This is a previous version of the article published in Optimization Letters. 2018. doi:10.1007/s11590-018-1311-5

Convexity and closedness in stable robust duality

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July 26, 2018

Abstract

The paper deals with optimization problems with uncertain constraints and linear perturbations of the objective function, which are associated with given families of perturbation functions whose dual variable depends on the uncertainty parameters. More in detail, the paper provides characterizations of stable strong robust duality and stable robust duality under convexity and closedness assumptions. The paper also reviews the classical Fenchel duality of the sum of two functions by considering a suitable family of perturbation functions.

Key words Stable robust duality - Stable strong robust duality - Fenchel duality of the sum - Deterministic conjugate duality

1 Introduction

Robust duality and stable duality have attracted in recent years the attention of many researchers. For instance, on the one hand, sufficient conditions for robust duality theorems in uncertain infinite linear optimization are provided in [8], and in uncertain convex optimization in [12] and [14], while a subdifferential constraint qualification has been used in [16] to characterize robust duality in the latter setting. On the one hand, stable duality theorems (without uncertainty) have been provided in [7], [11] and [13], among others, also in the framework of convex optimization. On the other hand, both types of duality are simultaneously studied in [2] and [5], where characterizations of robust and robust stable strong duality are given for non-convex and/or convex robust problems. It is worth mentioning that among the mentioned papers, some provide

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perturbational schemes covering optimization problems with uncertain constraints and linear perturbations of the objective functions, such as [3], [5], [14].

In this short paper, following [3], we consider a given family $\{F_u : u \in U\}$ of *perturbation functions,* where the index set U is called the *uncertainty set of the family*, $F_u : X \times Y_u \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, and the *decision space* X and the *parameter spaces* Y_u, $u \in U$, are locally convex Hausdorff topological vector spaces. In contrast with the "classical" robust duality scheme (as in [14]), where a unique parameter space Y is considered, in our model the parameter space Y_u depends on u (this dependence is illustrated with a realistic production planning problem in [3, Section 2, Case 3]). Here, we specialize the totally general results obtained in [3] to problems satisfying certain convexity and closedness properties. Applying this approach to a suitable family of perturbation functions, we also obtain calculus rules for conjugacy of the sum of functions in a non-standard way (the standard one being Fenchel conjugacy) which may recall the way that [15] obtains calculus rules for support functions from intersection formulas for normals to convex sets.

We associate with the family of perturbation functions $\{F_u : u \in U\}$ and a given continuous linear functional $x^* \in X^*$ a *robust* (or *pessimistic*) *primal problem*

$$(\mathrm{RP})_{x^*}: \qquad \inf_{x \in X} \left\{ \sup_{u \in U} F_u(x, 0_u) - \langle x^*, x \rangle \right\},$$

where 0_u is the null vector of Y_u, and its corresponding *robust* (or *optimistic*) *dual problem*

$$(\mathrm{ODP})_{x^*}: \qquad \sup_{(u, y_u^*) \in \Delta} -F_u^*(x^*, y_u^*),$$

where $\Delta := \{(u, y_u^*) : u \in U, y_u^* \in Y_u^*\}$ is the disjoint union of the parameter spaces Y_u^*, $u \in U$ and $F_u^* : X^* \times Y_u^* \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the Fenchel conjugate of F_u, i.e.,

$$F_u^*(x^*, y_u^*) := \sup_{(x, y_u) \in X \times Y_u} \{\langle x^*, x \rangle + \langle y_u^*, y_u \rangle - F_u(x, y_u)\}, \forall (x^*, y_u^*) \in X^* \times Y_u^*.$$

It is known that

$$\sup(\mathrm{ODP})_{x^*} \leq \inf(\mathrm{RP})_{x^*}, \tag{1.1}$$

where $\inf(\mathrm{RP})_{x^*}$ represents the optimal value of $(\mathrm{RP})_{x^*}$ and $\sup(\mathrm{ODP})_{x^*}$ is the optimal value of $(\mathrm{ODP})_{x^*}$ ($\max(\mathrm{ODP})_{x^*}$ when the supremum is attained).

