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Cassini's motions of the Moon and Mercury and possible excitations of free librations

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ABSTRACT

On the basis of conditionally-periodic solutions of Hamiltonian systems at resonance of main frequencies Cassini's motions, their stability, Cassini's angle and periods of free librations of the Moon and Mercury have been recently studied and determined. The generalized formulations of Cassini's laws for the motion of the Moon and Mercury, that are considered as absolutely rigid non-spherical bodies, have been determined. The study of the second approximation equations of the desired quasi-periodic solutions in the case of the Moon allows us to determine the constant components of the first order for six Andoyer variables and the constant component of the second order for the angular velocity of the Moon. These effects are caused by the influence of the third harmonic of selenopotential. In this paper, these effects are described by analytical formulas, the dynamic and geometric interpretations are given, and a new interpretation of Mercury's motion under the generalized Cassini's laws has been proposed. Predictions of the existence of free librations of significant amplitude in the Mercury longitude, that are confirmed by the radar measurements data of the Mercury angular velocity, and in its pole motion in the body and in space have been made. The mechanism describing free librations of celestial bodies and their pole oscillations has been proposed due to the forced relative oscillations and wobble of the core-mantle system of celestial bodies (Moon, Mercury, Earth and other bodies in the solar system) under gravitational action of the external celestial bodies. The paper shows that the ascending node of equator of Mercury (and the intermediate plane orthogonal to the angular momentum) of epoch 2000.0 on the ecliptic does not coincide with the ascending node of orbital plane of Mercury on the same plane, and is ahead of it at an angle 23°4'. Angular momentum vector of the rotational motion of Mercury forms a constant angle $\rho_G = 4'1 \pm 1'1$ with normal to the moveable plane of its orbit. The observed inclination of the angular velocity$\rho_\omega = 2'1 \pm 0'1$, can be considered as a possible evidence of a significant amplitude of the poles free motion of the Mercury rotation axis (c amplitude of about 2′- 4′).

1. Introduction

The motion of the Moon on the Cassini's laws has been studied by a number of scientists [1–5], and other followers. In recent decades, due to the rapid space explorations, a lot of progress in the Moon and Mercury exploration have been made. It is largely driven by an unprecedented accuracy of modern laser measurement to the Moon (about 1 mm), prominent on the accuracy of interferometric radar measurements of the Mercury rotational motion [6]. Increasingly higher demands are made to describe the translational and rotational motion of the Moon, its tidal deformations and non-

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tidal effects [7–9]. Developed and being implemented are the fantastic scientific projects on the study of the Moon's rotation method of cross-location of the satellites and the Moon, through a telescope that is mounted to the polar region of the Moon (Selena and Japanese projects ILOM). In this paper, we use high-precision results on the study of the gravitational field of the Moon [10] to study the basic laws of the Moon motion by the Cassini and subtle effects of constant angular displacements of the Moon axes of inertia and its angular velocity with relation to the base orbital coordinate system and with respect to Cassini's reference frame.

Precision results of the study of the Mercury rotation using the radar ground-based observations and new data on its gravitational field [6,11] are used to interpret the motion of Mercury by the generalized Cassini-Colombo laws, formulated for Mercury [1–3,12].

The closest theory of the Moon physical libration which isn't analytical and is based generally on old ideas of the Moon libration and the results of processing the laser observations statistical data of the Moon [13]. In connection with the planned space missions to the Moon, the demands to the theories of the Moon physical libration and to its models sharply increase. Precise measurements of the Moon orientation and its temporal changes open up new possibilities for studying the internal structure and internal dynamics of the Moon and its deformations. Investigation of the Moon inner dynamics as a system of shells (solid core, liquid core, mantle, crust) is of a great interest for the construction of this theory, as well as for explaining the observed non-tidal variations of the light-reflecting distances to the reflectors on the Moon, for studying the features of its tectonic structures, flow, etc. The work is aimed at creating a new analytical theory of the Moon physical librations, which satisfies modern requirements to the accuracy of determining the orientation and rotation of the Moon. The theory is intended for effective interpretation of observational data (both laser observations and observations that will be obtained during the implementation of future space missions to the Moon (Japanese project ILOM, etc.)). To solve the problems stated above, a new dynamic research of the perturbed Moon rotational motion is needed to determine the parameters of the Moon free librations.

Dynamic studies of these patterns and effects in the motion of the Moon will always be one of the central problems of lunar exploration. The research is based on the equations of rotational motion of the Moon in Andoyer's variables and analytical method for the construction of quasi-periodic solutions of Hamiltonian systems at the resonance of the fundamental frequencies [14]. In the paper, the Moon is modeled as non-spherical solid body. A more realistic models of the Moon (deformable by the terrestrial and solar tides, with a liquid core, with the displaced liquid core, etc.) will be discussed in subsequent papers. On relevant phenomena we have obtained the most accurate values of the parameters. We discuss the fundamental problem of excitating the free oscillations and the free librations in the rotational motion of the Moon, Mercury and other solar system bodies.

2. Cassini's motions of the Moon

2.1. Canonical equations of rotational motion of the Moon in Andoyer variables referred to the moveable ecliptic plane

Let C_MXYZ be the moving coordinate system connected with the ecliptic of date with the origin in the center of mass of the Moon C_M and with axis C_MX, directed upward to the mean node of lunar orbit on the ecliptic plane of date [15]. Let $C_M\overline{\xi}\overline{\eta}\overline{\zeta}$ be a system of principal axes of inertia of the Moon. The axes $C_M\overline{\xi}$, $C_M\overline{\eta}$ and $C_M\overline{\zeta}$ correspond to the principal moments of inertia A, B and C respectively. We will study the rotation of the Moon, considered as a rigid body.

The rotational motion of the Moon will be described by the Andoyer variables, classified in the main moveable coordinate system of ecliptic of date:

$$L, G, H, l, g, h(\theta, \rho) \tag{1}$$

$$\boldsymbol{x} = (L, G, H)^{\mathrm{T}}, \boldsymbol{y} = (l, g, h)^{\mathrm{T}} \tag{2}$$

($\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})^{\mathrm{T}}$ is matrix notation of variables).here G is the module of vector of angular momentum of rotational motion of the Moon $\mathbf{G}$, L is a projection of vector $\mathbf{G}$ on the axis of inertia of the Moon $C_M\overline{\zeta}$, H is a projection of vector $\mathbf{G}$ on normal $\mathbf{n_E}$ to the ecliptic plane of date C_MXY, l is an angle of own rotation of the Moon, measured from the plane Q_G, orthogonal to the vector $\mathbf{G}$, g is a longitude of ascending node of the equatorial plane of the Moon $C_M\overline{\xi}\overline{\eta}$ on a intermediate plane Q_G, h is a longitude of ascending node of the plane Q_G on the base coordinate plane C_MXY, measured from the axis C_MX. We introduce the angles of inclinations θ and ρ: θ is an inclination of the plane $C_M\overline{\xi}\overline{\eta}$ of the Moon to intermediate plane Q_G, an angle ρ is an inclination of the plane Q_G to ecliptic plane of date [15]. Therefore we have: $\cos\theta = L/G, \cos\rho = H/G$. Note also that the longitude h is measured from the moving point on ecliptic of date Ω_m, corresponding to the mean position of the node of the lunar orbit. We also introduce into consideration the longitude $\overline{h} = h + \Omega$ of the intermediate plane, measured from the mean point of the equinox of date along the ecliptic of epoch, and then along the ecliptic of date, as was the custom in the analytic theory of the rotational motion of the Moon [15].

Let the axes of inertia $C_M\overline{\xi}$, $C_M\overline{\eta}$ and $C_M\overline{\zeta}$ correspond to axial moments of inertia A, B and C, for which $B > C > A$ and, respectively,

$$J_2 = \frac{2B - C - A}{2mr_0^2}, C_{22} = \frac{C - A}{4mr_0^2}, \frac{B - A}{mr_0^2} = J_2 + 2C_{22},$$
$$\frac{B - C}{mr_0^2} = J_2 - 2C_{22}. \tag{3}$$

here m and r_0 is a mass and mean radius of the satellite.

