}} = 0 \tag{56}$$

and

$$\Delta u_1(0) = 0, \qquad \frac{\mathrm{d}\Delta u_1}{\mathrm{d}t}(0) = 0 \tag{57}$$

To obtain the second approximation to the exact solution, $\Delta u_1(t)$ in Eq. (55), which must satisfy the initial conditions in Eq. (57), takes the form

$$\Delta u_1 = c_1(\cos\omega t - \cos 3\omega t) \tag{58}$$

where c_1 is a constant to be determined. Substituting Eqs. (9), (55) and (58) into Eq. (56), expanding the expression in a trigonometric series and setting the coefficients of the resulting items $\cos\omega t$ and $\cos 3\omega t$ equal to zero, respectively, yield

$$\pi(B+c_1)B^4\omega^2 - 4(B^3 - 8c_1)E(-B^2) + 4(B^3 - 8c_1 - 4c_1B^2)K(-B^2) = 0 \tag{59}$$

and

$$\begin{aligned} &27\pi c_1 B^6\omega^2 - 4(8B^3 + B^5 - 128c_1 - 88c_1B^2)E(-B^2) \\ &\qquad + 4[(B^3(8+5B^2) - 4(32+38B^2+9B^4)c_1]K(-B^2) = 0 \end{aligned} \tag{60}$$

From Eq. (29) we can obtain the second analytical approximate frequency ω_{b2} as follows

$$\omega_{b2}(B)=\left(\frac{4(B^3-8c_1)E(-B^2)-4(B^3-8c_1-4c_1B^2)K(-B^2)}{\pi(c_1+B)B^4}\right)^{1/2} \tag{61}$$

Substituting Eq. (61) into Eq. (60) and solving for c_1, we obtain

$$c_1=\frac{B^5[(8+B^2)E(-B^2)-(8+5B^2)K(-B^2)]}{(\Delta_1+\Delta_2)^{1/2}B^2+(64B^2+40B^4+13B^6)E(-B^2)-(64B^2+72B^4+29B^6)K(-B^2)} \tag{61}$$

where

$$\Delta_1=[(-64-48B^2+13B^4)E(-B^2)+(64+80B^2+7B^4)K(-B^2)]^2 \tag{62}$$

$$\Delta_2=-432B^4[-(4+5B^2)E^2(-B^2)-2(4+5B^2+B^4)E(-B^2)K(-B^2)-(4+5B^2+B^4)K^2(-B^2)] \tag{63}$$

For small values of the amplitude A it is possible to take into account the following power series expansion of the second order approximate angular frequency

$$\omega_{b2}(A)\approx 1-\frac{3}{16}A^2+\frac{51}{1024}A^4-\frac{238.25}{16384}A^6+\ldots \tag{64}$$

where we have taken into account Eq. (34). As can be seen, in the expansion of the angular frequency $\omega_{b2}(A)$, the first three terms are the same as the first three terms of the equation obtained in the power-series expansion of the exact angular frequency, $\omega_{ex}(A)$ (Eq. (41)). For very large values of the amplitude A it is possible to take into account the following power series expansion

$$\omega_{b2}(A) \approx \frac{1.5659}{A} + \ldots \qquad (65)$$

Once again we can see than $\omega_{b2}(A)$ provides excellent approximations to the exact frequency $\omega_{ex}(A)$ for very large values of oscillation amplitude and the relative error for $\omega_{2b}(A)$ is lower than 0.31% for all the range of values of amplitude of oscillation A.