This paper continues our research in [3], specifying to a family of convex problems with certain closedness properties, to give characterizations of the two following desirable duality properties of the above primal-dual pair of problems:

- *Stable robust duality*, i.e., $\inf(\mathrm{RP})_{x^*} = \sup(\mathrm{ODP})_{x^*}$ for all $x^* \in X^*$,
- *Stable strong robust duality*, i.e., $\inf(\mathrm{RP})_{x^*} = \max(\mathrm{ODP})_{x^*}$ for all $x^* \in X^*$.

The following two functions, $p : X \longrightarrow \mathbb{R}_\infty$ and $q : X^* \longrightarrow \overline{\mathbb{R}}$ (in short, $p \in \mathbb{R}_\infty^X$ and $q \in \overline{\mathbb{R}}^{X^*}$), play a crucial role in this paper:

$$p := \sup_{u \in U} F_u(\cdot, 0_u) \quad \text{and} \quad q := \inf_{(u, y_u^*) \in \Delta} F_u^*(\cdot, y_u^*). \tag{1.2}$$

In fact, since

$$p^*(x^*) = -\inf{(\mathrm{RP})}_{x^*} \quad \text{and} \quad q(x^*) = -\sup{(\mathrm{ODP})}_{x^*},$$

stable robust duality holds if and only if $p^*(x^*) = q(x^*)$ for all $x^* \in X^*$, and *stable strong robust duality* holds if and only if $p^*(x^*) = q(x^*)$ with attainment at the second member for all $x^* \in X^*$.

The paper is organized as follows: Section 2 contains the necessary notations and basic results, while Sections 3 and 4 characterize stable strong robust duality and stable robust duality under the convexity of q and the closedness of certain sets, respectively. Moreover, Section 4 revisits the classical Fenchel duality of the sum of two functions by considering a suitable family of perturbation functions.

2 Preliminaries

Let us introduce the necessary notations. Given a locally convex Hausdorff topological vector space Z, we consider its dual space Z^* equipped with the weak* (w^* in short) topology. We denote by 0_Z and 0_Z^* the null vectors of Z and Z^*, respectively. Given a set V contained in either Z or Z^*, $\operatorname{co} V$ and $\overline{V}$ denote its convex hull and its closure w.r.t. the corresponding topology, respectively, while $\overline{\operatorname{co}}V := \overline{\operatorname{co} V}$.

Given an extended real-valued function $h \in \overline{\mathbb{R}}^Z$, by $\operatorname{epi} h$ and $\operatorname{epi}_s h$ we represent the epigraph and the strict epigraph of h, and by $[h \leq r]$ and $[h < r]$, $r \in \mathbb{R}$, the corresponding sublevel and strict sublevel sets. The domain of the function f is $\operatorname{dom} f := \{z \in Z : f(z) < +\infty\}$. Recall also that the Fenchel conjugate function of the function f, $f^* \in \overline{\mathbb{R}}^{Z^*}$, is the one defined by $f^*(z^*) := \sup\{\langle z^*, z\rangle - f(z) : z \in Z\}$ for any $z^* \in Z^*$ while the bi-conjugate of f, $f^{**} \in \overline{\mathbb{R}}^Z$, is defined by $f^{**}(z) = \sup\{\langle z^*, z\rangle - f^*(z^*) : z^* \in Z^*\}$ for all $z \in Z$. By $\partial^\varepsilon h(a)$, $\varepsilon \geq 0$, we represent the $\varepsilon-$subdifferential of h at a point $a \in Z$ such that $h(a) \in \mathbb{R}$ (if $h(a)$ is not real, $\partial^\varepsilon h(a) = \emptyset$):

$$\begin{aligned}\partial^\varepsilon h(a) &= \{z^* \in Z^* : h(z) \geq h(a) + \langle z^*, z - a\rangle - \varepsilon, \forall z \in Z\} \\ &= \left\{z^* \in (h^*)^{-1}(\mathbb{R}) : h^*(z^*) + h(a) \leq \langle z^*, a\rangle + \varepsilon\right\}.\end{aligned}$$

