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One-sided performance measures under Gram Charlier distributions

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Abstract

We derive closed-form expressions for risk measures based on partial moments by assuming the Gram-Charlier (GC) density for stock returns. As a result, the lower partial moments (LPM) can be expressed as linear functions on both skewness and excess kurtosis. Under this framework, we study the behaviour of portfolio rankings with performance measures based on partial moments, that is, both Farinelli-Tibiletti and Kappa ratios. Contrary to previous results, significant differences are found in ranking portfolios between the Sharpe ratio and the FT family. We also obtain closed-form expressions for LPMs under the semi non-parametric (SNP) distribution which allows higher flexibility (in terms of third- and fourth- order moments) than the GC distribution.

Keywords: Lower/Upper partial moment, Certainty Equivalent, Rank correlation, SNP distribution.
JEL classification: C10, C61, G11, G17.

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1 Introduction

An adequate risk-adjusted return performance measure (PM) is essential for selecting investment funds. The Sharpe ratio (Sharpe, 1966, 1994) has become the bechmark PM by adjusting the expected excess fund return by the symmetric risk measure or standard deviation. Although this ratio is still a reference indicator for assessing the accuracy of investment strategies, its use becomes rather doubtful when the fund return distribution is beyond the class of elliptical distributions (Owen and Rabinovitch, 1983) that include the normal distribution. As a result, several one-sided type measures of risk have been proposed and the associated PMs are known as one-sided PMs. In fact, some of these PMs are also characterized by one-sided reward measures.

Some examples of one-sided PMs are the adjusted for skewness Sharpe ratio (ASSR) proposed by Zakamouline and Kokebakker (2009), the Generalized Rachev family based on the conditional Value at Risk (Biglova et al. 2004), the Farinelli-Tibiletti (FT) family based on both upper and lower partial moments (Farinelli and Tibiletti, 2008) and the Kappa or S-S family (Sortino and Satchell, 2001) based on lower partial moments. Other alternative reward-to-variability ratios are well documeted in Caporin et al. (2014) and the references therein. We will also implement PMs based on the certainty equivalent amount as a function of both prudence and temperance coefficients. These coefficients are related to the investor's appetite for asymmetry and aversion to leptokurtocity of fund returns. For details, see Eeckhoudt and Schlesinger (2006), Ebert (2013) and references therein.

Some papers as Eling and Schuhmacher (2007), Eling (2008) and Auer (2015) find that chosing different PMs is not critical to the portfolio evaluation. More specifically, the PM choice does not matter because any PM generates the same rank ordering as the Sharpe ratio (SR). Guo and Xiao (2016) agree with this result whenever the selected PMs satisfy the monotonicity property regarding the SR and the fund return distributions belong to the location-scale (LS) family. Indeed, many PMs hold the monotonocity property and some popular elliptical multivariate distributions for modeling stock returns (Normal, t-Student, Logistic, Exponential, etc.) belong to the LS family.

In contrast, in this paper we show that some PMs like the FT family can generate different rank scores meaning that the selected PM matters. Particularly, we get a closed-form expression for FT measures by assuming a return distribution that does not belong to the LS family. To be more precise, we consider the Gram-Charlier (GC) expansion as the probability density function (pdf). The GC distribution has been implemented, among others, by Corrado and Su (1996), Jondeau and Rockinger (2001) and Jurczenko and Maillet (2006). The advantage of this distribution is that both skewness (s) and excess kurtosis (ek) appear directly as the pdf's parameters. We previously get the closed-form expressions for the LPMs as simple linear functions of both parameters. As a consequence, we can easily understand the behaviour of these risk measures regarding changes in these higher moments. By expressing the upper moments in terms of LPMs, we can focus just on this kind of downside risk measure[1] and analyze its properties under the GC distribution when studying the FT measures. Similarly, we obtain closed-form expressions for the Kappa measures under the previous distribution.[2]

Finally, the GC restriction to capture higher levels of s and ek suggests some other distributions to seize better these higher moments but, unfortunately, leading to more complex expressions for the PMs. For instance, a more flexible distribution to the restricted higher moments under the GC distribution can be the semi-nonparametric (SNP) density proposed by Gallant and Nychka (1987). We also obtain the LPM analytical expressions under the SNP distribution.

The rest of the paper is organized as follows. In Section 2 we present different PMs based on either the Expected Utility Theory (EUT) or the Prospect Theory/Cumulative Prospect Theory (PT/CPT), see Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Section 3 shows the GC distribution and some properties. In Section 4 we obtain closed-form expressions for LPM and UPM measures under GC and hence, the expressions for both FTR and Kappa ratios. We also analyze the behaviour of the Kappa ratios regarding the levels of s and ek and also obtain the iso-curves for the Kappa measures.In Section 5 we conduct a simulation study on the performance evaluation. Section 6 shows the SNP distribution and the corresponding LPM expressions. Finally, Section 7 summarizes and provides the main conclusions. The proofs of propositions and corollaries are deferred to a final technical Appendix.

2 Performance Measures (PMs)

Let $U(W)$ denote the investor's utility function where W is the amount of wealth. The investor faces a capital allocation problema that is solved by maximizing his expected utility of wealth $E[U(W)]$, where $E[\cdot]$ is the expectation operator. The market includes a risky asset and a risk-free one. Assume that the initial wealth is W_I and the capital allocation aims to invest an amount a in the risky asset and, hence, $W_I - a$ in the risk-free asset. Thus, the investor's final wealth is

$$W(r,a) = a(r - r_f) + W_I(1 + r_f), \tag{1}$$

where r is a random variable that denotes the return of the risky asset and r_f is the risk-free rate of return that is assumed to be a constant. Assuming that $a \geq 0$ (short-selling is not allowed), the investor's objective is selecting a to maximize the expected utility:

$$E[U(W(r,a^*))] = \max_a E[U(W(r,a))], \tag{2}$$

[1]Some seminal references on LPMs are Bawa (1975), Bawa and Lindenberg (1977), Fishburn (1977), Holthausen (1981), Harlow and Rao (1989) and Harlow (1991).

[2]In the same spirit, Passow (2005) obtains a closed-form expression for the Sharpe-Omega ratio (that belongs to the Kappa family) under the (more flexible) Johnson distribution family. The drawback is that the above ratio becomes more cumbersome and, then, more difficult to interpret than when assuming a GC distribution.

where a^* denotes the optimal amount invested in the risky asset from the maximization of the expected utility on the final wealth in (1). Besides EUT as the benchmark model of choice under uncertainty, we are interested in those models under PT/CPT where the utility function is defined over gains and losses relative to some reference point (kink), as opposed to wealth in EUT.

By using the *maximum principle* method,[3] we can rewrite (2) as $E\left[U\left(W\left(r,a^*\right)\right)\right] = h\left(\pi\left(r\right)\right)$ where $h\left(\cdot\right)$ is a strictly increasing function and $\pi\left(r\right)$ represents the PM.[4] More specifically, the investor prefers the risky portfolio r_1 to the risky portfolio r_2 if $\pi\left(r_1\right) > \pi\left(r_2\right)$. Hence, the aim at maximizing the investor's expected utility can alternatively be formulated as the maximization of a particular PM. In addition, a rational utility-based PM must be consistent with the stochastic-dominance principles that will be analyzed later.

Finally, a GC probability distribution for the returns of the risky asset will be assumed to obtain closed-form PM expressions under both EUT and PT/CPT. The reason for this specific distribution is because we can get a very easy interpretation in terms of the implied distribution parameters which are both skewness and kurtosis.

2.1 PMs based on EUT

These PMs will be obtained by implementing the maximum principle method and using the certainty equivalent (CE) amount corresponding to $E\left[U\left(W\left(r,a^*\right)\right)\right]$ in (2) for ranking portfolios. Thus,

$$E\left[U\left(W\left(r,a^*\right)\right)\right] = \max_a U\left(CE\right), \tag{3}$$

such that $CE = \mu_W - \xi$, where $\mu_W = E\left[W\left(r,a\right)\right]$ represents the expected final wealth and ξ denotes the risk premium.

2.1.1 Certainty Equivalent as PM

Let U be a utility function with desirable properties, that is, $U^{(1)} > 0$, $U^{(2)} < 0$, $U^{(3)} > 0$ and $U^{(4)} < 0$, where $U^{(i)}$ denotes the i-th derivative of the utility function. We start from the equation defining the CE amount given by

$$U\left(E\left[W\left(r,a\right)\right] - \xi\right) = E\left[U\left(W\left(r,a\right)\right)\right]. \tag{4}$$

First, on the right-hand side in (4), approximate the utility function $U\left(x\right)$ by a fourth-order Taylor expansion around the point $x_0 = \mu_W$, where $x = W$, and take expectations. Second, on the left-hand side in (4), apply a first-order Taylor

[3]This method is presented by Pedersen and Satchell (2002), and later in Zakamouline and Koekebakker (2009a, b) and Zakamouline (2014). Finally, under some conditions, we can find an explicit solution for the amount a.

[4]Note that the PM is not unique since any positive increasing transformation of a PM leads to an equivalent PM.

expansion around the same point x_0 but now $x = \mu_W - \xi$. Then, the maximum CE amount satisfying (3) is given by

$$CE^* \simeq \max_a \left\{ W_0 + a(\mu - r_f) - \frac{1}{2}\gamma\sigma^2 a^2 + \frac{1}{3!} s\psi_3\sigma^3 a^3 - \frac{1}{4!} k\psi_4\sigma^4 a^4 \right\}, \qquad (5)$$

where μ, σ, s, and k denote, respectively, the mean, standard deviation, skewness and kurtosis of the risky asset return and $W_0 = W_I(1 + r_f)$. Let $\omega_k = U^{(k)}(\mu_W)$, then $\gamma = -\omega_2/\omega_1$ is the traditional absolute risk aversion coefficient, $\psi_3 = \omega_3/\omega_1$ is the coefficient of *appetite toward asymmetry* and $\psi_4 = -\omega_4/\omega_1$ is the coefficient of *aversion to leptokurticity*.[5] Note that both ψ_3 and ψ_4 are connected with the popular coefficients of *prudence* and *temperance*. In particular, $\psi_3 = \gamma\delta_P$, where $\delta_P = -\omega_3/\omega_2$, is the coefficient of (absolute) prudence introduced by Kimball (1990), and $\psi_4 = \gamma\delta_P\delta_T$, where $\delta_T = -\omega_4/\omega_3$, is the coefficient of temperance presented in Kimball (1992).

2.1.2 Adjusted for Skewness Sharpe ratio ($ASSR$)

By applying the maximum principle method, Zakamouline and Koekebakker (2009) obtain a closed-form expression for $\pi(r)$ by using a third-order Taylor expansion of $U(\cdot)$ around the point W_0 and considering the HARA utility function. More specifically, the adjusted for skewness Sharpe ratio ($ASSR$) is defined as

$$ASSR = SR\sqrt{1 + \varphi\frac{s}{3}SR}, \qquad (6)$$

such that SR denotes the Sharpe ratio, defined as $SR = (\mu - r_f)/\sigma$, where μ and σ denote, respectively, the mean and standard deviation of the risky asset return, the coefficient $\varphi = \delta_P/\gamma$ is related to the investor's preferences through the coefficients of prudence and risk aversion, and s is the skewness coefficient for the return of the risky asset.