2.2. The rotational motion of the rigid Moon on Cassini's laws

In the Andoyer variables (1)–(2) the equations of the Moon rotational motion take the following standard canonical form:

$$\frac{d\boldsymbol{x}}{dt} = -\frac{\partial F}{\partial \boldsymbol{y}^{\mathrm{T}}}, \frac{d\boldsymbol{y}}{dt} = -\frac{\partial F}{\partial \boldsymbol{x}^{\mathrm{T}}} \tag{4}$$

$$F = F_0(G) + \mu F_1(\boldsymbol{x}, \boldsymbol{y}, t) + \mu^2 F_2(\boldsymbol{x}, \boldsymbol{y}, t) + \ldots, \tag{5}$$

here $F_0 = \frac{G^2}{2B} - n_\Omega G$ is the Hamiltonian of the Moon unperturbed rotational motion. $\frac{G^2}{2B}$ is a kinetic energy of rotational motion of the Moon, considered as homogeneous sphere. $\mu F_1(\boldsymbol{x}, \boldsymbol{y}, t) + \mu^2 F_2(\boldsymbol{x}, \boldsymbol{y}, t) + \ldots$ is a perturbed Hamiltonian, representing the terms of the first, second and higher orders with respect to small parameter μ characterizing the closeness of the inertia ellipsoid of the Moon to the sphere and a small rate of precession of the lunar orbit [15]. The terms of the first-order in (5) are determined by

$$\mu F_1 = \mu T_1 + \mu E_1 - \mu U_1, \tag{6}$$

Analyzing the terms of the Hamiltonian F we can see that the terms of the first order μF_1 with respect to μ include the second harmonic of the force function of the Moon $-\mu U_1 = -U_M^{(2)}$, caused by the gravitational attraction of the Earth, as well as the kinetic energy of rotational motion μT_1, due to its nonsphericity,

$$\mu T_1 = \frac{1}{2}\left(\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right)L^2 + \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} - \frac{1}{B}\right)G^2 \quad (7)$$

and additional term $\mu E_1 = n_\Omega(G - H)$.

The terms of the second order should include the third and higher harmonics of the force function of the Moon-Earth system $U_M^{(n)}$ and the second harmonic of the force function of the Moon-Sun system U_{MS}:$\mu^2 F_2 = -U_{MS} - \sum_{n\geq 3} U_M^{(n)}$. In this case all the terms of specified force functions should be represented by explicit functions of Andoyer variables. Similar work was done earlier in the construction of the analytic theory of the rotational motion of the Moon [15] and substantially developed in this work, especially in the study of constant displacements of axes of inertia of the Moon in the intermediate and in the basic spatial coordinate system with respect to their Cassini's positions.

Very extensive algebra on the construction of the trigonometric series of force functions in the Andoyer variables for brevity is not shown here. These constructions are made for two basic models of the Moon orbital motion: lunar orbit for high-precision, described by the modern theory of the Moon orbital motion and its simplified version - for a uniformly precessing mean plane of lunar orbit on ecliptic plane of date, saving a constant angle of inclination of the orbital plane. Both models allow us to explore as the basic laws of motion of the Moon, and so the subtle resonance effects in the vicinity of Cassini's motions. The following is a detailed description of these effects.

In this paper we have used two sets of coefficients of selenopotential (second and third harmonics), $\overline{C}_{nm}$, $\overline{S}_{nm}$ and C_{nm}, S_{nm}. The first of these corresponds to the standard representation of the force function in the coordinate system associated with the Moon, $C_M\overline{\xi}\overline{\eta}\overline{\zeta}$. It is noted that the axis of rotation corresponds to the axis of inertia of the Moon $C_M\overline{\eta}$, and the axis $C_M\overline{\xi}$ corresponds to the Moon geocentric radius vector. Coefficients C_{nm},S_{nm} are the common standard coefficients of selenopotential calculated in the main selenographic coordinate system. They are referred to selenographic coordinate system $C_M\xi\eta\zeta$. Between the axes of the two coordinate systems we have the simple relations: $\overline{\xi} = -\xi$, $\overline{\eta} = \zeta$, $\overline{\zeta} = \eta$. The simple relations between these two sets of coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$ and C_{nm}, S_{nm} are shown as (8).

The parameters of the gravitational field of the Moon C_{nm}, S_{nm} (for second, third and fourth harmonics) have approximately the same order of magnitude $\sim 10^{-4}$ [10]. Consequently, the opportunity to introduce a small parameter μ by the formulas:

$$C_{nm} = \mu C_{nm}^{(0)}, S_{nm} = \mu S_{nm}^{(0)}, \mu = 10^{-4}, \quad (n = 2, 3, \ldots; \quad m = 0, 1, 2, \ldots, n) \quad (8)$$

where $C_{nm}^{(0)}$, $S_{nm}^{(0)}$ are fixed numerical values of the order of unity. Conditions (8) actually means that the distribution of density of the Moon is close to the concentric. The remaining terms of the Hamiltonian F, representing the force functions of the gravitational interaction between the Moon and the planets U_i and the corresponding harmonics of the force function U_{MS} (of the Moon – the Sun system), the terms due to the mobility of the ecliptic of date, we refer to members of the third order with respect to μ [15].

The current state of study of the Moon rotational motion is described in detail in monograph [8]. In this paper, we continue to develop a method of constructing of the analytical theory of lunar physical librations, the basis of which was made by the author [14,15]. A method for constructing the analytical theory of lunar physical librations on the basis of the formulated model problem is to implement the following program of research of solutions of equations (4)–(8), which provides:

1. Investigation of generating periodic solutions of (4)–(8) (explanation of Cassini's laws, the determination of parameter of physical librations of the Moon ρ_0 - the angle of Cassini),
2. Generalized Cassini's laws and fine stationary dynamical resonance effects in the rotation of the Moon, due to the third and higher harmonics of the force function of the Moon-Earth system;
3. Construction of conditionally - periodic solution of system (4)–(8), which describes an intermediate rotational motion of the Moon in the vicinity of Cassini's motion,
4. Investigation of a neighborhood of a conditionally - periodic solutions on the basis of the variation equations (the stability of the intermediate rotary motion of the Moon, the determination of free librations, resonant librations of the Moon and subtle resonance effects);
5. Study of the influence of secular orbital perturbations in the motion of the Moon, the Sun on the rotational motion of the Moon (the influence of these orbital perturbations on the motion of the Moon in accordance with Cassini's laws, on the structure of short-period perturbations in the rotation of the Moon); construction of perturbations of mixed type,
6. Investigation of the effects of long-period secular perturbations in the orbital motion of the Moon and the Sun (with Milankovitch periods from tens to hundreds of thousands of years) for the rotation of the Moon;
7. Nonlinear analysis of the vicinity of a conditionally periodic solution (the determination of resonance effects in the rotation of the Moon of the second order with respect to amplitudes of resonant librations). In this paper we concentrate on the first two paragraphs.

2.3. Cassini's laws in case of an accurate theory of lunar orbital motion

If $\mu = 0$ the Hamiltonian equations (4) and (5) have a periodic solution:

$$L = L_0, G = G_0, H = H_0, l = l_0, g = F + g_0, h = h_0, \frac{G_0}{B} = n_F. \quad (9)$$

here L_0, G_0, H_0, l_0, g_0, h_0 are constants (initial values of corresponding variables). The theorem is valid due to Yu.V. Barkin [14,15].

Theorem 1. If generating values of variables (9) satisfy the conditions

$$\frac{\partial\langle F_1\rangle}{\partial\boldsymbol{\omega}_0} = 0, \quad (10)$$

$$\Delta_1 = \frac{\partial^2 F_0}{\partial G_0^2} \neq 0, \Delta_2 = \det\frac{\partial^2\langle F_1\rangle}{\partial\boldsymbol{\omega}_0\partial\boldsymbol{\omega}_0^{\mathbf{T}}} \neq 0, \quad (11)$$

where

$$\boldsymbol{\omega}_0 = (l_0, g_0, h_0, L_0, H_0)^{\mathrm{T}}$$

$$\langle F_1\rangle = \frac{1}{(2\pi)^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} F_1(L_0, G_0, H_0, l_0, F + g_0, l_M, l_S, F, D)\,\mathrm{d}l_M\mathrm{d}l_S\mathrm{d}F\mathrm{d}D, \quad (12)$$

then in the neighborhood of this solution by approximate method can make formal series in integer degrees of a small parameter μ, representing the conditionally-periodic solution of the lunar rotation problem (4)-(7).