6. Conclusions

The first order harmonic balance method was used to obtain two approximate frequencies for the relativistic oscillator in which the restoring force has an irrational form. An approximate frequency, ω_{a1}, was obtained by rewriting the nonlinear differential equation in a form that does not contain an irrational expression; while the second one, ω_{b1}, was obtained by solving the nonlinear differential equation containing a square-root expression approximately. We can conclude that formulas (35) and (36) are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Excellent agreement of the approximate frequencies with the exact one was demonstrated and discussed and the discrepancy between the first order approximate frequency, ω_{b1}, and the exact one never exceeds 1.6%. The first order approximate frequency, ω_{b1}, derived here is the best frequency that can be obtained using the first-order harmonic balance method, and the maximum relative error was significantly reduced as compared with the first approximate frequency, ω_{a1}. We have also shown that applying the first order harmonic balance method to Eqs. (6) and (8) we obtain the same results that those obtained by applying the first order homotopy perturbation method to Eqs. (6) and (8). We can conclude that this is a general result for conservative oscillators. Some examples have been presented to illustrate excellent accuracy of the approximate analytical solutions. We discussed the reason why the accuracy of the approximate frequency, ω_{b1}, is better than that of the frequency, ω_{a1}. This reason is related to the number of harmonics that application of the first-order harmonic balance method produces for each differential equation solved, two harmonics for the first case and the infinite set of harmonics for the second one. We also obtained the approximate frequency $\omega_{2b}(A)$ by applying the second order harmonic balance method to Eq. (6) and we obtained that the relative error for $\omega_{2b}(A)$ is lower than 0.31% for all the range of values of amplitude of oscillation A.

Finally, we can see that the method considered here is very simple in its principle, and is very easy to apply.

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FIGURE CAPTIONS

Figure 1.- Comparison of the analytical approximate solutions x_1 (black circles) and x_2 (white circles) with the exact solution (continuous line) for A = 1 $(\beta_0 = v_0 / c = 0.74536)$.

Figure 2.- Comparison of the analytical approximate solutions x_1 (black circles) and x_2 (white circles) with the exact solution (continuous line) for A = 10 $(\beta_0 = v_0 / c = 0.99981)$.

FIGURE 1

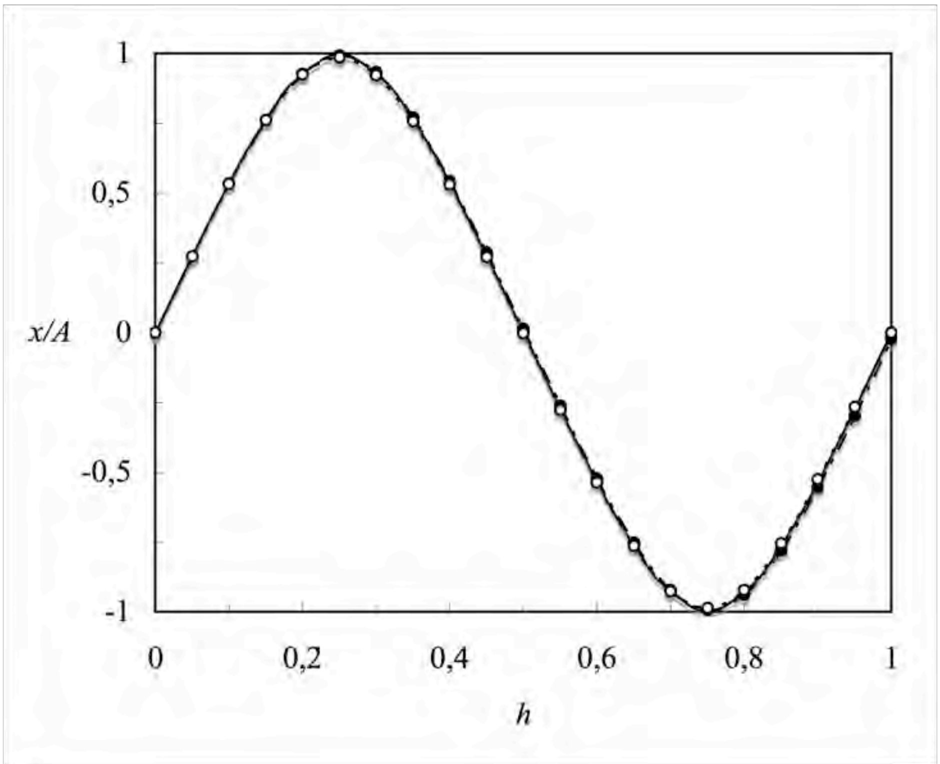

FIGURE 2