Some properties of the $ASSR$ are the following:

1. If all the investors have the same HARA utility (i.e., the parameter φ does not change), the PM value is the same for all of them.

2. Note that (6) is just SR times a skewness adjusted factor and so, the $ASSR$ nests the SR for symmetric return distributions.

3. It holds that $\varphi = 0$ for quadratic utility, $\varphi = 1$ for CARA utility (i.e., negative exponential utility), and $\varphi = 2$ for logarithmic utility. The case for CRRA utility (i.e., the power utility defined as $U(x) = \frac{x^{1-\lambda}}{1-\lambda}$ for $\lambda \neq 1$, where λ is the coefficient of relative risk aversion) leads to $\varphi = (\lambda + 1)/\lambda$. Note that $\partial\varphi/\partial\lambda < 0$ for any λ. These types of utility functions are nested in the HARA utility.

[5] The optimal amount a^* in (5) comes from solving the equation $\alpha_3 a^3 + \alpha_2 a^2 + \alpha_1 a + \alpha_0 = 0$ where $\alpha_0 = \mu - r_f$, $\alpha_1 = -\gamma\sigma^2$, $\alpha_2 = \frac{1}{2}s\psi_3\sigma^3$ and $\alpha_3 = -\frac{1}{3!}k\psi_4\sigma^4$. For more details, see Le Courtois (2012).

4. Finally, the $ASSR$ can be interpreted as a particular case of the generalized Sharpe ratio (GSR) introduced by Hodges (1998).[6]

2.2 PMs based on PT/CPT

As it was mentioned at the beginning of this Section, under PT/CPT, the utility function is defined over gain and losses relative to a reference point. It means that the utility function exhibits a kink at the reference point, with the slope of the loss function steeper than that of the gain function. This is called *loss aversion.* In this new framework, the investor can show three different types of aversion: aversion to loss and aversion to uncertainty either in gains or losses.

Let $\overline{W}$ denote the reference point for the wealth. Köbberling and Wakker (2005) proposes a measure for loss aversion (in a local sense) as given by $\lambda_{KW} = U^{(1)}\left(\overline{W}_-\right)/U^{(1)}\left(\overline{W}_+\right)$ where $U^{(1)}\left(\overline{W}_-\right)$ and $U^{(1)}\left(\overline{W}_+\right)$ denote, respectively, the left and right derivatives of U at $\overline{W}$. For a loss-neutral investor, we have $\lambda_{KW} = 1$ while loss aversion (seeking) implies $\lambda_{KW} > 1$ ($\lambda_{KW} < 1$). In particular, the utility investor has the generalized form of a piecewise linear function plus a power one:

$$U(W) = \begin{cases} \mathbf{1}_+\left(W - \overline{W}\right) - \left(\gamma_+/q\right)\left(W - \overline{W}\right)^q, & \text{if } W \geq \overline{W}, \\ -\lambda\left[\mathbf{1}_-\left(\overline{W} - W\right) + \left(\gamma_-/m\right)\left(\overline{W} - W\right)^m\right], & \text{if } W < \overline{W}, \end{cases} \tag{7}$$

where $\mathbf{1}_+$ and $\mathbf{1}_-$ take values in $\{0, 1\}$, γ_+ and γ_- are real numbers, and $\lambda,\ q,\ m > 0$. We can rewrite (7) as

$$U(W(r,a)) = \begin{cases} \mathbf{1}_+ a\left(r - \overline{\tau}\right) - \left(\gamma_+/q\right)a^q\left(r - \overline{\tau}\right)^q, & \text{if } r \geq \overline{\tau}, \\ -\lambda\left[\mathbf{1}_- a\left(\overline{\tau} - r\right) + \left(\gamma_-/m\right)a^m\left(\overline{\tau} - r\right)^m\right], & \text{if } r < \overline{\tau}, \end{cases} \tag{8}$$

where $\overline{\tau} = r_f - \left(W_0 - \overline{W}\right)/a$. See Zakamouline (2014) for more details. Equation (8) implies that $E\left[U\left(W(r,a)\right)\right]$ can be expressed in terms of lower and partial moments.

A lower partial moment (LPM) measures risk by negative deviations of the stock returns, r, with respect to a minimal acceptable return (return threshold), τ. Fishburn (1977) defines the LPM of order m for a stock return as

$$LPM(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr, \tag{9}$$

where $f(\cdot)$ denotes the probability density function (pdf) of r. Similarly, the upper partial moment (UPM) is defined as

$$UPM(\tau, q) = \int_{\tau}^{\infty} (r - \tau)^q f(r)\, dr. \tag{10}$$

If we set $\overline{W} = W_0$ in (8), then $\overline{\tau} = r_f$ and the investor's expected utility can be rewritten as

$$\begin{aligned} E\left[U\left(W(r)\right)\right] &= \mathbf{1}_+ aUPM(r_f, 1) - \left(\gamma_+/q\right)a^q UPM(r_f, q) \\ &\quad -\lambda\left[\mathbf{1}_- aLPM(r_f, 1) + \left(\gamma_-/m\right)a^m LPM(r_f, m)\right]. \end{aligned} \tag{11}$$

[6] $GSR = \sqrt{-2\log\left(-E\left[U\left(W(r, a^*)\right)\right]\right)}$ by assuming a negative exponential utility and zero initial wealth.

If we maximize (11) with respect to a, the first order condition is given by

$$\begin{aligned}&\mathbf{1}_+ UPM(r_f,1) - \lambda \mathbf{1}_- LPM(r_f,1)\\&-a^{q-1}\left[\gamma_+ UPM(r_f,q) + a^{m-q}\lambda\gamma_- LPM(r_f,m)\right] = 0.\end{aligned} \tag{12}$$

Next, some closed-form expressions for a are obtained by considering two particular cases in (8). By applying the maximum principle, the PM will be obtained as a function of $E\left[U\left(W\left(r,a^*\right)\right)\right]$ in (2).

2.2.1 The Kappa measures

Consider the following case in (8): $\mathbf{1}_+ = \mathbf{1}_- = 1$, $\lambda = 1$, $\gamma_+ = 0$, $\gamma_- > 0$ and $q = m > 1$. This setting leads to a piecewise linear (in both sides of the return threshold r_f) plus a power function (in one side, just for $r < r_f$). In this case, the investor exhibits loss aversion. Note that the excess expected return can be expressed as $\mu - r_f = UPM(r_f,1) - LPM(r_f,1)$. Then, (12) simplifies to $(\mu - r_f) - a^{m-1}\gamma_- LPM(r_f,m) = 0$. Solving for a in this equation and applying the maximum principle, we obtain the group of PMs known as the S-S or Kappa ratios (see Sortino and Satchell, 2001), defined for any return threshold τ as

$$K(\tau,m) = \frac{\mu-\tau}{\sqrt[m]{LPM(\tau,m)}}, \tag{13}$$

where $\mu-\tau$ is the excess expected return with respect to τ. Some popular PMs are:

1. The Omega-Sharpe ratio (see Kaplan and Knowles, 2004), which can be considered as the limit case of $K(r_f,m)$ in (13) for $m \to 1$ (see Zakamouline, 2014).
2. The Sortino ratio (see Sortino and Van der Meer, 1991) for $m = 2$.
3. The Kappa 3 ratio (see Kaplan and Knowles, 2004) for $m = 3$.

2.2.2 The Farinelli-Tibiletti (FT) measures

Consider now the following case in (8): $\mathbf{1}_+ = \mathbf{1}_- = 0$, $\gamma_+ = -q$, $\gamma_- = m$ such that $m > q > 0$. These restrictions lead to a piecewise power utility function. In this case, the equation ((12)) can be rewritten as $q\,UPM(r_f,q) - a^{m-q}\,m\,\lambda\,LPM(r_f,m) = 0$. Following the same steps as in the Kappa measures, we get the group of PMs known as the Farinelli-Tibiletti (FT) ratios, see Farinelli and Tibiletti (2008). For any return threshold τ, the FT is defined as

$$FT\left(\tau,q,m\right) = \frac{\sqrt[q]{UPM(\tau,q)}}{\sqrt[m]{LPM(\tau,m)}}. \tag{14}$$

Note that (14) nests two popular PMs:

1. The Omega ratio[7] which is obtained as the limit of $FT(\tau, 1, m)$ in (14) when $m \to 1$. This ratio can also be expressed (see Keating and Shadwick, 2002) as:
$$FT(\tau, 1, 1) = \frac{\int_{\tau}^{\infty} (1 - F(r))\, dr}{\int_{-\infty}^{\tau} F(r) dr}, \tag{15}$$
where $F(\cdot)$ is the cumulative distribution function of r.

2. The Upside Potential ratio (see Sortino et al., 1999) when $q = 1,\ m = 2$.

2.2.3 Relationship between Kappa and FT measures

If we rewrite $UPM(\tau, q)$ in terms of $LPM(\tau, m)$ (see equations (9)-(10)), we get an alternative expression for the FT ratio (that will be useful later), as shown in the following Corollary.

Corollary 1 *Let $\psi(\tau, q) = E[(r - \tau)^q]$ and let $f(\cdot)$ be the pdf of the portfolio stock return r. Then, (14) can be expressed as*
$$FT(\tau, q, m) = \frac{\sqrt[q]{\psi(\tau, q) + (-1)^{q+1} LPM(\tau, q)}}{\sqrt[m]{LPM(\tau, m)}}. \tag{16}$$
■

Using the relationship between $LPM(\tau, m)$ amd the Kappa measures (see (13)), equation (16) becomes
$$FT(\tau, q, m) = \frac{\sqrt[q]{\psi(\tau, q) + (-1)^{q+1} (\mu - \tau)^q K(\tau, q)^{-q}}}{(\mu - \tau) K(\tau, m)^{-1}}. \tag{17}$$
Note that, for $q = 1$, equation (17) simplifies to $FT(\tau, 1, m) = K(\tau, m)\left[1 + K(\tau, 1)^{-1}\right]$.

2.3 Properties of utility-based PMs

The above two groups of PMs (i.e., based on EUT and PT/CPT) will be consistent with the first-order stochastic dominance (FSD) when the underlying utility function is everywhere increasing. The Kappa and FT measures will satisfy FSD when $q,\ m > 0$ in (8). Note that this restriction holds since $q = m > 1$ for Kappa measures (see (13)) whereas $m > q > 0$ for FT measures (see (14)).

Regarding the second-order stochastic dominance (SSD), a specific PM will be consistent with SSD if the utility function is everywhere increasing and concave. Under the EUT framework, this property is held for PMs like $ASSR$ and GSR. The Kappa ratios are also consistent with SSD because the restriction $m > 1$ guarantees

[7] Note that $K(\tau, 1) = FTR(\tau, 1, 1) - 1$. The ratio introduced by Bernardo and Ledoit (2000) is just the Omega measure for $\tau = 0$ (or gain-to-loss ratio).

that the utility function (8) is concave. Nevetheless, under the utility function (8), the FT ratios (where $m > q$) are consistent with SSD when $q < 1 < m$ since (8) is concave everywhere. On the contrary, for other values of q and m, the stochastic dominance (SD) principles cannot be applied.