Calculating the average value of the perturbing function by the formula (12) and resolving the equation (10) we obtain generating solution:

$$l_0 = 0, g_0 = 0, h_0 = \pi, \theta_0 = \frac{\pi}{2}, \rho_0 = \rho_0\left(C_{20}, C_{22}, F_\Omega, A_\nu^{(j)}\right) \quad (13)$$

where the constant value of inclination $\rho = \rho_0$ (of angle of Cassini) is obtained from the Ref. [14] equation 14 and 15

$$\mathbf{B}\cos 2\rho + \mathbf{A}\sin 2\rho + \mathbf{Y}\cos\rho + \mathbf{U}\sin\rho = 0 \quad (14)$$

where

$$\mathbf{B} = -J_2\sum_{\nu_5}(-1)^{\nu_5}A^{(1)}_{0.0.0.0.\nu_5} + C_{22}\sum_{\nu_5}(-1)^{\nu_5}\left(A^{(1)}_{0.0.2.0.\nu_5} + A^{(1)}_{0.0.-2.0.\nu_5}\right),$$

$$\mathbf{Y} = C_{22}\sum_{\nu_5}(-1)^{\nu_5}\left(A^{(1)}_{0.0.2.0.\nu_5} - A^{(1)}_{0.0.-2.0.\nu_5}\right), \mathbf{U} = \frac{1}{3}I\frac{n_F n_\Omega}{n_0^2} + \frac{1}{2}C_{22}\sum_{\nu_5}(-1)^{\nu_5}\left(A^{(2)}_{0.0.2.0.\nu_5} - A^{(2)}_{0.0.-2.0.\nu_5}\right). \quad (15)$$

$$\mathbf{A} = \frac{1}{4}J_2\sum_{\nu_5}(-1)^{\nu_5}\left(2A^{(0)}_{0.0.0.0.\nu_5} - A^{(2)}_{0.0.0.0.\nu_5}\right) + \frac{1}{4}C_{22}\sum_{\nu_5}(-1)^{\nu_5}\left(A^{(2)}_{0.0.2.0.\nu_5} + A^{(2)}_{0.0.-2.0.\nu_5} - 2A^{(0)}_{0.0.2.0.\nu_5} - 2A^{(0)}_{0.0.-2.0.\nu_5}\right)$$

In equations 14 and 15 J_2 and C_{22} are general coefficients of the second harmonic of selenopotential, $n_0^2 = fm_E/a^3$ is a fundamental frequency, a is the mean value of semi-axis major, f is a gravitational constant, m_E is the Earth mass, $I = 0.392$ is a dimensionless moment of inertia of the Moon. In equations 14 and 15 we have used some new notations: $F_\Omega = n_F|n_\Omega|/n_0^2$ and $A_\nu^{(0)}$, $A_\nu^{(1)}$, $A_\nu^{(2)}$are numerical coefficients in standard trigonometric developments of some spherical functions of the lunar geocentric spherical coordinates, describing the perturbed orbital motion of the Moon [16]. The main coefficients that are involved in determining the angle of Cassini $\rho = \rho_0$, by (14) – (15), were obtained from the relevant tables of Kinoshita and are listed in Table 1.

For the fundamental frequencies of the problem have been taken the known values: $n_F = 17395266''10$ (1/yr), $n_0 = 17325643''7931$ (1/yr).

Equations 14 and 15 was first obtained and studied by the author [14,15]. On the basis of this equation the corresponding values ρ_0 have been calculated. Determined values are changed in domain [15]: $\rho_0 = 1^0 32' 41'', 35 \div 1^0 32' 42'', 41$. Modern calculations for the angle ρ_0 have confirmed the correctness of the analytical constructions of analytical theory [15] on the base of the new tables of values of the coefficients $A_\nu^{(0)}$, $A_\nu^{(1)}$ $A_\nu^{(2)}$[17,18], an appropriate choice of the ecliptic of date and initial epoch. Estimated value of the angle ρ_0 was $\rho = \rho_0 = 1^0 32' 40'' 0 = 1^0 54444$. This value is in good agreement with the values of the parameter obtained on the basis of the laser observation data [8].

Table 1
Main coefficients of perturbed orbital motion of the Moon.

$A^{(0)}_{0.0.2.0.0} = 59582 \times 10^{-7}$	$A^{(2)}_{0.0.2.0.2} = 9880167 \times 10^{-7}$
$A^{(0)}_{0.0.0.0.0} = 56 \times 10^{-7}$	$A^{(2)}_{0.0.-2.0.2} = 40 \times 10^{-7}$
$A^{(0)}_{0.0.0.0.0} = 4963033 \times 10^{-7}$	$A^{(2)}_{0.0.0.0.0} = -1 \times 10^{-7}$
$A^{(1)}_{0.0.-2.0.1} = 833 \times 10^{-7}$	$A^{(2)}_{0.0.0.0.1} = -41 \times 10^{-7}$
$A^{(1)}_{0.0.2.0.2} = 190 \times 10^{-7}$	$A^{(2)}_{0.0.0.0.2} = 40439 \times 10^{-7}$
$A^{(1)}_{0.0.2.0.1} = -443828 \times 10^{-7}$	$A^{(1)}_{0.0.0.0.1} = 448718 \times 10^{-7}$
	$A^{(1)}_{0.0.0.0.0} = -207 \times 10^{-7}$

The first condition (11) is satisfied, as $\Delta_1 = 1/B \neq 0$ and the second was written in explicit form and it was shown that for the solution (13) – (15) $\Delta_2 \neq 0$. Thus, in the neighborhood of the found generating solution a conditionally periodic solution of the problem (4) – (7) can be constructed in the form:

$$\mathbf{z} = \mathbf{z}^{(0)} + \mu\mathbf{z}^{(1)} + \mu^2\mathbf{z}^{(2)} + \ldots, \quad (16)$$

where $\mathbf{z}^{(0)}$ is a generating solution described by the formulas (13)–(15) and $z^{(\sigma)}(\sigma = 1, 2, \ldots)$ are conditionally - periodic time functions with the determined frequency basis.

Note, that the analysis of the solution neighborhood (13) – (15) indicates that all the necessary conditions of stability of discussed solution are satisfied. The results of the mechanical and geometric interpretation of the already obtained solution are formulated as a theorem.

Theorem 2. The generating periodic solution for conditionally - periodic solution corresponding to the Moon perturbed rotation is determined by formulas (13)–(15) and describes the following fundamental regularities in the Moon motion:

1. within a mean period of time (equal to the draconic month $T_{draconic} = 27.22$days) between the two consecutive passages of the center of mass of the Moon through a mean ascending node of its orbit on the ecliptic of date the Moon makes one revolution about its polar axis of inertia $C_M\overline{\eta}$ in the coordinate reference system $C_M XYZ$ connected with mean moveable node of the lunar orbit;
2. the axis $C_M\overline{\eta}$ corresponds to the maximal moment of inertia B, and at the moment of the Moon center of mass passing of the mean orbit node of the Moon another axis of inertia $C_M\overline{\xi}$, corresponding to a minimal moment of inertia A, is directed along the line of nodes;
3. in Cassini's motion an angular momentum vector of the Moon rotation $\mathbf{G}$ and its angular velocity vector $\boldsymbol{\omega}$ coincide with the inertia axis of the Moon $C_M\overline{\eta}$;
4. the vector $\mathbf{G}$ forms a constant angle ρ_0 with the normal vector to the ecliptic plane of date $\mathbf{n_E}$, about which it describes in the space a cone with half-anglea ρ_0 with the time period $T_\Omega = 2\pi/|n_\Omega| = 18.5995$ years;
5. the orbit plane, the intermediate plane Q_G normal to the angular momentum vector $\mathbf{G}$, and equatorial plane of the Moon (plane $C_M\overline{\xi}\overline{\zeta}$ in our notations) have the same line of nodes on the ecliptic of date NN', with the longitude of the general descending nodes of the planes Q_G and $C_M\overline{\xi}\overline{\zeta}$ being equal to the mean longitude of the ascending node of the orbit plane of the center of mass of the Moon;
6. the inertia axis of the Moon $C_M\overline{\eta}$, the vector of normal to the ecliptic plane of date $\mathbf{n_E}$, the angular momentum vector of the Moon rotation $\mathbf{G}$ and the angular velocity vector of the Moon

rotation **ω** lie in the same plane, orthogonal to the line of nodes NN' on ecliptic of date;