If h is convex and $h(a) \in \mathbb{R}$, then $\partial^\varepsilon h(a) \neq \emptyset$ for all $\varepsilon > 0$ if and only if h is lower semicontinuous (lsc, in brief) at a. The inverse of the set-valued mapping $\partial^\varepsilon h : Z \rightrightarrows Z^*$ is denoted by $M^\varepsilon h : Z^* \rightrightarrows Z$. We have:

$$M^\varepsilon h(z^*) = \varepsilon\text{–argmin}(h - z^*) = \begin{cases} \{z \in Z : h(z) - \langle z^*, z\rangle \leq -h^*(z^*) + \varepsilon\}, & \text{if } h^*(z^*) \in \mathbb{R}, \\ \emptyset, & \text{if } h^*(z^*) \notin \mathbb{R}. \end{cases}$$

We note that $M^\varepsilon h(z^*) \neq \emptyset$ whenever $h^*(z^*) \in \mathbb{R}$ and $\varepsilon > 0$.

The *lsc hull* of h is the function $\overline{h} \in \overline{\mathbb{R}}^Z$ defined by

$$\overline{h}(z) := \inf\{t : (z, t) \in \overline{\operatorname{epi} h}\} = \liminf_{y \to z} h(y).$$

We have $\operatorname{epi} \overline{h} = \overline{\operatorname{epi} h}$ and $\overline{h}$ is the greatest lsc minorant of h. One has $h^{**} \leq \overline{h} \leq h$. If h is convex and has a continuous affine minorant (that means $\operatorname{dom} h^* \neq \emptyset$), then $h^{**} = \overline{h}$.

We need to introduce for each $\varepsilon \geq 0$ the (ε-active indexes) set-valued mapping $I^\varepsilon : X \rightrightarrows U$ with

$$I^\varepsilon(x) = \begin{cases} \{u \in U \ : \ F_u(x, 0_u) \geq p(x) - \varepsilon\}, & \text{if} \quad p(x) \in \mathbb{R}, \\ \emptyset, & \text{if} \quad p(x) \notin \mathbb{R}. \end{cases} \tag{2.1}$$

For each $(\varepsilon, x) \in \mathbb{R}_+ \times X$, let us define

$$C^\varepsilon(x) := \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta \\ \varepsilon_1 \geqslant 0, \ \varepsilon_2 \geqslant 0}} \bigcup_{u \in I^{\varepsilon_1}(x)} \operatorname{proj}^u_{X^*}(\partial^{\varepsilon_2} F_u)(x, 0_u), \tag{2.2}$$

where $\operatorname{proj}^u_{X^*} : X^* \times Y^*_u \longrightarrow X^*$ is the projection mapping $\operatorname{proj}^u_{X^*}(x^*, y^*_u) = x^*$.

Theorem 2.1 (Stable robust duality I) *[3, Theorem 6.1] Assume that* $\operatorname{dom} p \neq \emptyset$. *The next statements are equivalent:*

(i) $\inf(\mathrm{RP})_{x^*} = \sup(\mathrm{ODP})_{x^*}$ *for all* $x^* \in X^*$,

(ii) $\partial^\varepsilon p(x) = C^\varepsilon(x)$ *for all* $(\varepsilon, x) \in \mathbb{R}_+ \times X$,

(iii) *there exists* $\bar{\varepsilon} > 0$ *such that* $\partial^\varepsilon p(x) = C^\varepsilon(x)$ *for all* $(\varepsilon, x) \in]0, \bar{\varepsilon}[\times X$.