Levy and Levy (2002) extend the classical SD principles by developing Prospect and Markowitz SD (PSD and MSD, respectively) theories with S-shaped and reverse S-shaped utility functions. Thus, a PM is consistent with second-order PSD (MSD) when the utility function is increasing, convex (concave) below the reference point and concave (convex) above the reference point. In particular, FT measures with $0 < q,\ m < 1$ are consistent with second-order PSD since (8) becomes a S-shaped utility function. FT measures with $q,\ m > 1$ verify the second-order MSD where (8) is a reverse S-shaped utility function. In brief, it is straightforward to prove the following Corollary that summarizes the main results for the above PMs based on PT/CPT.

Corollary 2 *Consider the utility function $U(\cdot)$ in (8) where $q,\ m > 0$, $\lambda > 0$, and $a > 0$ (i.e., short selling is not allowed). Then, the following properties are verified:*

1. *$U(\cdot)$ is increasing everywhere and, hence, Kappa and FT measures are consistent with FSD.*

2. *Kappa measures are consistent with SSD when $m > 1$.*

3. *By setting $m > q$, then FT measures are consitent with*

   (a) *SSD when $q < 1 < m$.*
   (b) *Second-order PSD when $q,\ m < 1$ (S-shaped utility function).*
   (c) *MSD when $q,\ m > 1$ (reverse S-shaped utility function).*

∎

3 GC density and properties

We assume that the stock return r is a random variable defined as

$$r = \mu + \sigma z, \qquad z \sim GC(0, 1, s, ek), \tag{18}$$

such that the pdf of the (standardized) random variable z is the Gram-Charlier expansion with zero mean, unit variance, skewness s and excess kurtosis ek. Hence, the return in (18) is just an affine transformation of a random variable with GC expansion as pdf. The GC pdf, denoted as $g(z)$, is defined as[8]

$$g(z) = p(z)\,\phi(z)\,; \quad p(z) = 1 + \frac{s}{\sqrt{3!}} H_3(z) + \frac{ek}{\sqrt{4!}} H_4(z), \tag{19}$$

[8]The pdf of r in (18) is obtained as $f(r) = g(z)/\sigma$.

where $\phi(\cdot)$ denotes the pdf of the standard normal variable and $H_k(z)$ is the normalized Hermite polynomial of order k. These polynomials can be defined recursively for $k \geq 2$ as

$$H_k(z) = \frac{zH_{k-1}(z) - \sqrt{k-1}H_{k-2}(z)}{\sqrt{k}}, \tag{20}$$

with initial conditions $H_0(z) = 1$ and $H_1(z) = z$. It holds that $\{H_k(z)\}_{k\in\mathbb{N}}$ constitutes an orthonormal basis with respect to the weighting function $\phi(z)$, that is,

$$E_\phi[H_k(z)H_l(z)] = \mathbf{1}_{(k=l)}, \tag{21}$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function and the operator $E_\phi[\cdot]$ takes the expectation of its argument regarding $\phi(\cdot)$. Note that $E_\phi[H_k(z)] = 0$ for $k \geq 1$ and then, (21) is just the covariance between $H_k(z)$ and $H_l(z)$.

The pdf $g(z)$ in (19) can lead to negative values for certain values of both centered moments. Jondeau and Rockinger (2001) obtain numerically a restricted space Γ for possible values of (ek, s) that guarantees the positivity of $g(z)$. The constrained GC expansion restricted to Γ will be referred as the true GC density. Figure 1 exhibits Γ with a frontier (the envelope) delimiting the oval domain. Γ is a compact and convex set. Note that $ek \in [0, 4]$ while $|s| \leq 1.0493$ verifying that the range of s depends on the level of ek. For instance, if $|s| = 0.6$ then ek ranges from 0.6908 to 3.7590, while for $s = 0$, ek ranges from 0 to 4. The maximum size for skewness is reached for $ek = 2.4508$. Obviously, the case for the normal distribution corresponds to the origin.

[INSERT FIGURE 1 AROUND HERE]

3.1 Moments of r and z

The first four (non-central) moments of z with pdf $g(\cdot)$ in (19) can be obtained by using the relationship between the powers of z and the Hermite polynomials in (20) and the condition in (21):

$$E_g[z] = 0, \quad E_g\left[z^2\right] = 1, \quad E_g\left[z^3\right] = s, \quad E_g\left[z^4\right] = ek + 3.$$

The following Proposition provides a general expression for $E_g\left[z^k\right]$ where $k \in \mathbb{N}_+$.

Proposition 1 *The general expression for $E_g\left[z^k\right]$, where $k \geq 5$ and pdf $g(\cdot)$ in* (19)*, is given as*

$$E_g\left[z^k\right] = \begin{cases} \lambda_{k,0} + \lambda_{k,1}\, ek, & k \text{ is even}, \\ \lambda_{k,2}\, s, & k \text{ is odd}, \end{cases} \tag{22}$$

where $\lambda_{k,l} \in \mathbb{R}$ can be seen in the Appendix. Since r in (18) *is an affine transformation of z, the non-central moments of r are obtained as*

$$E_f[r^k] = \sum_{n=0}^{k} \binom{k}{n} \mu^{k-n}\sigma^n E_g[z^n], \tag{23}$$

where $\binom{k}{n} = \frac{k!}{n!(k-n)!}$. ∎

Proof. See the Appendix. ■

Note that, if k is even (odd), $E_g\left[z^k\right]$ depends only on the excess kurtosis (skewness).

3.2 Cumulative density function

For $m = 0$, the LPM expression (see (9)) becomes the distribution function for the standardized return z in (18).

Proposition 2 *The cumulative density function of r in* (18) *is given by*

$$F(r) = \Phi(r^*) - \frac{s}{3\sqrt{2!}} H_2(r^*)\,\phi(r^*) - \frac{ek}{4\sqrt{3!}} H_3(r^*)\,\phi(r^*), \tag{24}$$

where $r^* = (r-\mu)/\sigma$. ■

Proof. See the Appendix. ■

The following Corollary shows the behaviour of $F(r)$ with respect to the parameters s, ek, μ and σ.

Corollary 3 *Let $F(r)$ in* (24) *and $r^* = (r-\mu)/\sigma$. Then, it holds that*

1. $\partial F(r)/\partial s > 0 \Leftrightarrow |r^*| < 1$.
2. $\partial F(r)/\partial ek > 0 \Leftrightarrow r^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$.
3. *For $r^* \in \left(-\sqrt{3-\sqrt{6}}, 0\right) \cup \left(\sqrt{3+\sqrt{6}}, +\infty\right)$, then $\partial F(r)/\partial r^* > 0$ and so,*
   (a) $\partial F(r)/\partial \mu < 0$.
   (b) $\partial F(r)/\partial \sigma > 0$ *iff* $\mu > r$. ■

Proof. See the Appendix. ■

Hence, for $r^* \in (0,1)$, we have that both $\partial F(r)/\partial s$ and $\partial F(r)/\partial ek$ are positive.

4 LPMs and related PMs with GC distribution

This Section starts providing the closed-form expressions of $LPM_f(\tau, m)$, where the stock return is driven by (18), and the related expressions for $K_f(\tau, m)$ and $FTR_f(\tau, q, m)$. Later, we show that these LPMs are linear functions of both s and ek and analyze the behaviour of the above PMs. Finally, we obtain some Kappa iso-curves.

4.1 Closed-form expressions of $LPM_f\,(\tau, m)$

Proposition 3 *Let z be the standardized return of r in (18). The lower partial moment of order $m \in \mathbb{N}_+$ for the stock return r can be expressed as*

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}, \tag{25}$$

where $LPM_n(\tau, m)$ is the LPM by assuming a normal distribution $n\,(\cdot)$ for r in (18) with

$$LPM_n(\tau, m) = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k}\, \sigma^k B_k, \tag{26}$$

and

$$\theta_{j,m} = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k}\, \sigma^k A_{kj}, \tag{27}$$

such that $B_k = B_k\,(\tau^*)$ and $A_{kj} = A_{kj}\,(\tau^*)$, with $\tau^* = (\tau - \mu)\,/\sigma$, can be seen in the Appendix. ■

Proof. See the Appendix. ■

The Kappa measures are easily obtained by using (13) and (25). It can be seen that the behaviour of $\theta_{j,m}$ depends on the expected return, the volatility and the return threshold, i.e., $\theta_{j,m} = \theta_{j,m}\,(\mu, \sigma, \tau)$. Note that $\theta_{2,m}$ and $\theta_{3,m}$ quantify the sensitivity of $LPM_f\,(\tau, m)$ to changes in s and ek, respectively. The following Corollary shows the general expression for the FT measures in (16) under the GC density for the standardized return.

Corollary 4 *Let z be the standardized return of r in (18). The performance measure $FTR_f(\tau, q, m)$ in (16) for $q, m \in \mathbb{N}_+$ can be expressed as*

$$FTR_f\,(\tau, q, m) = \frac{\sqrt[q]{\psi_f\,(\tau, q) + (-1)^{q+1}\, LPM_f(\tau, q)}}{\sqrt[m]{LPM_f(\tau, m)}}, \tag{28}$$

where $LPM_f(\tau, \cdot)$ is given in (25) and $\psi_f\,(\tau, q) = E_f[(r - \tau)^q]$ is obtained as

$$\psi_f\,(\tau, q) = \sum\nolimits_{k=0}^{q} \binom{q}{k} (\mu - \tau)^{q-k}\, \sigma^k E_g\left[z^k\right], \tag{29}$$

where $E_g\left[z^k\right]$ is given in (22). ■

Many studies about performance evaluation focus on some popular Kappa measures such as the Omega-Sharpe, Sortino, and Kappa 3 ratios and the Upside Potential ratio from the FT family. As a consequence, we are very interested in the expressions of $LPM_f\,(\tau, m)$ for $m = 1, 2, 3$. This is shown in the following Corollary.