7. the value of the angle between the normal to ecliptic plane of date $\mathbf{n_E}$ and vectors **G**, **ω** coinciding with the Moon axis $C_M\overline{\eta}$ is obtained from equations 14 and 15 depending on the dynamic parameters J_2 и C_{22} (oblatenesses of the Moon), from the parameter of mobility of the Moon's orbital plane F_Ω (its precession) and on corresponding parameters $A_\nu^{(j)}$ characterizing the perturbed lunar orbital motion (numerical value of the parameter $\rho_0 = 1^0 32' 41''$).

The regularities mentioned 1 - 7 in particular thoroughly explain all statements of classical empirical Cassini's laws, and contain some additions and specifications related to the orientation of the vectors **G**, **ω**, of the lunar inertia ellipsoid orientation, stability of Cassini's motions. Last aspects of the problem and generally dynamics of the Moon did not considered by Cassini.

3. Fine dynamical regularities in the Moon rotation

3.1. Fine dynamical effects in the rotation of the Moon, caused by the third harmonic of the force function of the Moon-Earth system

The equations for second-order perturbations $\boldsymbol{x}_2$ and $\boldsymbol{y}_2$ are obtained from the general formulas of the method of constructing of conditionally - periodic solutions of Hamiltonian systems at resonances [15] and have the form:

$$\begin{aligned}
\frac{d\mathbf{x_2}}{dt} &= -\frac{\partial^2 F_1}{\partial l \partial \mathbf{y}^{\mathrm{T}}} l_1 - \frac{\partial^2 F_1}{\partial g \partial \mathbf{y}^{\mathrm{T}}} g_1 - \frac{\partial^2 F_1}{\partial h \partial \mathbf{y}^{\mathrm{T}}} h_1 - \frac{\partial^2 F_1}{\partial L \partial \mathbf{y}^{\mathrm{T}}} L_1 - \frac{\partial^2 F_1}{\partial g \partial \mathbf{y}^{\mathrm{T}}} G_1 - \frac{\partial^2 F_1}{\partial h \partial \mathbf{y}^{\mathrm{T}}} H_1 - \frac{\partial F_2}{\partial \mathbf{y}^{\mathrm{T}}} \\
\frac{d\mathbf{y_2}}{dt} &= \frac{\partial^2 F_1}{\partial l \partial \mathbf{x}^{\mathrm{T}}} l_1 + \frac{\partial^2 F_1}{\partial g \partial \mathbf{x}^{\mathrm{T}}} g_1 + \frac{\partial^2 F_1}{\partial h \partial \mathbf{x}^{\mathrm{T}}} h_1 + \frac{\partial^2 F_1}{\partial L \partial \mathbf{x}^{\mathrm{T}}} L_1 + \frac{\partial^2 F_1}{\partial g \partial \mathbf{x}^{\mathrm{T}}} G_1 + \frac{\partial^2 F_1}{\partial h \partial \mathbf{x}^{\mathrm{T}}} H_1 + \frac{\partial F_2}{\partial \mathbf{x}^{\mathrm{T}}} + \left(0, \frac{\partial^2 F_0}{\partial G^2} G_2, 0\right)^{\mathrm{T}}
\end{aligned} \tag{17}$$

In this paper we confine ourselves to the effects of permanent displacements directly from the third harmonic of the force function. This means that, formally, here we do not consider the following small terms on the right sides of equation (18):

$$\left\langle \frac{\partial^2 \tilde{F}_1}{\partial l \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{l}_1 \right\rangle, \left\langle \frac{\partial^2 \tilde{F}_1}{\partial g \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{g}_1 \right\rangle, \left\langle \frac{\partial^2 \tilde{F}_1}{\partial h \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{h}_1 \right\rangle, \left\langle \frac{\partial^2 \tilde{F}_1}{\partial L \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{L}_1 \right\rangle, \left\langle \frac{\partial^2 \tilde{F}_1}{\partial G \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{G}_1 \right\rangle, \left\langle \frac{\partial^2 \tilde{F}_1}{\partial H \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{H}_1 \right\rangle$$

Sign $\tilde{f}$ means that in the function f computed by generating values of Andoyer variables (13)–(15) only the purely conditionally - periodic terms are saved. The sign $\langle f \rangle$ indicates the average value of function of Andoyer variables f calculated by generating their values and taking into account the resonance relations between the corresponding frequencies. For the second partial derivatives of the averaged function $\langle F_1 \rangle$, we can use standard concepts previously used [15]:

$$\frac{\partial \langle F_1 \rangle}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\mathbf{T}}} = \frac{B n_0^2}{G^2} f_1^{(\boldsymbol{x},\boldsymbol{x})}, \ \frac{\partial \langle F_1 \rangle}{\partial \boldsymbol{y} \partial \boldsymbol{x}^{\mathbf{T}}} = \frac{B n_0^2}{G} f_1^{(\boldsymbol{x},\boldsymbol{y})}, \ \frac{\partial \langle F_1 \rangle}{\partial \boldsymbol{y} \partial \boldsymbol{y}^{\mathbf{T}}} = B n_0^2 f_1^{(\boldsymbol{y},\boldsymbol{y})}. \tag{19}$$

The result is a simplified system of algebraic equations:

Partial derivatives F_1 are computed by generating the values of variables (13), corresponding to the motion of the Moon by Cassini's laws. In Ref. (17) $\boldsymbol{x}_1 = (L_1, G_1, H_1)$ and $\boldsymbol{y}_1 = (l_1, g_1, h_1)$ are first-order perturbations of the canonical Andoyer variables. They contain the constant components $\overline{\boldsymbol{x}}_1 = (\overline{L}_1, \overline{G}_1, \overline{H}_1)$ and $\overline{\boldsymbol{y}}_1 = (\overline{l}_1, \overline{g}_1, \overline{h}_1)$, and so pure conditionally - periodic components: $\tilde{\boldsymbol{x}}_1 = (\tilde{L}_1, \tilde{G}_1, \tilde{H}_1)$ and $\tilde{\boldsymbol{y}}_1 = (\tilde{l}_1, \tilde{g}_1, \tilde{h}_1)$. First-approximation equations allowed us to identify components $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{y}}_1$ and only one constant component $\overline{G}_1 = 0$.