For each $\varepsilon \geq 0$, defining

$$D^\varepsilon(x) \ = \ \bigcup_{\substack{\varepsilon_1 + \varepsilon_2 = \varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \bigcup_{u \in I^{\varepsilon_1}(x)} \operatorname{proj}^u_{X^*}(\partial^{\varepsilon_2} F_u)(x, 0_u), \ \forall (\varepsilon, x) \in \mathbb{R}_+ \times X, \tag{2.3}$$

we have

$$C^\varepsilon(x) = \bigcap_{\eta > 0} D^{\varepsilon + \eta}(x), \ \forall (\varepsilon, x) \in \mathbb{R}_+ \times X, \tag{2.4}$$

and $D^\varepsilon(x) = C^\varepsilon(x) = \partial^\varepsilon p(x) = \emptyset$ whenever $p(x) \notin \mathbb{R}$. The next result characterizes the stable strong robust duality in terms of ε-subdifferential formulas.

Theorem 2.2 (Stable strong robust duality) *[3, Theorem 7.1] Assume that* $\operatorname{dom} p \neq \emptyset$. *The next statements are equivalent:*

(i) $\inf(\mathrm{RP})_{x^*} = \max(\mathrm{ODP})_{x^*} = \max\limits_{\substack{u \in U \\ y^*_u \in Y^*_u}} -F^*_u(x^*, y^*_u)$ *for all* $x^* \in X^*$,

(ii) $\partial^\varepsilon p(x) = D^\varepsilon(x)$ *for all* $(\varepsilon, x) \in \mathbb{R}_+ \times X$.

3 Using convexity and closedness in robust duality

Given $F_u : X \times Y_u \to \mathbb{R}_\infty,\ u \in U$, recall that we defined $p = \sup_{u\in U} F_u(\cdot, 0_u)$, and $q = \inf_{(u,y_u^*)\in\Delta} F_u^*(\cdot, y_u^*)$. As general facts we observe that

$$q^* = \sup_{(u,y_u^*)\in\Delta} \left(F_u^*\left(\cdot, y_u^*\right)\right)^* = \sup_{u\in U} F_u^{**}\left(\cdot, 0_u\right), \tag{3.1}$$

and by this,

$$q^* \leq p \text{ and } p^* \leq q^{**} \leq q,$$

while stable robust duality means that $p^* = q$.

Defining

$$\mathbb{E} := \bigcup_{u\in U} \text{proj}^u_{X^*\times\mathbb{R}} \text{epi} F_u^*,$$

where $\text{proj}^u_{X^*\times\mathbb{R}} : X^* \times Y_u^* \times \mathbb{R} \longrightarrow X^* \times \mathbb{R}$ is $\text{proj}^u_{X^*\times\mathbb{R}}(x^*, y_u^*, \alpha) = (x^*, \alpha)$, and

$$\mathbb{E}_s := \bigcup_{u\in U} \text{proj}^u_{X^*\times\mathbb{R}} \text{epi}_s F_u^*,$$

we have straightforwardly:

$$\text{epi}_s q = \mathbb{E}_s \subset \mathbb{E} \subset \text{epi } q. \tag{3.2}$$

Let us equip X^* (resp. $X^*\times\mathbb{R}$) with the w^*-topology and denote by $\overline{q}$ (resp. q^{co}, resp. $q^{\overline{\text{co}}} = \overline{q^{\text{co}}}$) the w^*-lsc hull (resp. the convex hull, resp. the w^*-lsc convex hull) of q. From (3.2) we get that

$$\text{epi } \overline{q} = \overline{\mathbb{E}}, \quad \text{epi } q^{\overline{\text{co}}} = \overline{\text{co}}\, \mathbb{E}, \tag{3.3}$$

and, consequently,

$$\mathbb{E} \text{ convex} \implies q \text{ convex} \implies \overline{q} \text{ convex} \iff \overline{\mathbb{E}} \text{ convex}.$$

Lemma 3.1 *Assume that* $\text{dom}\, p \neq \emptyset$, *and*

$$p^{**} = \sup_{u\in U} F_u^{**}(\cdot, 0_u). \tag{3.4}$$

Then $p^* = q^{\overline{\text{co}}}$.