Corollary 5 *The expressions of $\theta_{j,m}$ for $j = 2, 3$ in (27) and $LPM_n(\tau, m)$ for $m = 1, 2, 3$ in (25) are given by*

$$\begin{aligned}
\theta_{j,1} &= (\tau-\mu)A_{0j} - \sigma A_{1j}, \\
\theta_{j,2} &= (\tau-\mu)^2 A_{0j} - 2(\tau-\mu)\sigma A_{1j} + \sigma^2 A_{2j}, \\
\theta_{j,3} &= (\tau-\mu)^3 A_{0j} - 3(\tau-\mu)^2\sigma A_{1j} + 3(\tau-\mu)\sigma^2 A_{2j} - \sigma^3 A_{3j},
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
LPM_n(\tau,1) &= (\tau-\mu)\,\Phi(\tau^*) + \sigma\phi(\tau^*), \\
LPM_n(\tau,2) &= (\tau-\mu)^2\,\Phi(\tau^*) + (\tau-\mu)\,\sigma\phi(\tau^*) + \sigma^2\Phi(\tau^*), \\
LPM_n(\tau,3) &= (\tau-\mu)^3\,\Phi(\tau^*) + (\tau-\mu)^2\,\sigma\phi(\tau^*) + 3(\tau-\mu)\,\sigma^2\Phi(\tau^*) + 2\sigma^3\phi(\tau^*).
\end{aligned} \tag{31}$$

where the values for $A_{kj} = A_{kj}(\tau^*)$, $\tau^* = (\tau-\mu)/\sigma$, can be seen in the Appendix.[9] ■

Proof. See the Appendix. ■

4.2 Behaviour of Kappa measures with respect to s and ek

We analyze the effects of the higher moments on the performance ratios. We set the parameter vector (μ, σ, τ) equal to $(\mu_0, \sigma_0, \tau_0)$. Then, $LPM_f(\tau_0, m)$=LPM_m is a function, g_m, on both s and ek. Let ΔLPM_m and $dLPM_m$ denote, respectively, the increment and the total differential of LPM_m with respect to its arguments.[10] The next Corollary inmediately arises.

Corollary 6 *If we approximate ΔLPM_m by $dLPM_m$, we get*

$$\Delta LPM_m = \frac{\partial g_m}{\partial s}\Delta s + \frac{\partial g_m}{\partial ek}\Delta ek$$

holding that

$$\Delta LPM_m > 0 \Leftrightarrow \Delta s < \varphi_m \Delta ek$$

where

$$\varphi_m = -\frac{1}{2}\frac{\theta_{3,m}}{\theta_{2,m}}. \tag{32}$$

■

Table 1 exhibits the behaviour of the popular Kappa measures by changing either s or ek, that is, we provide these measures for alternative portfolios with the same μ and σ but different values for s and ek such that $(ek, s) \in \Gamma$. Consider, for instance, the monthly return values of $\mu = 0.86\%$, $\sigma = 2.61\%$, $\tau = r_f$ ($r_f = 0.39\%$), and three possible values of skewness ($s = -0.7,\ 0,\ 0.4$). The Sharpe ratio is equal to 0.1796. Plugging these parameters in (32), we get $\varphi_1 = -1.3471$, $\varphi_2 = 0.0499$ and $\varphi_3 = 0.2289$.

[9] $LPM_n(\tau, m)$ is the same as $\theta_{1,m}$ when $j = 1$ in (30). We decide to use $LPM_n(\tau, m)$ instead of $\theta_{1,m}$ for an easier interpretation of (25).

[10] That is, $\Delta LPM_m = g_m(s + \Delta s, ek + \Delta ek) - g_m(s, ek)$, where Δx represents a small increment in x.

[INSERT TABLE 1 AROUND HERE]

Columns 2 to 10 show the three Kappa measures for the different combinations (ek, s) according to different levels of skewness. The main results can be summarized as follows:

1. Consider the portfolio π_1 with $(ek, s) = (0.8996, 0)$ and build new portfolios by only increasing ek. We see that $K_f(r_f, 1)$ increases but $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease.

2. Take a new portfolio π_2 with $(ek, s) = (0.8996, 0.4)$. It holds that $K_f(r_f, m)$ increases if we only increase s by changing π_1 for π_2. The same behaviour holds for alternative values of ek.

3. Suppose now that ek increases and s decreases. Consider either portfolio π_3, with $(ek, s) = (1.2048, -0.7)$, or π_4, with $(ek, s) = (2.1205, -0.7)$. It is verified that both $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease when going from π_1 to either π_3 or π_4. Meanwhile, there are opposite effects about the behaviour of $LPM_f(r_f, 1)$. Note that $K_f(r_f, 1)$ decreases (increases) when moving to π_3 (π_4).

4. Finally, we can see that (32) holds under these examples and so, the behaviour of their related Kappa measures is verified. Thus, changing π_1 for π_2 leads to $\Delta s = 0.4, \Delta ek = 0$. The case of changing π_1 for π_3 leads to $\Delta s = -0.7, \Delta ek = 0.3052$. Finally, changing π_1 for π_4 leads $\Delta s = -0.7, \Delta ek = 1.2209$.

In short, from the above results, we can suggest that $\partial LPM_f(r_f, m)/\partial s < 0$ for $m = 1, 2, 3$ and that $\partial LPM_f(r_f, 1)/\partial ek < 0$ and $\partial LPM_f(r_f, m)/\partial ek > 0$ for $m = 2, 3$.[11]

4.3 Iso-curves for performance measures

We obtain now the points (ek, s) that provide the same value for the selected Kappa measure given fixed levels of τ, μ and σ. To shorten, let Ψ denote the vector (μ, σ, τ) and let Ψ_0 be a fixed value for Ψ. Thus, the iso-curve associated for any Kappa measure, or iso-Kappa, corresponds to the set of points Π defined as

$$\Pi(m, \Psi_0) = \left\{ (ek, s) \in \Gamma : \overline{K}_f(\tau_0, m) = \frac{\mu_0 - \tau_0}{\sqrt[m]{LPM_n(\tau_0, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}}} \right\}, \tag{33}$$

where $\overline{K}_f(\tau_0, m)$ denotes a fixed value for the Kappa ratio given by equations (13) and (25). These spaces are easily obtained according to the following Corollary.

[11] These results are also supported by studying the behaviour of $\theta_{2,m}$ and $\theta_{3,m}$ in (30) from many simulated parameters of μ and σ. The simulation results confirm the previous conclusions.

Corollary 7 *The iso-Kappa* (33) *implies a linear relation between* s *and* ek. *Thus,* $s = a_m + \varphi_m\, ek$ *such that the slope* φ_m *is defined in* (32) *and*

$$a_m = \frac{\sqrt{6}}{\theta_{2,m}} \left[\xi_{0,m} - LPM_n(\tau_0, m)\right], \quad \xi_{0,m} = \left[\frac{\mu_0 - \tau_0}{\overline{K}_f(\tau_0, m)}\right]^m, \tag{34}$$

with $LPM_n(\tau_0, m)$ *as in (26).* ■

The iso-Kappa in (34) will be labeled as 'iso-Omega-Sharpe', 'iso-Sortino' and 'iso-Kappa 3' respectively for $m = 1, 2, 3$. Since $\partial\xi_{0,m}/\partial\overline{K}_f(\tau_0, m) < 0$ for $\mu_0 > \tau_0$, then $\partial a_m/\partial\overline{K}_f(\tau_0, m) > 0$ iff $\theta_{2,m} < 0$. Let $\Psi_0 = (0.86\%, 2.61\%, 0.39\%)$ be the parameter set used to obtain Table 1. Then, the slopes φ_m for the different iso-Kappas (see the values of φ_m in Subsection 4.2) verify that $\varphi_1 < 0, \varphi_2 > 0, \varphi_3 > 0$. So, an increase in ek leads to a decrease (increase) in s when moving along the iso-Omega-Sharpe (iso-Sortino or iso-Kappa 3) curve.

By setting $s = -0.7$ and taking higher levels of ek, Table 1 shows that $K_f\,(0.39\%, 1)$ increases ($\xi_{0,1}$ decreases) but $K_f\,(0.39\%, m)$ decreases ($\xi_{0,m}$ increases) for $m = 2, 3$. This means that the iso-Omega-Sharpe curves with negative slopes move in paralell to the right with higher levels of $\overline{K}_f(0.39\%, 1)$ since a_1 in (34) increases because $\theta_{2,1} < 0$. Nevertheless, both the iso-Sortino and iso-Kappa3 curves with positive slopes move in paralell to the right with lower levels of $\overline{K}_f(0.39\%, m)$ since a_1 decreases because $\theta_{2,m} < 0$.[12]

Note that, on one hand, the iso-Kappas from Corollary 7 become very restrictive since we are fixing both the mean and volatility parameters for the portfolio returns but, on the other hand, we obtain linear equations which can help to understand the behaviour of the iso-Kappas.[13] Finally, the iso-curves under FT measures do not imply a linear relation between s and ek but we can obtain a linear approximation.

5 Simulation analysis

We implement a simulation analysis based on the closed-form expressions for the PMs by assuming a GC distribution for the standardized stock returns in (18). We start our analysis by obtaining the Spearman's rank correlation between the SR and alternative PMs for different portfolios. The higher this rank correlation, the lower difference in ranking between the SR and the selected PM. A deeper study will also be carried out by assessing how skewness and excess kurtosis can affect the portfolio composition.

[12]The values for $\theta_{2,m}$ are, respectively, $\theta_{2,1} = -7.53 \times 10^{-4}, \theta_{2,2} = -2.18 \times 10^{-4}$ and $\theta_{2,3} = -1.87 \times 10^{-5}$. The curves for the iso-Kappas are not exhibited here for the sake of brevity.

[13]Suppose that we change the value for μ in Corollary 7. Then, the iso-Kappas depend on μ, s and ek. By using the implicit function Theorem, we can obtain the corresponding partial derivatives (evaluated at a certain point) to analyze the behavior of μ with respect to s and ek. This extension is not shown here but it is available upon request.

5.1 Simulation of parameters and performance ratios

To begin with, we simulate (for monthly returns) the parameter vectors $\vartheta_i = (\mu_i, \sigma_i, s_i, ek_i)$, $i = 1, \ldots, N_T$.[14] Let $x_{\min}$ and $x_{\max}$ denote the extreme values of σ, s and SR. In particular, we set $\sigma_{\min} = 0.963\%$, $\sigma_{\max} = 2.163\%$, $s_{\min} = -0.798$, $s_{\max} = 0.987$, $SR_{\min} = 1\%$, and $SR_{\max} = 22.3\%$. We need to generate four independent uniform random variables $U_j, j = 1, \ldots, 4$ on the interval $(0, 1)$, each with sample size $N = 10,000$. The realizations of these variables will be denoted as $u_{ji}, i = 1, \ldots, N$. We implement the following two steps for each portfolio i:

1. We get $\sigma_i = \sigma_{\min} + (\sigma_{\max} - \sigma_{\min})\, u_{1i}$ and $SR_i = SR_{\min} + (SR_{\max} - SR_{\min})\, u_{2i}$. Then, the mean is obtained as $\mu_i = r_f + \sigma_i SR_i$.

2. The skewness is obtained as $s_i = s_{\min} + (s_{\max} - s_{\min})\, u_{3i}$. Hence, $ek_i = ek_{i,\min} + (ek_{i,\max} - ek_{i,\min})\, u_{4i}$ such that $(ek_{i,\min}, s_i)$ and $(ek_{i,\max}, s_i)$ belong to the restricted space Γ, that is, $(ek_i, s_i) \in \Gamma$.[15]

Finally, we obtain the values for the PMs presented in Section 2 by inserting ϑ_i into the corresponding formulas and fixing $\tau = r_f$.