The basic equations for the determination of the constant components of the first approximation for the other variables are obtained from the equations for the second approximation (17) and have the following form [15]:

$$\frac{\partial^2 \langle F_1 \rangle}{\partial l \partial \boldsymbol{z}^{\mathrm{T}}} \overline{l}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial g \partial \boldsymbol{z}^{\mathrm{T}}} \overline{g}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial h \partial \boldsymbol{z}^{\mathrm{T}}} \overline{h}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial L \partial \boldsymbol{z}^{\mathrm{T}}} \overline{L}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial G \partial \boldsymbol{z}^{\mathrm{T}}} \overline{G}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial H \partial \boldsymbol{z}^{\mathrm{T}}} \overline{H}_1 + \frac{\partial \langle F_2 \rangle}{\partial \boldsymbol{z}^{\mathrm{T}}} = 0 \tag{20}$$

$$\boldsymbol{z} = (L, H, l, g, h)^{\mathrm{T}}.$$

This paper uses a combined method for calculating of derivatives in equation (20). Namely, the first derivatives $\frac{\partial \langle \tilde{F}_2 \rangle}{\partial \boldsymbol{z}^{\mathrm{T}}}$ calculated for the model of the orbital motion of the Moon on a

$$\begin{aligned}
&\frac{\partial^2 \langle F_1 \rangle}{\partial l \partial \boldsymbol{y}^{\mathrm{T}}} \overline{l}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial g \partial \boldsymbol{y}^{\mathrm{T}}} \overline{g}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial h \partial \boldsymbol{y}^{\mathrm{T}}} \overline{h}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial L \partial \boldsymbol{y}^{\mathrm{T}}} \overline{L}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial g \partial \boldsymbol{y}^{\mathrm{T}}} \overline{G}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial h \partial \boldsymbol{y}^{\mathrm{T}}} \overline{H}_1 + \\
&+ \left\langle \frac{\partial^2 \tilde{F}_1}{\partial l \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{l}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial g \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{g}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial h \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{h}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial L \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{L}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial G \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{H}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial H \partial \boldsymbol{y}^{\mathrm{T}}} \tilde{H}_1 \right\rangle + \frac{\partial \langle F_2 \rangle}{\partial \boldsymbol{y}^{\mathrm{T}}} = 0 \\
&\frac{\partial^2 \langle F_1 \rangle}{\partial l \partial \boldsymbol{x}^{\mathrm{T}}} \overline{l}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial g \partial \boldsymbol{x}^{\mathrm{T}}} \overline{g}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial h \partial \boldsymbol{x}^{\mathrm{T}}} \overline{h}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial L \partial \boldsymbol{x}^{\mathrm{T}}} \overline{L}_1 + \frac{\partial^2 \langle F \rangle_1}{\partial g \partial \boldsymbol{x}^{\mathrm{T}}} \overline{G}_1 + \frac{\partial^2 \langle F_1 \rangle}{\partial h \partial \boldsymbol{x}^{\mathrm{T}}} \overline{H}_1 + \left(0, \frac{\partial^2 F_0}{\partial G^2} G_2, 0\right)^{\mathrm{T}} + \\
&+ \left\langle \frac{\partial^2 \tilde{F}_1}{\partial l \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{l}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial g \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{g}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial h \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{h}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial L \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{L}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial G \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{G}_1 \right\rangle + \left\langle \frac{\partial^2 \tilde{F}_1}{\partial H \partial \boldsymbol{x}^{\mathrm{T}}} \tilde{H}_1 \right\rangle + \frac{\partial \langle F_2 \rangle}{\partial \boldsymbol{x}^{\mathrm{T}}} = 0
\end{aligned} \tag{18}$$

uniformly precessing elliptical orbit, and the second derivatives $\frac{\partial^2\langle F_1\rangle}{\partial \mathbf{z}\partial \mathbf{z}^T}$ are calculated as specified for the precessing elliptical orbit, and by accurate description of the orbital motion of the Moon.

Without going into details, and omitting the complicated calculations, we present the final formula for the constant components of the first-order perturbations of the canonical Andoyer variables $\overline{\mathbf{x}}_1 = (\overline{L}_1, \overline{G}_1, \overline{H}_1)$ and $\overline{\mathbf{y}}_1 = (\overline{l}_1, \overline{g}_1, \overline{h}_1)$ and inclinations θ and $\rho(\overline{\theta}_1$ and $\overline{\rho}_1)$, and components of the angular velocity corresponding to the original coordinate axes of the Moon $C_M\overline{\xi}\overline{\eta}\overline{\zeta}$ p, q and $r(\overline{p}_1, \overline{q}_1$ and $\overline{r}_1)$:

$$\begin{aligned}
\mu\frac{\overline{L}_1}{G} &= -\frac{n_0^2}{If_0^{(L,L)}\omega^2}\cdot\frac{r_0}{a}\left[\left(11\overline{S}_{31}+20\overline{S}_{32}\right)PX_1^{-4.1}+\left(\overline{S}_{31}+4\overline{S}_{32}\right)QX_3^{-4.3}\right]\\
\mu\frac{\overline{H}_1}{G} &= \frac{n_0^2}{\sin\rho If_0^{(H,H)}\omega^2}\cdot\frac{r_0}{a}\left[\left(\overline{C}_{31}+30\overline{C}_{33}\right)RX_1^{-4.1}+\left(\overline{C}_{31}-2\overline{C}_{33}\right)SX_3^{-4.3}\right]\\
\mu\overline{l}_1 &= -\frac{1}{If^{(l,l)}}\cdot\frac{r_0}{a}\left[\left(\overline{S}_{31}+90\overline{S}_{33}\right)PX_1^{-4.1}+\left(\overline{S}_{31}-6\overline{S}_{33}\right)QX_3^{-4.3}\right]\\
\mu\overline{g}_1 &= -\frac{1}{I}\cdot\frac{r_0}{a}\cdot\frac{f_0^{(g,h)}-f_0^{(h,h)}}{f_0^{(g,g)}f_0^{(h,h)}-\left(f_0^{(g,h)}\right)^2}\left[\left(\overline{C}_{30}+10\overline{C}_{32}\right)PX_1^{-4.1}+\left(\overline{C}_{30}-6\overline{C}_{32}\right)QX_3^{-4.3}\right]\\
\mu\overline{h}_1 &= -\frac{1}{I}\cdot\frac{r_0}{a}\cdot\frac{f_0^{(g,h)}-f_0^{(g,g)}}{f_0^{(g,g)}f_0^{(h,h)}-\left(f_0^{(g,h)}\right)^2}\left[\left(\overline{C}_{30}+10\overline{C}_{32}\right)PX_1^{-4.1}+\left(\overline{C}_{30}-6\overline{C}_{32}\right)QX_3^{-4.3}\right]\\
\mu\overline{\theta}_1 &= \frac{n_0^2}{If_0^{(L,L)}\omega^2}\cdot\frac{r_0}{a}\left[\left(11\overline{S}_{31}+20\overline{S}_{32}\right)PX_1^{-4.1}+\left(\overline{S}_{31}+4\overline{S}_{32}\right)QX_3^{-4.3}\right]\\
\mu\overline{\rho}_1 &= -\frac{n_0^2}{\sin^2\rho If_0^{(H,H)}\omega^2}\cdot\frac{r_0}{a}\left[\left(\overline{C}_{31}+30\overline{C}_{33}\right)RX_1^{-4.1}+\left(\overline{C}_{31}-2\overline{C}_{33}\right)SX_3^{-4.3}\right]\\
\overline{p}_1 &= \omega\overline{l}_1, \overline{q}_1 = \omega\frac{\overline{G}_1}{G} = 0, \overline{r}_1 = -\omega\overline{\theta}_1, \overline{G}_1 = 0.
\end{aligned} \tag{21}$$

where $n_0^2 = fm_E/a^3$ (f is a gravitational constant, m_E is a mass of the Earth), $\omega = n_F$ is a frequency of argument $F$of the lunar orbital motion. For the second partial derivatives of the function $\langle F_1\rangle$ we will use the standard notations in accordance with equation (19):

$$\frac{\partial^2\langle F_1\rangle}{\partial L^2} = \frac{1}{B}f_0^{(L,L)}, \frac{\partial^2\langle F_1\rangle}{\partial H^2} = \frac{1}{B}f_0^{(H,H)}, \frac{\partial^2\langle F_1\rangle}{\partial l^2} = Bn_0^2f_0^{(l,l)}, \frac{\partial^2\langle F_1\rangle}{\partial g^2} = Bn_0^2f_0^{(g,g)}, \frac{\partial^2\langle F_1\rangle}{\partial g\partial h} = Bn_0^2f_0^{(g,h)}, \frac{\partial^2\langle F_1\rangle}{\partial h^2} = Bn_0^2f_0^{(h,h)} \tag{23}$$

To formulae (21) we add formula for the constant component of the angular momentum of the Moon of the second order $\overline{G}_2$, which also follows from the algebraic equation (18):

$$\mu^2\frac{\overline{G}_2}{G} = \cot\rho\frac{n_0^2}{I\omega^2}\frac{r_0}{a}\left[\left(\overline{C}_{31}+30\overline{C}_{33}\right)RX_1^{-4.1}+\left(\overline{C}_{31}-2\overline{C}_{33}\right)SX_3^{-4.3}\right] \tag{22}$$