Proof. By (3.1) and (3.4), $p^{**} = q^* = (q^{\text{co}})^*$, and $p^* = p^{***} = (q^{\text{co}})^{**}$. Since $\text{dom}\,(q^{\text{co}})^* = \text{dom}\, p^{**} \supset \text{dom}\, p \neq \emptyset$, we have $\text{dom}\,(q^{\text{co}})^* \neq \emptyset$ and $(q^{\text{co}})^{**} = \overline{q^{\text{co}}} = q^{\overline{\text{co}}}$. Hence, $p^* = q^{\overline{\text{co}}}$. ∎

Proposition 3.1 *Assume that* $\text{dom}\, p \neq \emptyset$ *and* (3.4) *holds. Then, stable robust duality holds if and only if q is convex and w^*-lsc.*

Proof. Thanks to Lemma 3.1, q is convex and w^*-lsc amounts to say $p^* = q$, that is stable robust duality. ∎

Let us recall that $A \subset X^* \times \mathbb{R}$ is said to be w^*-closed and convex regarding $B \subset X^* \times \mathbb{R}$ if (see [6])

$$(\overline{\text{co}}A) \cap B = A \cap B.$$

Proposition 3.2 *Assume that* $\operatorname{dom} p \neq \emptyset$ *and that (3.4) holds and let* $x^* \in X^*$*. Then, strong robust duality holds at* x^* *if and only if* $\mathbb{E}$ *is* w^*-*closed and convex regarding* $\{x^*\} \times \mathbb{R}$*.*

Proof. We start by proving the necessity. Let $(x^*, r) \in \overline{\text{co}}\ \mathbb{E}$. By (3.3) and Lemma 3.1, $p^*(x^*) = (q^{\overline{\text{co}}})(x^*) \leq r$. Since strong robust duality holds at x^*, there will exist $(u, y_u^*) \in \Delta$ such that $p^*(x^*) = F_u^*(x^*, y_u^*) \leq r$ and, by definition of $\mathbb{E}$, $(x^*, r) \in \mathbb{E}$.

For the sufficiency, we proceed with the following discussion. If $p^*(x^*) = +\infty$ we are done as, then, $q(x^*) = +\infty = F_u^*(x^*, y_u^*)$ for all $(u, y_u^*) \in \Delta$. Since $\operatorname{dom} p \neq \emptyset$, we have $p^*(x^*) \neq -\infty$. Assume that $p^*(x^*) \in \mathbb{R}$. Then, $(x^*, p^*(x^*)) \in \operatorname{epi} p^*$ and, by Lemma 3.1 and (3.3), $(x^*, p^*(x^*)) \in \overline{\text{co}}\mathbb{E}$. Since $\mathbb{E}$ is w^*-closed and convex regarding $\{x^*\} \times \mathbb{R}$, one concludes that $(x^*, p^*(x^*)) \in \mathbb{E}$ and, by the own definition of $\mathbb{E}$, there must exist $(u, y_u^*) \in \Delta$ such that $((x^*, y_u^*), p^*(x^*)) \in \operatorname{epi} F_u^*$, and consequently

$$q(x^*) \leq F_u^*(x^*, y_u^*) \leq p^*(x^*) \leq q(x^*),$$

and this entails that strong robust duality holds at x^*. ∎

Corollary 3.1 *Assume that* $\operatorname{dom} p \neq \emptyset$ *and that (3.4) holds. Then, stable strong robust duality holds if and only if* $\mathbb{E}$ *is* w^*-*closed and convex.*

Proof. This is due to the fact that $\mathbb{E}$ is w^*-closed and convex if and only if $\mathbb{E}$ is w^*-closed and convex regarding $\{x^*\} \times \mathbb{R}$ for any $x^* \in X^*$. ∎