5.2 Rank correlations

We obtain the average of one hundred Spearman's rank correlations between π_i and SR_i, such that each correlation is obtained through N vectors (π_i, SR_i) computed for portfolios characterized by the vector ϑ_i, as explained in Subsection 5.1. The mains results are as follows:

1. We compute the correlation between SR and $FTR\,(r_f, q, m)$ (see (28)) for integer values $q, m \leq 6$ such that $m > q$ (reverse S-shaped utility function, see Corollary 2). Accordingly, the correlations never exceed 25%. For instance, the correlations for $FTR\,(r_f, 2, 3)$ and $FTR\,(r_f, 3, 4)$ are, respectively, 24.81% and 17.46%. Therefore, these PMs lead to quite different portfolio rankings with respect to the Sharpe ratio. These results are also supported by Eling et al. (2011) who analyze, among others, the behaviour of the FTRs. Nevertheless, for the upside potential ratio $(FTR\,(r_f, 1, 2))$ which does not verify that $m > q$, the rank correlation is 62.95%.

2. The Sharpe-Omega ratio, $K\,(r_f, 1)$, exhibits a very high correlation of 97.90%. The same happens to both Sortino ratio, $K\,(r_f, 2)$, and Kappa 3, $K\,(r_f, 3)$, with high correlations of, respectively, 94.29% and 91.18%. This evidence may suggest no ranking difference with respect to the Sharpe ratio. The following Subsection provides a more robust analysis by splitting the total sample in

[14] $N_T = N \times T$ where, as we will see later, N denotes the sample size per regression and T is the number of regressions. We set $N = 10,000$ and $T = 100$.

[15] We use the notation $ek_{i,\min}$ and $ek_{i,\max}$ to emphasize that the range for the possible values of ek depends on s_i.

two subsamples depending on the SR value. For instance, it will be shown that the ranking difference between $K(r_f, 1)$ and the SR increases with the level of SR.

3. The correlation between the $ASSR$ (with $\varphi = 1$ in (6) assuming a CARA utility) and the SR is extremely high (99.75%). Similar results are also obtained by assuming a CRRA utility function.

4. Assuming a CRRA utility (with λ as the relative risk aversion parameter), the correlation between the PMs based on the CE amount in (5) and the SR is 66.52% when $\lambda = 1$, 71.80% for $\lambda = 2$, 87.48% for $\lambda = 5$, and 94.35% for $\lambda = 7$. Similar results hold under the CARA utility function. In brief, for low risk averse investors, we find significant differences in ranking portfolios when comparing the CE and the SR.

5.3 Effects of skewness and kurtosis on portfolio evaluation

We start studying the PMs from the S-S and FT families with higher rank correlations according to the results in Subsection 5.2. Later, we will consider the PMs based on CE and analyze the effects of the higher moments on ranking portfolios for different types of risk-averse investors.

5.3.1 Results for PMs based on ASSR, both S-S and FT families

Following Zakamouline (2011), we propose two models:

1. The first model is defined as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi}, \quad \alpha_\pi, \beta_\pi > 0, \quad i = 1, \cdots, N, \tag{35}$$

where π is a specific PM (for all the portfolios i) such that $\pi_i > 0$.[16] The portfolio i is characterized by the parameter vector ϑ_i. We estimate by ordinary least squares (OLS) the logarithm of equation (35) and obtain $R^2_{\pi,0}$, the (adjusted) R^2 statistic. If the estimates of α_π and β_π are positive, π_i is equivalent to SR_i in the sense that both produce the same ranking. A high equivalence between both measures will be indicated by a high value of $R^2_{\pi,0}$.

2. The second model is given as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi} \exp\left(\beta_\pi^s s_i + \beta_\pi^{ek} ek_i + \varepsilon_{\pi,i}\right), \quad i = 1, \ldots, N, \tag{36}$$

where $\varepsilon_{\pi,i}$ is the error term according to π and ϑ_i. Note that β_π^s and β_π^{ek} are, respectively, the (relative) sensitivity of π to the skewness and excess kurtosis of the portfolio return distribution. That is, $\beta_\pi^s = (\partial\pi/\partial s)/\pi$ and

[16] We just focus on the portfolios with positive PMs as the relevant ones in our study.

$\beta_\pi^{ek} = (\partial\pi/\partial ek)/\pi$. We estimate by OLS the following expression, obtained after applying logarithms to (36):

$$\log(\pi_i) = \log(\alpha_\pi) + \beta_\pi \log(SR_i) + \beta_\pi^s s_i + \beta_\pi^{ek} ek_i + \varepsilon_{\pi,i}, \quad i = 1,\ldots,N. \quad (37)$$

Let $R^2_{\pi,1}$ denote the (adjusted) R^2 statistics of the OLS regression in (37). If the estimates of β_π^s and β_π^{ek} were statistically significant, then both s and ek can affect the behaviour of the PM given by π. Thus, this measure would produce a different portfolio ranking than that from the SR. It also means that $R^2_{\pi,1} - R^2_{\pi,0}$ would become large and the Spearman's rank correlation coefficient between π and SR, denoted as $R_S(\pi, SR)$, would be small. Otherwise, if the estimates of both parameters were not significant, then $R^2_{\pi,0}$ would be similar to $R^2_{\pi,1}$ and the Spearman's rank correlation would be larger.

Table 2 provides the OLS estimates of β_π, β_π^s and β_π^{ek}, the rank correlation $R_S(\pi, SR)$ and the statistics $R^2_{\pi,0}$ and $R^2_{\pi,1}$ by considering five different PMs. Zakamouline (2011) shows[17] that a larger SR implies a lower $R_S(\pi, SR)$. A possible reason might be that the larger the Sharpe ratio, the larger the adjustment for non-normality of the portfolio return distribution by the selected PM. Hence, our simulation analysis aims to test this behaviour by using two non-overlapping ranges for SR for each PM. Specifically, we take $SR_{\min}$, $SR_{\max}$ (see Subsection 5.1), we compute the mean for both values and set the intervals $J_1 \equiv [1\%, 11.65\%]$ and $J_2 \equiv (11.65\%, 22.3\%]$. We split each sample size N in two parts, N_1 and N_2, and run two regressions with π_i as dependent variable:

1. In the first regression, the independent variables are the vectors $(1, SR_i, s_i, ek_i)$ such that both s_i and ek_i come from ϑ_i and $SR_i = (\mu_i - r_f)/\sigma_i \in J_1$ where μ_i and σ_i also belong to ϑ_i.
2. The second regression is based on the remaining N_2 points such that each $SR_i \in J_2$.

The last column of this Table displays the Chow test (and its p-value). That is, testing the null hypothesis of no structural break (one regression) against the alternative one of structural break (two regressions). This experiment is repeated 100 times and Table 2 exhibits the *mean* values of the parameter estimates, rank correlations, etc.

[INSERT TABLE 2 AROUND HERE]

The main results from this Table are as follows:

[17] A simulation analysis is implemented by assuming that the return follows a Normal-Inverse-Gaussian distribution.

- **R^2 statistics, rank correlation and Chow test**

  Running two regressions is better than just one since the p-values for the Chow test are null for all the PMs. In other words, the larger the SR, the larger the sensitivity of any π to the higher moments. For the PMs with very high rank correlations in Subsection 5.2 ($ASSR$, Omega-Sharpe ratio, Sortino, and Kappa 3), their rank correlations are lower under J_2. It holds that the $ASSR$ (not reported in the Table) is the PM with the lowest difference between J_1 and J_2. Finally, $R^2_{\pi,1} - R^2_{\pi,0}$ becomes higher under J_2. On the contrary, as expected, the PMs with low rank correlation (such as $FTR\,(r_f, 1, 2)$ and $FTR\,(r_f, 2, 3)$) show correlations that are quite small and similar in both intervals. Furthermore, $R^2_{\pi,1} - R^2_{\pi,0}$ is now extremely large.

- **Behaviour of OLS beta estimates**

  First, for all the PMs, the OLS estimates for the three betas are statistically significant at the 1% level and so, both skewness and excess kurtosis play significant roles in these measures. Note that we only study the effects of skewness (but not kurtosis) when the $ASSR$ is considered. Second, $\widehat{\beta}_\pi$ and $\widehat{\beta}^s_\pi$ are always positive and larger in the regression under J_2. So, we can conclude that (a) a higher value for the SR leads to a higher value for π and (b) all the PMs are, as expected, in favour of positive skewness. Third, $\widehat{\beta}^{ek}_\pi$ is negative in the Sortino and Kappa 3 measures. Finally, $|\widehat{\beta}^s_\pi|$ is much higher than $|\widehat{\beta}^{ek}_\pi|$ in all the PMs except for the Omega-Sharpe ratio.

5.3.2 Results for PMs based on CE

We assume a CRRA utility function and analyze the impact of both higher moments on the behaviour of the percentage change of the CE measure in (5) with respect to CE_0, a benchmark CE obtained under a normal distribution. We consider several types of risk-averse investors by setting $\lambda = 1, 2, \ldots, 8$ and denote the related CE as $CE\,(\lambda)$. In summary, we consider the following equation for any portfolio i, characterized by the vector ϑ_i:

$$\frac{CE_i\,(\lambda) - CE_{0,i}\,(\lambda)}{CE_{0,i}\,(\lambda)} = \gamma_0 + \gamma_s s_i + \gamma_k k_i + \varepsilon_i, \quad i = 1, \cdots, N, \tag{38}$$

where $CE_i\,(\lambda)$ is obtained by considering both s_i and k_i ($=ek_i+3$) from the simulated portfolio i in Subsection 5.1 and $CE_{0,i}$ is obtained by assuming $s_i = ek_i = 0$ in ϑ_i. Finally, ε_i denotes the error component. We estimate by OLS the equation (38), a different one to each type of investor (eight regressions). In the same way as in Subsection 5.3.1, we also test if there is a structural change, the experiment is repeated 100 times, we compute the mean values, etc. We obtain the following results:[18]

[18]The details are not reported here but available upon request.

- The OLS estimates for γ_s and γ_k are always, respectively, positive and negative. Thus, a higher level of skewness (kurtosis) implies a higher (lower) difference between $CE_i(\lambda)$ and $CE_{0,i}(\lambda)$.

- The Chow test is rejected when $\lambda \geq 6$. In contrast, the p-value for $\lambda = 5$ is 9.84% and the p-values for lower values of λ exhibit even more evidence of no structural change. In conclusion, the size of the Sharpe ratio really matters to higher risk-averse investors when ranking portfolios.

- The previous results can be reinforced as follows. We show that the mean of the ratio γ_s^1/γ_s^2 over the one hundred regressions goes from 0.83 ($\lambda = 6$) to 0.55 ($\lambda = 8$). Thus, the impact of skewness is higher in the interval J_2 where SR is larger and it becomes higher when the investor is more risk-averse. A similar analysis shows that the ratio γ_k^1/γ_k^2 goes from 0.76 ($\lambda = 6$) to 0.43 ($\lambda = 8$) and we get the same conclusions as before.

6 LPMs under the SNP density

We aim to provide closed-form expressions for LPMs under the semi-nonparametric (SNP) distributions introduced by Gallant and Nychka (1987).[19] Both SNP and GC densities are just the product of a standard normal density times a finite number of Hermite polynomials. We can be interested in this alternative distribution for two main reasons:

1. The SNP density shares the analytical tractability of the GC density, but it is always positive. By comparison, the GC density is restricted to be positive into the parameter set Γ (see Section 3).