In the case of the model of uniformly precessing orbit of the Moon dimensionless coefficients, $f_0^{(L,L)}$, $f_0^{(H,H)}$, …, $f_0^{(h,h)}$, are determined by:

$$\begin{aligned}
\mu f_0^{(L,L)} &= \frac{1}{I}(J_2-2C_{22})\left\{1-\frac{3}{8}\frac{n_0^2}{\omega^2}\left[-2\left(2-3\sin^2\rho\right)X_0^{(-3.0)}+(1+\cos\rho)^2X_2^{(-3.2)}\right]\right\}\\
\mu f_0^{(H,H)} &= \frac{n_\Omega\sin i}{\omega\sin^3\rho}-\frac{3}{2I}\cdot\frac{n_0^2}{\omega^2}\left(J_2X_0^{(-3.0)}+C_{22}X_2^{(-3.2)}\right)\\
\mu f_0^{(l,l)} &= \frac{\omega^2}{In_0^2}(J_2+2C_{22})\left\{1+\frac{3}{8}\frac{n_0^2}{\omega^2}\left[2\left(2-3\sin^2\rho\right)X_0^{(-3.0)}+(1+\cos\rho)^2X_2^{(-3.2)}\right]\right\}\\
\mu f_0^{(g,g)} &= \mu f_0^{(g,h)} = \frac{3}{I}C_{22}(1+\cos\rho)^2X_2^{(-3.2)}\\
\mu f_0^{(h,h)} &= \frac{n_\Omega\omega}{n_0^2}\sin\rho\sin i+\frac{3}{I}C_{22}(1+\cos\rho)^2X_2^{(-3.2)}
\end{aligned} \tag{24}$$

here $i = 5^{\circ}16$ is the constant angle of inclination of the plane of lunar orbit to the ecliptic plane. $n_{\Omega}$is the constant rate of precession of the lunar orbit with a period of 18.599 years. Frequencies n_0 and n_F have values $n_F = 17395266''10(1/\text{yr})$ and $n_0 = 17325643''7931(1/\text{yr})$. $\rho = 6^{\circ}782$ is a unperturbed value of the angle of inclination of axis of the Moon rotation relatively to normal to the plane of its mean orbit. Note that in this case the Andoyer variables are referred to precessing orbital plane. In the case of high-precision lunar orbit to study the rotational motion of the Moon we have used Andoyer variables referred to the ecliptic plane, as it was assumed in the analytic theory of the author [15].

In the case of high-precision lunar orbit the values of the dimensionless coefficients, $f_0^{(L,L)}, f_0^{(H,H)}, \ldots, f_0^{(h,h)}$(21) are calculated by more complex formulas obtained in Ref. [15], and depend on all coefficients from the Table 1:

$$\mu f_0^{(g,h)} = 673.1132 \times 10^{-6}, \quad \mu f_0^{(g,g)} = 672.2999 \times 10^{-6}$$
$$\mu f_0^{(H,H)} = -4.331321, \quad \mu f_0^{(h,h)} = 671.6517 \times 10^{-6}$$
$$\mu f_0^{(L,L)} = \mu f_0^{(\theta,\theta)} = 410.4448 \times 10^{-6}, \quad \mu f_0^{(l,l)} = 2506.765 \times 10^{-6} \tag{25}$$

For the analysis of the constant components of perturbations of the first order, we have used two models of the orbital motion of the Moon and in both cases obtained results were in good agreement. Here we give only sample estimates of these constants including accurate model of selenopotential (its second and third harmonics), obtained by Japanese scientists on the basis of data of a space mission “Selena” [10].

In the formulae (21) and (22) $r_0/a = 4.521 \times 10^{-3}$ is a ratio of mean radius of the Moon to unperturbed value of semi-axis of lunar orbit. Eccentricity functions $X_1^{(-4,1)}(e), X_3^{(-4,3)}(e)$ and values of others functions from (21) are given by:

$$f_0^{(l,l)} = \frac{4}{I}(J_2 + 2C_{22}), f_0^{(g,g)} = f_0^{(g,h)} = \frac{12}{I}C_{22}, f_0^{(h,h)} = \frac{n_\Omega}{n_0}\sin\rho\sin i + \frac{12}{I}C_{22},$$
$$f_0^{(L,L)} = \frac{1}{I}(J_2 - 2C_{22}), f_0^{(H,H)} = \frac{n_\Omega \sin i}{\omega \sin^3\rho} - \frac{3}{2I}(J_2 + C_{22}),$$
$$f_0^{(h,h)} - f_0^{(g,h)} = -\frac{n_\Omega \omega}{n_0^2}\cdot\frac{\sin\rho\sin i}{\cos h} = \frac{n_\Omega}{n_0}\sin\rho\sin i. \tag{28}$$

We emphasize that in (24), (25) and (28) we have used the standard notations for the coefficients of selenopotential J_2 and C_{22}. Simplified formulae (21) and (22) take the form:

$$\mu\frac{\overline{L}_1}{G} = \frac{9n_0^2}{4If_0^{(L,L)}\omega^2}\overline{S}_{31}\frac{r_0}{a}, \mu\overline{\theta}_1 = -\frac{9n_0^2}{4If_0^{(L,L)}\omega^2}\overline{S}_{31}\frac{r_0}{a}$$
$$\mu\frac{\overline{H}_1}{G} = \frac{9n_0^2}{4If_0^{(H,H)}\omega^2}\left(\overline{C}_{31} - 10\overline{C}_{33}\right)\frac{r_0}{a},$$
$$\mu\overline{\rho}_1 = -\frac{9n_0^2}{4\sin\rho If_0^{(H,H)}\omega^2}\left(\overline{C}_{31} - 10\overline{C}_{33}\right)\frac{r_0}{a}$$
$$\mu\overline{l}_1 = \frac{3}{2If^{(l,l)}}\left(\overline{S}_{31} - 30\overline{S}_{33}\right)\frac{r_0}{a},$$
$$\mu\overline{g}_1 = -\frac{3}{2I}\frac{f_0^{(g,h)} - f_0^{(h,h)}}{f_0^{(g,g)}f_0^{(h,h)} - \left(f_0^{(g,h)}\right)^2}\left(\overline{C}_{30} - 10\overline{C}_{32}\right)\frac{r_0}{a}$$
$$\mu\overline{h}_1 = -\frac{3}{2I}\frac{f_0^{(g,h)} - f_0^{(g,g)}}{f_0^{(g,g)}f_0^{(h,h)} - \left(f_0^{(g,h)}\right)^2}\left(\overline{C}_{30} - 10\overline{C}_{32}\right)\frac{r_0}{a}. \tag{29}$$

$$X_0^{-3,0}(e) = 1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \frac{35}{16}e^6, \quad X_2^{-3,2}(e) = 1 - \frac{5}{2}e^2 + \frac{39}{48}e^4 - \frac{175}{1440}e^6$$
$$X_1^{-4,1}(e) = 1 + 2e^2 + \frac{239}{64}e^4 + \frac{3323}{576}e^6, \quad X_3^{-4,3}(e) = 1 - 6e^2 + \frac{423}{64}e^4 - \frac{10000}{5120}e^6$$
$$P = \frac{3}{64}(1+\cos\rho)\left(1 + 10\cos\rho - 15\cos^2\rho\right), \quad Q = \frac{15}{64}(1+\cos\rho)^3$$
$$R = \frac{3}{64}\sin\rho\left(11 - 10\cos\rho - 45\cos^2\rho\right), \quad S = \frac{45}{64}\sin\rho(1+\cos\rho)^2 \tag{26}$$

If we neglect the small values of the eccentricity of the orbit and take into account the small values of angles ρ, for example, for some other's synchronous satellites of planets, then we have:

$$X_1^{(-4,1)}(e) = 1, X_3^{(-4,3)}(e) = 1, P = -\frac{3}{8}, Q = \frac{15}{8}, R = -\frac{33}{16}\sin\rho,$$
$$S = \frac{45}{16}\sin\rho A \tag{27}$$

Simplified expressions for (24) take the form:

In the case of a uniformly precessing orbit $f_0^{(g,h)} = f_0^{(g,g)}$ and we obtain $\overline{h}_1 = 0$,

$$\overline{g}_1 = \frac{3}{2If_0^{(g,g)}}\left(\overline{C}_{30} - 10\overline{C}_{32}\right)\frac{r_0}{a}.$$

In (21), (29) coefficients $\overline{J}_2, \overline{C}_{22}$ and $\overline{C}_{30}, \overline{C}_{31}, \overline{C}_{32}, \overline{C}_{33}, \overline{S}_{31}, \overline{S}_{32}, \overline{S}_{33}$ are the coefficients of the standard representation of selenopotential, but recorded in oblique coordinate system $C_M\overline{\xi}\overline{\eta}\overline{\zeta}$ (with non-standard orientation). They are connected with the generally accepted values of the coefficients of selenopotential J_2, C_{22} and $C_{30}, C_{31}, C_{32}, C_{33}, S_{31}, S_{32}, S_{33}$ by the simple relations:

$$\overline{C}_{20} = -\frac{1}{2}(C_{20} + 6C_{22}), \overline{C}_{21} = -2S_{22}, \overline{S}_{21} = S_{21}, \overline{S}_{22} = -\frac{1}{2}C_{21}, \overline{C}_{22} = \frac{1}{4}(-C_{20} + 2C_{22})$$
$$\overline{C}_{30} = -\frac{3}{2}(S_{31} + 10S_{33}), \overline{C}_{31} = \frac{1}{4}(C_{31} - 30C_{33}), \overline{C}_{32} = \frac{1}{4}(-S_{31} + 6S_{33}), \overline{C}_{33} = \frac{1}{8}(C_{31} - 2C_{33}) \tag{30}$$
$$\overline{S}_{31} = -\frac{1}{4}(10C_{32} + C_{30}), \overline{S}_{32} = -S_{32}, \overline{S}_{33} = \frac{1}{24}(6C_{32} - C_{30}).$$

We will use the high-precision values of selenopotential coefficients obtained in implementing Selena lunar mission (values are given in units 1×10^{-6}) [10]:

$$\begin{array}{lll} C_{20} = -203.4495 \pm 0.0037, & C_{21} = 0.0147 \pm 0.0014, & S_{21} = -0.0205 \pm 0.0014, \\ C_{22} = 22.3722 \pm 0.0008, & C_{30} = -8.4328 \pm 0.0046, & C_{31} = 28.4646 \pm 0.0019, \\ S_{31} = 5.8777 \pm 0.0015, & C_{32} = 4.8428 \pm 0.0006, & S_{32} = 1.6754 \pm 0.0005, \\ C_{33} = 1.7132 \pm 0.0002, & S_{33} = -0.2428 \pm 0.0002. & \end{array} \tag{31}$$

As a result, by formulae (29)–(31) we obtain the following estimates of the constant components of Andoyer variables and constant components of the angular velocity of the Moon:

$$\begin{array}{lll} \mu\overline{\theta}_1 = 130''391 \pm 0''035, & \mu\overline{\rho}_1 = -7''3276 \pm 0''0006, & \mu\overline{l}_1 = 80''9399 \pm 0''0151 \\ \mu\overline{g}_1 = 44''8621 \pm 0''0256, & \mu\overline{h}_1 = 24''9649 \pm 0''0143, & \mu\left(\overline{g}_1 + \overline{h}_1\right) = 69''8270 \pm 0''0399 \\ \mu\overline{p}_1/\omega = 80''9399 \pm 0''0151, & \mu^2\overline{q}_2\big/\omega = (-4.1606 \pm 0.0003) \times 10^{-6}, & \mu\overline{r}_1/\omega = -130''391 \pm 0''035 \end{array} \tag{32}$$

The solution (32) describes a number of interesting dynamic effects in the rotational motion of the Moon, which in spite of the fine structure can be considered as clarifies of Cassini's laws, which describe the main features and characteristics of the resonant rotation of the Moon, regarded as some average rotational motion in Cassini's style.

In this paper, we omit a detailed derivation of the formulas for the frequencies and periods of free librations of the Moon and formulas for solving variational equations in the Anduaye variables, and give the final results for their subsequent analysis and the transition to variations of the classical variables of the theory of physical libration of the Moon, which were obtained in the work [19]. In this paper, in the beginning, the solution of the variational equations was obtained in Anduaye variables $\mathbf{z} = (l, g, h, l_c; L, G, H, L_c)$ and then it was transformed to known classical variables $\boldsymbol{Z} = (P_1, P_2, \tau, \rho, I\sigma)$ [13] (see Table 2).

The table shows a fairly close agreement between the analytical theory being developed here [19,20], and the empirical theory constructed from laser observations [13].

Theorem 3. The motion according to Cassini's laws is described by the generating solution (13) − (15). Mean values of the variables **z** describe the perturbed mean intermediate rotational Moon motion close to the Cassini's motion having the following regularities:

1. the mean longitude of the descending node of intermediate plane $\mathbf{Q}_G$ orthogonal to the vector of angular momentum **G** in the ecliptic date differs from the mean longitude of the ascending node of the lunar orbit by $\mu\overline{h}_1 = 24''9649 \pm 0''0143$;
2. the mean value of the angle between the ecliptic date and intermediate plane differs from the generating value $\boldsymbol{\rho}_0$ by $\mu\overline{\rho}_1 = -7''3276 \pm 0''0006$;
3. in the average position the Moon angular velocity vector $\boldsymbol{\omega}$ does not coincide with the polar inertia axes $O\eta$ and its orientation is determined by the angles: $\mu\overline{\theta}_1 = 130''391 \pm 0''035$, $\mu\overline{l}_1 = 80''9399 \pm 0''0151$;
4. the mean values of equatorial projections of the Moon rotational angular velocity $\boldsymbol{\omega}$ on its principal inertia axes are different from zero and consist: $\mu\overline{p}_1/\omega = 80''9399 \pm 0''0151$, $\mu\overline{r}_1/\omega = -130''391 \pm 0''035$;
5. the mean Moon rotational angular velocity in the coordinate system $C_M XYZ$ differs from its unperturbed value $\omega = n_F$; and its value is determined by the formula $\mu^2\overline{q}_2/\omega = (-4.1606 \pm 0.0003) \times 10^{-6}$.

Table 2
Free and resonant librations of the Moon. Amplitudes, periods, variations of the corresponding classical variables.

N	Variables	Theory analytical	Theory empirical	Periods, days
1	δP_1	−3″3060	−3″306	27257.273
2	δP_1	−0″0320	−0″032	27.296
3	δP_1	0″0250	0″025	27.932
4	δP_2	8″1830	8″183	27257.273
5	δP_2	0″0320	0″032	27.296
6	δP_2	0″001	0″002	27.312
7	$\delta \tau_K$	1″7570	1″735	1056.210
8	$\delta \tau_K$	0″0774	0″077	27.185
9	$\delta \tau_K$	−0″0328	−0″032	27.239
10	$\delta \rho_K$	5″7402	5″753	27.185
11	$\delta \rho_K$	2″4330	2″437	27.239
12	$\delta \rho_K$	0″0320	0″029	8822.883
13	$I\delta\sigma_K$	5″7402	5″758	27.185
14	$I\delta\sigma_K$	−2″4330	−2″443	27.239
15	$I\delta\sigma_K$	−0″0320	0″033	8822.883

All dynamic effects described in 1 - 5 are caused by the influence of the third harmonic of the Earth-Moon force function U_M. Define contributions in discussed phenomena will give also another perturbing factors (more high harmonics of force functions, gravitational attraction of the Sun and planets and oth.).