Lemma 3.2 *Let* $\varepsilon \geq 0$*,* $x \in X$*, and* $D^\varepsilon(x)$ *be as in* (2.3)*. We have*

$$D^\varepsilon(x) = \Big\{ x^* \in X^* \ : \ \exists\, (u, y_u^*) \in \Delta \quad \text{s.t.} \quad F_u^*(x^*, y_u^*) - \langle x^*, x \rangle + p(x) \leq \varepsilon \Big\}. \tag{3.5}$$

Proof. Let us denote by $E^\varepsilon(x)$ the right-hand side of (3.5), and let $x^* \in D^\varepsilon(x)$. Then, there exist $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $u \in I^{\varepsilon_1}(x)$, and $y_u^* \in Y_u^*$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$, and $(x^*, y_u^*) \in (\partial^{\varepsilon_2} F_u)(x, 0_u)$. Consequently,

$$\begin{aligned} F_u^*(x^*, y_u^*) - \langle x^*, x \rangle + p(x) &= F_u^*(x^*, y_u^*) + F_u(x, 0_u) - \langle x^*, x \rangle + p(x) - F_u(x, 0_u) \\ &\leq \varepsilon_2 + \varepsilon_1 = \varepsilon, \end{aligned}$$

that means that $x^* \in E^\varepsilon(x)$, and hence $D^\varepsilon(x) \subset E^\varepsilon(x)$.

Let us prove the opposite inclusion. Let $x^* \in E^\varepsilon(x)$. Then, there will exist $u \in U$, $y_u^* \in Y_u^*$ such that

$$F_u^*(x^*, y_u^*) - \langle x^*, x \rangle + p(x) \leq \varepsilon.$$

If $p(x) \in \mathbb{R}$, it follows that $F_u(x, 0_u) \in \mathbb{R}$ and we have $\alpha_1 + \alpha_2 \leq \varepsilon$, where $\alpha_1 := p(x) - F_u(x, 0_u) \in \mathbb{R}_+$ and $\alpha_2 := F_u^*(x^*, y_u^*) + F_u(x, 0_u) - \langle x^*, x \rangle \in \mathbb{R}_+$. Thus, there exist $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ such that $\alpha_1 \leq \varepsilon_1$ $\alpha_2 \leq \varepsilon_2$, $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Now $\alpha_1 \leq \varepsilon_1$ means that $u \in I^{\varepsilon_1}(x)$, and $\alpha_2 \leq \varepsilon_2$ means that $(x^*, y_u^*) \in (\partial^{\varepsilon_2} F_u)(x, 0_u)$. According to this, $x^* \in D^\varepsilon(x)$. In case $p(x) \notin \mathbb{R}$, by our convention (page 4), $D^\varepsilon(x) = \emptyset$ and, in this case, it is clear that $E^\varepsilon(x) = \emptyset$ and the proof is complete. ∎

Lemma 3.3 *For each $\varepsilon \geq 0$ and each $x \in X$, one has*

$$\overline{D^\varepsilon(x)} \subset \partial^\varepsilon p(x).$$

Proof. Since $\partial^\varepsilon p(x)$ is w^*-closed, it suffices to check that $D^\varepsilon(x) \subset \partial^\varepsilon p(x)$. Let $x^* \in D^\varepsilon(x)$. By (2.3), there exist $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $u \in U$, such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$, $u \in I^{\varepsilon_1}(x)$ and $x^* \in \operatorname{proj}_{X^*}^u (\partial^{\varepsilon_2} F_u)(x, 0_u)$. Then, we have $p(x) \in \mathbb{R}$ (by definition of $I^{\varepsilon_1}(x)$), and there exists $(x^*, y_u^*) \in (\partial^{\varepsilon_2} F_u)(x, 0_u)$, $F_u(x, 0_u) \in \mathbb{R}$, $F_u^*(x^*, y_u^*) \in \mathbb{R}$, and