2. The SNP nests the GC density in the sense that it allows a higher flexibility in terms of skewness and excess kurtosis. Thus, Γ is contained into a higher space under the SNP.[20]

However, in contrast to the GC density, the parameters implied in the SNP pdf do not correspond directly to the skewness and kurtosis of the distribution. In fact, this drawback is shared by other non-Gaussian distributions.

The SNP density of a random variable x is defined as

$$h(x) = \frac{\phi(x)}{v'v}\left(\sum_{i=0}^{p} v_i H_i(x)\right)^2, \tag{39}$$

where $v = (v_0, v_1, \ldots, v_p)' \in \mathbb{R}^{p+1}$, $\phi(\cdot)$ denotes the pdf of a standard normal variable and $H_i(x)$ are the normalized Hermite polynomials from (20). Since $h(\cdot)$ is

[19] See also Fenton and Gallant (1996) and Gallant and Tauchen (1999). León et al. (2009) analyze the statistical properties of the SNP distributions.

[20] See the skewness-kurtosis frontiers under the SNP distribution in Figure 1 in León et al. (2009).

homogeneous of degree zero in v, we can either impose $v'v = 1$ or $v_0 = 1$ to solve the scale indeterminacy. By expanding the square term expression in (39), we arrive at Proposition 1 in León et al. (2009). For $p = 2$, we get

$$h(x) = \phi(x) \sum_{k=0}^{4} \gamma_k(v) H_k(x), \tag{40}$$

$$\begin{aligned}
\gamma_0(v) &= 1, \quad \gamma_1(v) = \frac{2v_1\left(v_0 + \sqrt{2}v_2\right)}{v'v}, \quad \gamma_2(v) = \frac{\sqrt{2}\left(v_1^2 + 2v_2^2 + \sqrt{2}v_0v_2\right)}{v'v}, \\
\gamma_3(v) &= \frac{2\sqrt{3}v_1v_2}{v'v}, \quad \gamma_4(v) = \frac{\sqrt{6}v_2^2}{v'v}.
\end{aligned} \tag{41}$$

We are interested in an affine transformation $z^* = a(v) + b(v)x$ with $g(\cdot)$ as the density of z^* verifying that $E_g[z^*] = 0$, $E_g[z^{*2}] = 1$. Hence, the location and scale parameters $a(v)$ and $b(v)$ are obtained as

$$a(v) = -\frac{E_h[x]}{\sqrt{V_h[x]}}, \quad b(v) = \frac{1}{\sqrt{V_h[x]}}, \tag{42}$$

where $E_h[x]$ and $V_h[x]$ denote, respectively, the mean and variance of x with $h(\cdot)$ (defined in (40)) as pdf. Finally, we can express the stock return r as

$$r = \mu + \sigma z^* = \mu + a\sigma + b\sigma x, \tag{43}$$

such that $f(r) = h(x)/(b\sigma)$ is the pdf of r. The mean and variance of r are, respectively, $E_f[r] = \mu$ and $V_f[r] = \sigma^2$. The next Proposition shows the general LPM expression of r with the SNP density.

Proposition 4 *Let r be the stock return in* (43) *with pdf $f(r) = h(x)/(b\sigma)$ such that $h(x)$ is the SNP density in* (40)*. The lower partial moment $LPM_f(\tau, m)$ of r is given as*

$$LPM_f(\tau, m) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j} \kappa_1^j C_j, \tag{44}$$

where $\kappa_0 = \tau - \mu - a\sigma$, $\kappa_1 = b\sigma$, the parameters a and b are defined in (42) *and*

$$C_j = \sum_{i=0}^{4} \xi_i(v) B_{j+i}, \tag{45}$$

such that $B_l = \int_{-\infty}^{\tau_+} x^l \phi(x) dx$, with general solution in (60) (see the Appendix), where $\tau_+ = \kappa_0/\kappa_1$ and $\xi_i(v)$ is given by the following expressions:

$$\begin{aligned}
\xi_0(v) &= 1 - \gamma_2(v)/\sqrt{2} + 3\gamma_4(v)/\sqrt{4!}, \\
\xi_1(v) &= \gamma_1(v) - 3\gamma_3(v)/\sqrt{3!}, \\
\xi_2(v) &= \gamma_2(v)/\sqrt{2} - 6\gamma_4(v)/\sqrt{4!}, \\
\xi_3(v) &= \gamma_3(v)/\sqrt{3!}, \\
\xi_4(v) &= \gamma_4(v)/\sqrt{4!},
\end{aligned} \tag{46}$$

where the coefficients $\gamma_k(v)$ are defined in (41). ■

Proof. See the Appendix. ■

Given this result, it is easy to obtain the expressions of the S-S and FT families when the stock returns are driven by the SNP distribution.

7 Conclusions

We derive closed-form expressions for lower partial moments (LPM) and upper partial moments (LPM) under the Gram-Charlier (GC) density for stock returns. Since LPMs can be expressed as linear functions of both skewness and excess kurtosis (s and ek), the behaviour of the related performance measures (PM) can be easily understood in most cases. Both the Farinelli-Tibiletti (FT) and Kappa measures are studied here.

A simulation analysis is also carried out for portfolio evaluation. We show that one-sided PMs can affect the portfolio ranking differently to the Sharpe ratio (SR) because of the PM sensitivity to the levels of s and ek implied in the portfolio returns. Note that the SR is fully compatible with normally distributed returns and, more general, with elliptical distributions of returns. Hence, the SR can lead to incorrect evaluations due to asymmetry and heavy tails under the GC distribution, see Jondeau and Rockinger (2001).

Finally, the semi-nonparametric (SNP) distribution by Gallant and Nychka (1987) is introduced here as a more flexible density than the GC distribution in terms of capturing more levels of s and ek. Closed-form expressions for LPMs are also obtained under this distribution.

Several issues are left for further research. We could obtain efficient frontiers based on LPMs as alternative risk measures, see Cumova and Nawrocki (2011, 2014), and compare them with the Markowitz (1991) mean-variance approach. Alternative copula models, see Cherubini (2004), could be assumed for dependence among the different stock returns and either GC or SNP for their marginal distributions. Finally, in a similar way to Farinelli et al. (2008), we could also obtain optimal asset allocations with different PMs under a multivariate distribution framework.

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References

[1] Auer, B.R., 2015. Does the choice of performance measure influence the evaluation of commodity investments? International Review of Financial Analysis, 38, 142-150.

[2] Bawa, V.S., 1975. Optimal rules for ordering uncertain prospects. Journal of Financial Economics, 2 (1), 95-121.

[3] Bawa, V.S., Lindenberg, E.B., 1977. Capital market equilibrium in a mean-lower partial moment framework. Journal of Financial Economics, 5 (2), 189-200.

[4] Bernardo, A.E., Ledoit, O., 2000. Gain, loss and asset pricing. Journal of Political Economy, 108 (1), 144-172.

[5] Biglova, A., Ortobelli, S., Rachev, S., Stoyanov, S., 2004. Different approaches to risk estimation in portfolio theory. Journal of Portfolio Management, 31 (1), 103-112.

[6] Caporin, M., Jannin, G.M., Lisi, F., Maillet, B.B., 2014. A survey on the four families of performance measures. Journal of Economic Surveys, 28 (5), 917-942.

[7] Cherubini, U., Luciano, E., Vecchiato, W., 2004. Copula methods in finance, John Wiley & Sons.

[8] Corrado, C., Su, T., 1996. Skewness and kurtosis in S&P 500 index returns implied by option prices. Journal of Financial Research, 19 (2), 175-192.

[9] Cumova, D., Nawrocki, D., 2011. A symmetric LPM model for heuristic mean-semivariance analysis. Journal of Economics and Business, 63 (3), 217-236.

[10] Cumova, D., Nawrocki, D., 2014. Portfolio optimisation in an upside potential and downside risk framework. Journal of Economics and Business, 71, 68-89.

[11] Ebert, S., 2013. Moment characterization of higher-order risk preferences. Theory and Decision, 74 (2), 267-284.

[12] Eeckhoudt, L., Schlesinger, H., 2006. Putting risk in its proper place. American Economic Review, 96 (1), 280-289.

[13] Eling, M., Farinelli, S., Rossello, D., Tibiletti, L., 2011. One-size or tailor-made performance ratios for ranking hedge funds. Journal of Derivatives and Hedge Funds, 16 (4), 267-277.

[14] Eling, M., Schuhmacher, F., 2007. Does the choice of performance measure influence the evaluation of hedge funds? Journal of Banking and Finance, 31 (9), 2632-2647.

[15] Eling, M., 2008. Does the measure matter in the mutual fund industry? Financial Analysts Journal, 64 (3), 54-66.

[16] Farinelli, S., Ferreira, M., Rossello, D., Thoeny, M., Tibiletti, L., 2008. Beyond Sharpe ratio: Optimal asset allocation using different performance ratios. Journal of Banking and Finance, 32 (10), 2057-2063.

[17] Farinelli, S., Tibiletti, L., 2008. Sharpe thinking in asset ranking with one-sided measures. European Journal of Operational Research, 185 (3), 1542-1547.

[18] Fenton, V., Gallant, A. R., 1996. Qualitative and asymptotic performance of SNP density estimators. Journal of Econometrics, 74 (1), 77-118.

[19] Fishburn, P.C., 1977. Mean-risk analysis with risk associated with below-target returns. American Economic Review, 67 (2), 116-126.

[20] Gallant, A.R., Nychka, D.W., 1987. Semi-nonparametric maximum likelihood estimation. Econometrica, 55 (2), 363-390.

[21] Gallant, A.R., Tauchen, G., 1999. The relative efficiency of method of moment estimators. Journal of Econometrics, 92 (1), 149–172.

[22] Guo, B., Xiao, Y., 2016. A note on why doesn't the choice of performance measure matter? Finance Research Letters, in press.

[23] Harlow, W.V., Rao, R.K.S., 1989. Asset pricing in a generalised mean-lower partial moment framework: Theory and evidence. Journal of Financial and Quantitative Analysis, 24 (3), 285-311.

[24] Harlow, W., 1991. Asset allocation in a downside risk framework. Financial Analysts Journal, 47 (5), 28-40.

[25] Hodges, S., 1998. A generalization of the Sharpe ratio and its applications to valuation bounds and risk measures. Working paper Financial Options Research Centre, 1998, 88, University of Warwick.

[26] Holthausen, D.M., 1981. A risk-return model with risk and return measured as deviations from a target. American Economic Review, 71 (1), 182-188.

[27] Jondeau, E., Rockinger, M., 2001. Gram-Charlier densities. Journal of Economic Dynamics and Control, 25 (10), 1457-1483.

[28] Jurczenko, E., Maillet, B., 2006. Multi-moment asset allocation and Pricing models, Wiley Finance.

[29] Kahneman, D., Tversky, A., 1979. Prospect theory: An analysis of decision under risk. Econometrica, 47 (2), 263-291.