Constant components (32) were calculated for the most modern model of the gravitational potential of the Moon, obtained on the base of Selena mission data [10]. For comparison, we have calculated the values of (29) obtained for the model of selenopotential LURE2 [8] (1 unit=10^{-6}):

$$C_{20} = -203.8, C_{22} = 22.4; C_{30} = -10.44, C_{31} = 28.6, \quad S_{31} = 8.8, C_{32} = 4.8, S_{32} = 1.7, C_{33} = 2.7, S_{33} = -1.1. \tag{33}$$

Coefficients (33) allow us to describe these effects in the rotation of the Moon:

$$\overline{\theta}_1 = 124''326, \quad \overline{\rho}_1 = -8''4103, \quad \overline{l}_1 = 79''376 \quad \overline{g}_1 = 142''477, \quad \overline{h}_1 = 79''2742, \quad \overline{g}_1 + \overline{h}_1 = 221''751 \tag{34}$$

Calculated values of the constants (32) and (34) obtained for different models of selenopotential are in good agreement with each other. But the constants $\mu\overline{g}_1$, $\mu\overline{h}_1$ and $\mu(\overline{g}_1 + \overline{h}_1)$ from equations (32) and (34) have a significant distinguish. Therefore, additional studies are necessary for constant components of the various dynamical and geometric characteristics of the rotational motion of the Moon, and the further development and refinement of models of selenopotential.

4. About the angles of inclination of the rotational axis and the angular momentum of Mercury

4.1. Model

Resonant motion of Mercury on Cassini have been studied well-known authors: Colombo, Beletskii, Peale, etc. [21,22]]. Moreover, as a rule for the perturbed orbital motion was taken on the motion of a uniformly precessing orbit (with a constant angular velocity $n_\Omega < 0$) with constant angle of inclination of orbit plane i relatively to the base plane (ecliptic plane or Laplace plane). The orbit is elliptical and is characterized by constant eccentricity e. In this study, the rotational motion of Mercury (as a celestial body with nonspherical rigid mantle and liquid core) on an evolving orbit, referred not to the Laplace plane and to ecliptic plane of the given epoch. We take into account not only the uniform precession of the orbit plane (the secular change in longitude of the ascending node of the orbit Ω), but the slow change in orbital inclination (i) with small angular velocity n_i.

As the base model of Mercury's orbit in the study of its rotational motion in this paper we take the mean orbit of this planet, whose parameters are given in the famous website http://ssd.jpl.nasa.gov/ (J2000) (epoch = J2000 = 2000 January 1.5):

$$i = 7^0 0028806, n_\Omega = \frac{d\Omega}{dt} = -446''30(1/\text{cy}), n_i = \frac{di}{dt} = -23''57(1/\text{cy}). \tag{35}$$

Period of orbital motion of Mercury is $T_n = 2\pi/n = 87.969$ days (n in the mean orbital motion) and period of progressive precession of the line of node of the orbit plane on the Laplacian plane is $T_\Omega = 2\pi/|n_\Omega| = 278898$ years.

An estimation of the inclination angle of the angular momentum relatively to the normal to the mean orbital plane ρ_G, is made on the basis of non-normalized values of the coefficients of the second harmonic of the gravitational field of Mercury J_2 and C_{22} (gravity model HgM001 MESSENGER) [11] on parameters of physical liberations and internal structure of this planet:

$$J_2 = (1.92 \pm 0.65) \times 10^{-5}, C_{22} = (0.81 \pm 0.08) \times 10^{-5}, C_m/mr^2 = 0.160 \pm 0.018, \tag{36}$$

derived from satellite observations by apparatus Messenger, ground-based radar observations [6–14]: $(B-A)/C_m = (2.03 \pm 0.12) \times 10^{-4}$, $\rho_\omega = 2'1 \pm 0'1$ and based on theoretical estimates [23]: $C/(mR^2) = 0.35, C_m/C = 0.5 \pm 0.07$. Here C and C_m are polar moments of inertia of Mercury and its mantle, m and R are mass and mean radius of Mercury.

4.2. Cassini's motion of Mercury

Based on the method of investigation of resonant rotational motion of Mercury, developed in [21,22,24], for considered here the model of an evolving orbit, we obtain the following analytical expressions and numerical characteristics of the generalized Cassini's motion of Mercury (for the unperturbed values of the ascending node h_0 and inclination angle ρ_G of the vector angular momentum of the planet relative to the ecliptic of 2000.0):

$$h_0 = \arctan(-n_i/n_\Omega \sin i) = 23^\circ 3677, \tag{37}$$

$$\rho_G = -\frac{n_\Omega}{n_0}\frac{\sin i}{\cos h_0}\left[\frac{n_\Omega}{n_0}\cos i + \frac{1}{I}\left(J_2 C_0^{(-3.0)} + 2C_{22} X_N^{(-3.2)}\right)\right]^{-1} = 4'2 \pm 1'4. \tag{38}$$

From these formulas it follows that due to influence of the angle h_0 the value of Cassini's angle ρ_G increases at 8.94%, compared with an earlier value for $h_0 = 0$ [21,22,24].

5. Conclusion

This paper gives the fullest and most detailed analytical description of the effects of constant angular displacements of axes of inertia of the Moon relatively to their positions and angular displacements of the axis of rotation of the Moon relatively to the pole of the polar axis of inertia. We have identified a subtle effect of the second order, which manifests itself in reducing the mean angular velocity of the Moon's due to gravitational influence of gravity through the third harmonic of selenopotential. The generalized Cassini's laws describing as the main regularities in the rotation of the Moon, and subtle patterns and dynamic effects of the first and second order of smallness with respect to dynamic oblatenesses of the Moon have been formulated and revised.

On the basis of a special approach the basic laws of the resonant rotation of Mercury, when as a primary coordinate plane shall not Laplace plane and the ecliptic of the given epoch have been studied. If the secular change in inclination of the orbit of Mercury to exclude from consideration, then the equations (37) and (38) can be simplified and its solution will be $h_0 = 0$ and $\rho_G = 3'9 \pm 1'3$. That solution with $h_0 = 0$ (for others values of Mercury parameters) before, starting with the pioneering works of Colombo, Beletskii, Peale et al., have been studied actively. We emphasize here that the rotational motion is not attributable to the Laplace plane for the Mercury, and with respect to the mean ecliptic and equinox of epoch 2000.0. In [22] we have assumed the existence of large amplitude free librations of Mercury in longitude with a period of 12 years despite the claim that the free vibrations of the poles of Mercury have completely subside within a relatively short period of time [23]. As the main mechanism of excitation of free oscillations was proposed mechanism of forced relative oscillations of the core and mantle of the planet (which are non-spherical bodies and occupy the eccentric positions relative to each other) under the gravitational attraction of the Sun and the planets and due to perturbations in orbital motion. Naturally, this same mechanism is responsible for the excitation of free motion of the pole and the free oscillations of angular momentum vector in space. The results of this study can be regarded as a confirmation of the existence of long-period (not Euler) oscillations of the rotation axis of Mercury with an amplitude of about 2′- 4′. And if the prediction of the free librations in longitude [21,22] has already received confirmation [6], to confirm the free oscillations of the pole axis of rotation of Mercury in the body and the free oscillations of angular momentum vector in space should be a new and more accurate data on the gravitational field of Mercury and its librations. This issue should contribute to research on the MESSENGER spacecraft on mercurial orbit in 2011. The action of the mechanism of forced swing and wobble of the core and mantle of Mercury (and of others solar system bodies) should lead to the formation of active geological structures with an asymmetric distribution with respect to the northern and southern hemisphere and especially pronounced in polar regions. The first position was clear evidence in the asymmetric distribution of scarps and ridges of Mercury. We expect that the second assumption will be confirmed in studies of the polar regions of the MESSENGER spacecraft in 2011. In conclusion remarks that the phenomenon revealed the lines shift of nodes $h_0 = 23^\circ 4$ can make some adjustments in the calculation according to radar observations of Mercury's rotation carried out in Ref. [6]. The new objectives and new proposals for the study of rotation, the internal structure and internal dynamics of the Moon and Mercury and another's planets and satellites have been formulated [8,9,25].

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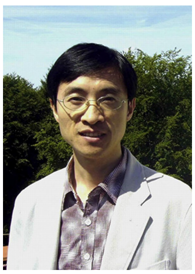

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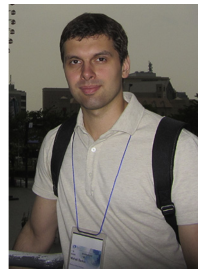

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