$$\begin{aligned} p(x) + p^*(x^*) - \langle x^*, x \rangle &\leq p(x) + q(x^*) - \langle x^*, x \rangle \\ &\leq F_u(x, 0_u) + \varepsilon_1 + q(x^*) - \langle x^*, x \rangle \\ &\leq F_u(x, 0_u) + F_u^*(x^*, y_u^*) - \langle x^*, x \rangle + \varepsilon_1 \\ &\leq \varepsilon_2 + \varepsilon_1 = \varepsilon, \end{aligned}$$

where the first inequality comes from $p^* \leq q$, the second one from the definition of $I^{\varepsilon_1}(x)$, the third one from the definition of q, and the last one from $(x^*, y_u^*) \in (\partial^{\varepsilon_2} F_u)(x, 0_u)$. Thus, we conclude that $x^* \in \partial^\varepsilon p(x)$. ∎

Proposition 3.3 *Assume that q is convex,* $\operatorname{dom} p \neq \emptyset$, *and*

$$p = \sup_{u \in U} F_u^{**}(\cdot, 0_u). \tag{3.6}$$

Then for each $\varepsilon > 0$ and each $x \in X$ one has

$$\overline{D^\varepsilon(x)} = \partial^\varepsilon p(x).$$

Proof. By (3.1) and (3.6) we have $p = q^*$, and so $p^* = q^{**}$. Since q is convex and $\operatorname{dom} q^* = \operatorname{dom} p \neq \emptyset$ we get $p^* = \overline{q}$. Now,

$$\partial^\varepsilon p(x) = \Big[\overline{q} - \langle \cdot, x \rangle + p(x) \leq \varepsilon\Big] = \Big[\overline{q - \langle \cdot, x \rangle + p(x)} \leq \varepsilon\Big]. \tag{3.7}$$

Let us observe that

$$\inf_{X^*} \Big\{ q - \langle \cdot, x \rangle + p(x) \Big\} = -q^*(x) + p(x) = 0.$$

Since the function $q - \langle \cdot, x \rangle + p(x)$ is convex and $\varepsilon > \inf_{X^*} \Big\{ q - \langle \cdot, x \rangle + p(x) \Big\}$, we thus have (see [10, Lemma 1.1])

$$\Big[\overline{q - \langle \cdot, x \rangle + p(x)} \leq \varepsilon\Big] = \overline{\Big[q - \langle \cdot, x \rangle + p(x) \leq \varepsilon\Big]} = \overline{\Big[q - \langle \cdot, x \rangle + p(x) < \varepsilon\Big]}. \tag{3.8}$$

It now follows from (3.7) and (3.8) that

$$\partial^{\varepsilon} p(x) = \overline{\left[q - \langle \cdot, x\rangle + p(x) < \varepsilon\right]}. \tag{3.9}$$

By definition of q and Lemma 3.2 we have

$$[q - \langle \cdot, x\rangle + p(x) < \varepsilon] \subset D^{\varepsilon}(x)$$

and, by (3.9), $\partial^{\varepsilon} p(x) \subset \overline{D^{\varepsilon}(x)}$. We conclude the proof by applying Lemma 3.3. ∎

Remark 3.1 *It is worth noticing that the condition* (3.6) *holds under some mild condition such as convexity in some concrete cases. For instance, let* $f, g : X \to \overline{\mathbb{R}}$, $k : Z \to \overline{\mathbb{R}}$ *be proper functions and, for each* $u \in U$, $H_u : \operatorname{dom} H_u \subset X \to Z$ *be a map with* $\emptyset \neq \operatorname{dom} H_u \subset X$. *Let* $Y_u = Z$, *for each* $u \in U$, *and consider the perturbation function, for each* $(u, x, z) \in U \times X \times Z$,

$$F_u(x, z) := \begin{cases} f(x) + g(x) + k\left(H_u(x) + z\right), & \textit{if } x \in \operatorname{dom} H_u, \\ +\infty, & \textit{else}, \end{cases}$$

where

$$(k \circ H_u)(x) := \begin{cases} k(H_u(x)), & \textit{if } x \in \operatorname{dom} H_u, \\ +\infty, & \textit{if } x \in X \setminus (\operatorname{dom} H_u), \end{cases}$$

which means that we consider the robust optimization problem

$$\inf_{x \in X} \sup_{u \in U} [f(x) + g(x) + k\left(H_u(x)\right)]. \tag{3.10}$$