[30] Kaplan, P.D., Knowles, J.A., 2004. Kappa: A generalized downside risk adjusted performance measure. Journal of Performance Measurement, 8 (3), 42-54.

[31] Keating, C., Shadwick, W.F., 2002. A universal performance measure. Journal of Performance Measurement, 6 (3), 59-84.

[32] Kendall, M., Stuart, A., 1977. The advanced theory of statistics, Volume 1: Distribution Theory, fourth edition, Griffin.

[33] Kimball, M. S., 1990. Precautionary saving in the small and in the large. Econometrica, 58 (1), 53-73.

[34] Kimball, M. S., 1992. Precautionary motives for holding assets. The New Palgrave Dictionary of Money and Finance.

[35] Köbberling, V., Wakker, P.P., 2005. An index of loss aversion. Journal of Economic Theory, 122 (1), 119-131.

[36] Le Courtois, O., 2012. On prudence and temperance. Illustration with monoperiodic portfolio optimization. Cahier de Recherche Working Paper 2012/01, EMLYON Business School.

[37] León, A., Mencía, J., Sentana, E., 2009. Parametric properties of seminonparametric distribution, with applications to option valuation. Journal of Business and Economic Statistics, 27 (2), 176-192.

[38] Levy, M., Levy, H., 2012. Prospect theory: Much ado about nothing? Management Science, 48 (10), 1334-1349.

[39] Markowitz, H., 1991. Portfolio selection. Efficient diversification of investments, second edition, John Wiley & Sons.

[40] Owen, J., Rabinovitch, R., 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice. Journal of Finance, 38 (3), 745-752.

[41] Passow, A., 2005. Omega portfolio construction with Johnson distributions. Risk, 18 (4), 85-90.

[42] Pedersen, C.S., Satchell, S.E., 2002. On the foundation of performance measures under asymmetric returns. Quantitative Finance, 2 (3), 217-223.

[43] Sharpe, W.F., 1966. Mutual fund performance. Journal of Business, 39 (1), 119-138.

[44] Sharpe, W.F., 1994. The Sharpe ratio. Journal of Portfolio Management, 21 (1), 49-58.

[45] Sortino, F.A., Satchell, S. 2001. Managing downside risk in financial markets, Butterworth Heinemann, Oxford.

[46] Sortino, F.A., Van der Meer, R., 1991. Downside risk. Journal of Portfolio Management, 17 (4), 27-31.

[47] Sortino, F.A., Van der Meer, R., Platinga, A., 1999. The Dutch triangle. Journal of Portfolio Management, 26 (1), 50-57.

[48] Tversky, A., Kahneman, D., 1992. Advances in prospect theory: cumulative representation of uncertainty. Journal of Risk and Uncertainty, 5 (4), 297-323.

[49] Zakamouline, V., Koekebakker, S., 2009. Portfolio performance evaluation with generalized Sharpe ratios: Beyond the mean and variance. Journal of Banking and Finance, 33 (7), 1242-1254.

[50] Zakamouline, V., Koekebakker, S., 2009. A generalization of the mean-variance analysis. European Financial Management, 15 (5), 934-970.

[51] Zakamouline, V., 2011. The performance measure you choose influences the evaluation of hedge funds. Journal of Performance Measurement, 15 (3), 48-64.

[52] Zakamouline, V., 2014. Portfolio performance evaluation with loss aversion. Quantitative Finance, 14 (4), 699-710.

Appendix. Proofs

Proof of Proposition 1

Kendall and Stuart (1977) shows that

$$H_k(z) = \sum_{n=0}^{[k/2]} a_{k,n}\, z^{k-2n}, \tag{47}$$

where $[\cdot]$ rounds its argument to the nearest (smaller) integer and

$$a_{k,n} = \left(-\frac{1}{2}\right)^n \frac{\sqrt{k!}}{(k-2n)!n!}.$$

Taking expectations in (47) with pdf $g(\cdot)$, see equation (19), we obtain

$$E_g\left[H_k(z)\right] = \sum_{n=0}^{[k/2]} a_{k,n}\, E_g\left[z^{k-2n}\right]. \tag{48}$$

Given (19) and (21), we get $E_g\left[H_k(z)\right] = 0$ for $k \geq 5$ and so, we obtain recursively the expression for $E_g\left[z^k\right]$ in (22). Finally, we obtain (23) by using the binomial expansion. ∎

Proof of Proposition 2

Let z be the standardized return of r in (18) with pdf $g(z)$ in (19). Its distribution function is

$$\begin{aligned} F_{GC}(z; s, ek) &= \int_{-\infty}^{z} g(x)\,dx = \int_{-\infty}^{z} p(x)\phi(x)\,dx \\ &= \Phi(z) + \frac{s}{\sqrt{3!}} \int_{-\infty}^{z} H_3(x)\,\phi(x)\,dx + \frac{ek}{\sqrt{4!}} \int_{-\infty}^{z} H_4(x)\,\phi(x)\,dx \\ &= \Phi(z) - \frac{s}{3\sqrt{2!}} H_2(z)\,\phi(z) - \frac{ek}{4\sqrt{3!}} H_3(z)\,\phi(z). \end{aligned} \tag{49}$$

where the last equality arises from the following relationship (see León et al. (2009) for details):

$$\int_{-\infty}^{a} H_k(x)\phi(x)dx = -\frac{1}{\sqrt{k}} H_{k-1}(a)\phi(a), \quad k \geq 1. \tag{50}$$

Let $f(r) = g(z)/\sigma$ be the pdf of r and $F(r)$ be the associated cdf, then $F(r) = F_{GC}(r^*; s, ek)$ where $r^* = (r-\mu)/\sigma$ completes the proof. ∎

Proof of Corollary 3

We can rewrite (24) as

$$F(r) = B_0 + \frac{s}{\sqrt{3!}} A_{02} + \frac{ek}{\sqrt{4!}} A_{03} \tag{51}$$

where

$$B_0 = \Phi(r^*), \quad A_{02} = -\frac{1}{\sqrt{3}} H_2(r^*)\phi(r^*), \quad A_{03} = -\frac{1}{\sqrt{4}} H_3(r^*)\phi(r^*). \tag{52}$$

It holds that $B_0 > 0 \; \forall r^*$, $A_{02} > 0$ iff $|r^*| < 1$ and $A_{03} > 0$ iff $r^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$. Moreover, using the relations (a) $d\phi\,(x)\,/dx = -x\phi\,(x)$ and (b) $dH_k\,(x)\,/dx = \sqrt{k} H_{k-1}\,(x)$, we get

$$\begin{aligned}
\frac{\partial A_{02}}{\partial r^*} &> 0 \Leftrightarrow r^* \in \left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, +\infty\right), \\
\frac{\partial A_{03}}{\partial r^*} &> 0 \Leftrightarrow r^* \in \left(-\infty, -r_1^*\right) \cup \left(-r_2^*, r_2^*\right) \cup \left(r_1^*, +\infty\right),
\end{aligned}$$

where $r_1^* = \sqrt{3 + \sqrt{6}}$ and $r_2^* = \sqrt{3 - \sqrt{6}}$. Hence,

$$\begin{aligned}
\frac{\partial F\,(r)}{\partial s} &> 0 \Leftrightarrow |r^*| < 1, \\
\frac{\partial F\,(r)}{\partial ek} &> 0 \Leftrightarrow r^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right),
\end{aligned}$$

and $\partial F\,(r)\,/\partial r^* > 0$ if $r^* \in \left(-r_2^*, 0\right) \cup \left(r_1^*, +\infty\right)$. Finally, the signs of the partial derivatives of $F\,(r)$ with respect to μ and σ are obtained by applying the chain rule and using that $\partial r^*/\partial\mu < 0$, $\partial r^*/\partial\sigma > 0$ iff $\mu > r$. ∎

Proof of Proposition 3

Let f and g denote, respectively, the pdfs for r and z in (18). So, it is verified that $f\,(r) = g\,(z)\,/\sigma$ where $z = (r - \mu)\,/\sigma$. Then, we can rewrite (9) as

$$LPM_f(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\,dr = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m g(z)\,dz \tag{53}$$

where $\tau^* = (\tau - \mu)\,/\sigma$. If we apply the binomial expansion to $(\tau - \mu - \sigma z)^m$ in (53), then

$$LPM_f(\tau, m) = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k \boldsymbol{I}_k, \tag{54}$$

where $\boldsymbol{I}_k = E_g\left[z^k \,|z < \tau^*\right]$ denotes the conditional expected value. Thus, for $k \geq 1$, we get

$$\boldsymbol{I}_k = \int_{-\infty}^{\tau^*} z^k g(z)\,dz = B_k + \frac{s}{\sqrt{3!}} A_{k2} + \frac{ek}{\sqrt{4!}} A_{k3}, \tag{55}$$

where

$$B_k = \int_{-\infty}^{\tau^*} z^k \phi(z) dz, \quad A_{k2} = \int_{-\infty}^{\tau^*} z^k H_3(z)\phi(z) dz, \quad A_{k3} = \int_{-\infty}^{\tau^*} z^k H_4(z)\phi(z) dz. \tag{56}$$

By using (50), A_{k2} and A_{k3} can be expressed as

$$A_{k2} = \frac{1}{\sqrt{3!}}(B_{k+3} - 3B_{k+1}),\ A_{k3} = \frac{1}{\sqrt{4!}}(B_{k+4} - 6B_{k+2} + 3B_k). \tag{57}$$

By plugging (55) into (54), we get

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m},$$

where

$$\theta_{j,m} = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k A_{kj},$$

where $LPM_n(\tau, m){=}E_n[(\tau{-}r)_+^m]$ such that $n(\cdot)$ is the pdf of the normal distribution with mean μ and standard deviation σ. Thus,

$$LPM_n(\tau, m) = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m \phi(z)\, dz = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k B_k. \tag{58}$$

Finally, to get the expression for B_k in (56), we need the following result (available upon request):

$$z^k = \sum_{n=0}^{[k/2]} c_{k,n} H_{k-2n}(z), \tag{59}$$

where $c_{k,n} \in \mathbb{R}$ and $H_i(\cdot)$ is a Hermite polynomial. Note that (59) is just the inversion formula of (47). If we take (50) and (59), then

$$\begin{aligned} B_k &= \int_{-\infty}^{\tau^*} z^k \phi(z) dz = \sum_{n=0}^{[k/2]} c_{k,n} \int_{-\infty}^{\tau^*} H_{k-2n}(z)\, \phi(z) dz \\ &= \sum_{n=0}^{[k/2]} c_{k,n} \left[\Phi(\tau^*) \mathbf{1}_{(k-2n=0)} + \frac{1}{\sqrt{k-2n}} H_{k-2n-1}(\tau^*)\, \phi(\tau^*)\, \mathbf{1}_{(k-2n\geq 1)} \right], \end{aligned} \tag{60}$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function. ∎

Proof of Corollary 5

The expressions of $\theta_{j,m}$ in (30) and $LPM_n(\tau, m)$ in (31) are easily obtained for $m \leq 3$ by using, respectively, the equations (27) and (26) from Proposition 3. Since we need to obtain A_{k2} and A_{k3} in (57) for $k = 0, 1, 2, 3, 4$, then we must previously get B_j in (56) for $j = 1, \ldots, 7$. This is straightforward by applying the condition (50). ∎

Proof of Proposition 4

Let h and f denote, respectively, the pdfs for x and r in (40) and (43). So, it is verified that $f(r) = h(x)/(b\sigma)$. Then, we can rewrite (9) as

$$LPM_f(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr = \int_{-\infty}^{\tau_+} (\kappa_0 - \kappa_1 x)^m h(x)\, dx, \tag{61}$$

where $\kappa_0 = \tau - \mu - a\sigma, \kappa_1 = b\sigma > 0$ and $\tau_+ = \kappa_0/\kappa_1$. The expressions of a and b in (42) can be obtained from the first two unconditional moments of x, denoted as $\mu'_x(1)$ and $\mu'_x(2)$, from Lemma 1 in León et al. (2009). Thus, $E_h[x] = \mu'_x(1)$ is just $\gamma_1(v)$ in (41) and

$$\mu'_x(2) = \frac{2\left(v_1^2 + 2v_2^2 + \sqrt{2}v_2v_0\right)}{v'v} + 1.$$

Then, $V_h[x] = \mu'_x(2) - \mu'_x(1)^2$. If we apply the binomial expansion to $(\kappa_0 - \kappa_1 x)^m$ in (61) and consider (40), we have

$$LPM_f(\tau, m) = \sum\nolimits_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j}\kappa_1^j \left[B_j + \sum\nolimits_{k=1}^{4} \gamma_k(v)\, G_{jk}\right] \qquad (62)$$

where $B_j = \int_{-\infty}^{\tau_+} x^j \phi(x)dx$ and $G_{jk} = \int_{-\infty}^{\tau_+} x^j H_k(x)\phi(x)\, dx$. If we consider (47), then the expressions of G_{jk}, for $0 \leq j \leq m$ and $1 \leq k \leq 4$, are:

$$\begin{array}{ll} G_{j1} = B_{j+1}, & G_{j2} = \frac{1}{\sqrt{2}}\left(B_{j+2} - B_j\right), \\ G_{j3} = \frac{1}{\sqrt{3!}}(B_{j+3} - 3B_{j+1}), & G_{j4} = \frac{1}{\sqrt{4!}}(B_{j+4} - 6B_{j+2} + 3B_j). \end{array} \qquad (63)$$

Note that, in the above expressions, B_j are evaluated at the point τ_+. By plugging (63) into (62), we arrive at expressions (44)-(46). ∎

Appendix of Tables

Table 1: Sensitivity analysis for the Kappa measures under a GC distribution. Effects of skewness (s) and excess kurtosis (ek)

	Panel A: Omega			**Panel B: Sortino**			**Panel C: Kappa 3**		
ek	$s = -0.7$	$s = 0$	$s = 0.4$	$s = -0.7$	$s = 0$	$s = 0.4$	$s = -0.7$	$s = 0$	$s = 0.4$
0.8996	0.5801	0.5959	0.6054	0.2622	0.2923	0.3150	0.1825	0.2070	0.2292
1.2048	0.5893	0.6057	0.6154	0.2617	0.2916	0.3141	0.1806	0.2040	0.2246
1.5100	0.5988	0.6157	0.6258	0.2612	0.2909	0.3133	0.1788	0.2011	0.2204
1.8153	0.6086	0.6261	0.6365	0.2607	0.2903	0.3124	0.1771	0.1983	0.2165
2.1205	0.6187	0.6368	0.6476	0.2602	0.2896	0.3116	0.1754	0.1957	0.2128
2.4257	0.6292	0.6479	0.6591	0.2598	0.2889	0.3107	0.1738	0.1932	0.2094
2.7309	0.6400	0.6594	0.6710	0.2593	0.2882	0.3099	0.1723	0.1909	0.2062
3.0361	0.6513	0.6713	0.6833	0.2588	0.2876	0.3091	0.1708	0.1887	0.2031
3.3414	0.6629	0.6836	0.6961	0.2583	0.2869	0.3083	0.1693	0.1865	0.2003
3.6466	0.6749	0.6964	0.7094	0.2578	0.2863	0.3074	0.1680	0.1845	0.1976

This Table exhibits the values of the closed-form formulas for the Kappa measures (Omega-Sharpe, Sortino and Kappa 3) by using the LPM expressions from Corollary 5 for monthly returns. All the portfolios in this Table have $\mu = 0.86\%$ and $\sigma = 2.61\%$ but different values for s and ek such that $(ek, s) \in \Gamma$. The Sharpe ratio is SR=0.1796 and the return threshold is $r_f = 0.39\%$.

Table 2: Some results and statistics from the regression analysis by using samples of simulated monthly returns from a GC distribution.

log (Perf. Measure)	Sharpe ratio interval	beta log(Sharpe)	beta Skewness	beta Exc. Kurt.	Rank Corr.	R2	R20	Chow Test (p-value)
Omega-Sharpe	$[1\%, 11.65\%]$	1.1888	0.0286	0.0536	0.9509	0.9996	0.8999	10484.41
	$(11.65\%, 22.3\%]$	1.2855	0.0458	0.0570	0.8965	0.9993	0.7941	(0.00)
Sortino	$[1\%, 11.65\%]$	1.0993	0.1672	-0.0046	0.8728	0.9984	0.7540	1012.26
	$(11.65\%, 22.3\%]$	1.1480	0.1820	-0.0077	0.7489	0.9966	0.5625	(0.00)
Kappa 3	$[1\%, 11.65\%]$	1.0685	0.1974	-0.0417	0.8059	0.9916	0.6439	95.29
	$(11.65\%, 22.3\%]$	1.1017	0.2060	-0.0457	0.6583	0.9853	0.4429	(0.00)
FTR (1,2)	$[1\%, 11.65\%]$	0.2802	0.1365	0.0498	0.3842	0.9979	0.1642	9798.84
	$(11.65\%, 22.3\%]$	0.4199	0.1437	0.0520	0.3736	0.9985	0.1558	(0.00)
FTR (2,3)	$[1\%, 11.65\%]$	0.1634	0.3150	0.0222	0.1277	0.9984	0.0179	1823.73
	$(11.65\%, 22.3\%]$	0.2426	0.3228	0.0164	0.1286	0.9986	0.0182	(0.00)

This Table exhibits the results of estimating by OLS the model in (37). All the regressions contain the same explanatory variables: a constant, the skewness, the excess kurtosis and the log of the Sharpe ratio. All the beta estimates (except the constant) are shown, respectively, in columns 3 to 5. Consider different performance measures (PM) (dependent variable) in column 1, then the regression is run twice. The total sample size (N=10,000) is divided in two parts having one regression per subsample. The criterion followed to split N depends on the size of the Sharpe ratio (SR) of the portfolio $i \in \{1, \cdots, N\}$, henceforth SR_i. The column 2 is the SR interval, $SR_i \in [1\%, 11.65\%]$ or $SR_i \in (11.65\%, 22.3\%]$. There is a total of 10 different regressions. The column 6 is the Spearman's rank correlation between a certain PM and the SR for each subsample. Columns 7 and 8 correspond, respectively, to the adjusted R^2 statistics of equaiton (37) and the logarithm of equation (35). The last column represents both the value and the p-value (in parenthesis) for the Chow-test with null hypothesis of no structural break (one regression) against the alternative one of structural break (two regressions). Each portfolio is characterized by the vector $(\mu_i, \sigma_i, s_i, ek_i)$ and simulated according to the procedure described in Subsection 5.1. Note that we repeat the above process a total of 100 times, so any value (or estimation) of the Table (from columns 3 to 9) is indeed the *mean* computed over 100 estimates.

Appendix of Figures

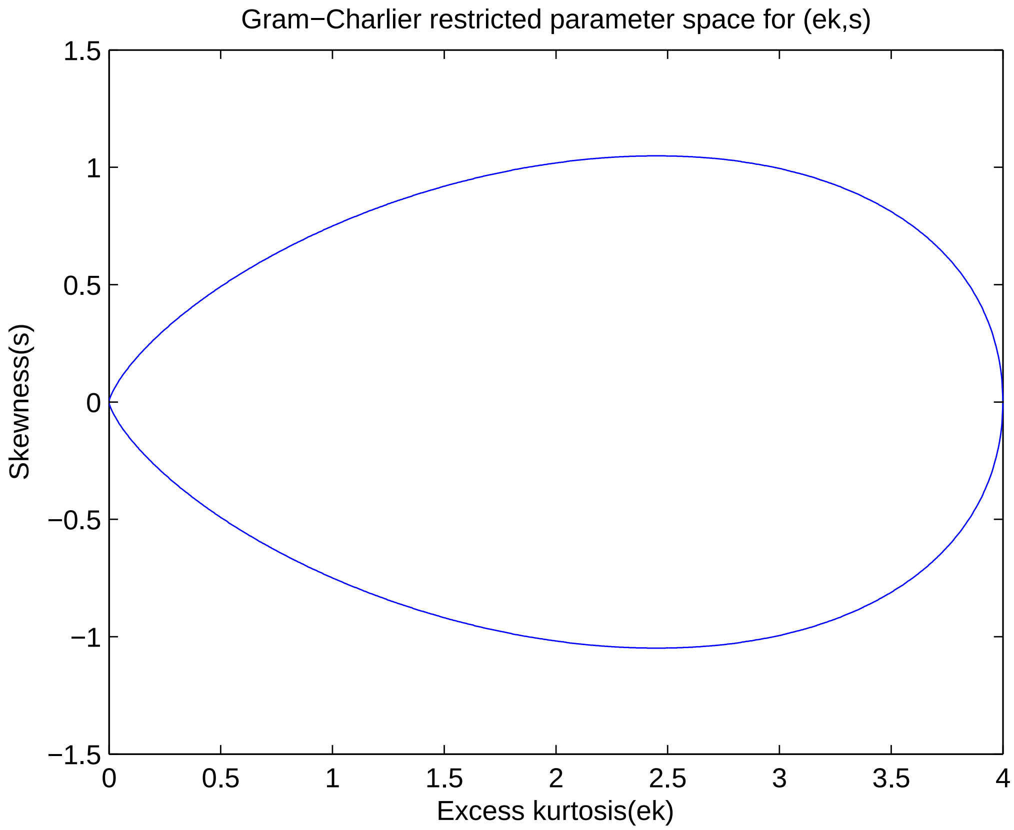

Figure 1: Space containing for stock returns the points (ek, s) with the excess kurtosis level, ek, in the x-axis and the skewness level, s, in the y-axis. This space is limited by a frontier (envelope) verifying that the Gram-Charlier (GC) density is well defined for the points on and inside the envelope. Thus, the GC density will be restricted to this space for (ek, s). Note that $ek \in [0, 4]$ while $s \in [-1.0493, 1.0493]$. The range of s depends on the level of ek. See Jondeau and Rockinger (2001) for more details about how to obtain this frontier.