Under the condition that $f + g + \lambda H_u$ *is a proper convex lsc function, for all* $(u, \lambda) \in U \times \operatorname{dom} k^*$, *and* $k^{**}(H_u(x)) = k(H_u(x))$, *for all* $u \in U$ *and for all* $x \in \bigcap_{u \in U} \operatorname{dom} H_u$, *then* (3.6) *holds (see the proof of [5, Theorem 4.2]), i.e.,*

$$p := \sup_{u \in U} F_u(x, 0_Z) = \sup_{u \in U} F_u^{**}(x, 0_Z).$$

A special case of (3.10) *is the general robust problem with cone constraints*

$$\inf_{x \in X} \left\{f(x) : \; H_u(x) \in -S, \quad u \in U, \quad x \in C\right\},$$

where S *is a convex cone in* Y *and* C *is a closed convex subset in* X. *In this case* g *and* k *are the indicator functions of* C *and* $-S$, *respectively.*

The next result provides another characterization of stable robust duality that does not require any convexity assumption.

Theorem 3.1 (Stable robust duality II) *Assume that* $\operatorname{dom} p \neq \emptyset$. *The next statements are equivalent:*

(i) *Stable robust duality holds, i.e.,* $p^* = q$,

(ii) *There exists* $\delta > 0$ *such that* $\partial^{\varepsilon} p(x) \subset D^{\delta\varepsilon}(x)$ *for all* $\varepsilon > 0$ *and all* $x \in X$.

Proof. [(i) $\Longrightarrow$ (ii)] Let $\varepsilon > 0$ and $x \in X$. By Theorem 2.1 we have

$$\partial^{\varepsilon} p(x) = C^{\varepsilon}(x) = \bigcap_{\eta > 0} D^{\varepsilon+\eta}(x) \subset D^{2\varepsilon}(x),$$

and (ii) holds with $\delta = 2$.

[(ii) $\Longrightarrow$ (i)] Take $x^* \in X^*$. The implication is trivial if $p^*(x^*) = +\infty$. Since $\operatorname{dom} p \neq \emptyset$ it remains to study the case when $p^*(x^*) \in \mathbb{R}$. Let $\varepsilon > 0$ and pick $x \in (M^{\varepsilon}p)(x^*) = \varepsilon - \operatorname{argmin}(p - x^*)$, which is non-empty. We have $x^* \in \partial^{\varepsilon} p(x)$ and, by (ii), there exists $\delta > 0$ such that $x^* \in D^{\delta\varepsilon}(x)$. By Lemma 3.2 there exist $u \in U$ and $y_u^* \in Y_u^*$ such that

$$F_u^*(x^*, y_u^*) - \langle x^*, x\rangle + p(x) \leq \delta\varepsilon.$$

Hence,

$$q(x^*) - p^*(x^*) \leq q(x^*) - \langle x^*, x\rangle + p(x) \leq F_u^*(x^*, y_u^*) - \langle x^*, x\rangle + p(x) \leq \delta\varepsilon.$$

Letting $\varepsilon \to 0_+$ we get $q(x^*) - p^*(x^*) \leq 0$ and, by (1.1), $q(x^*) = p^*(x^*)$. ∎

In the case when q is convex we have:

Theorem 3.2 (Stable robust duality under convexity) *Assume that q is convex, $\operatorname{dom} p \neq \emptyset$, and (3.6) holds. The next statements are equivalent:*

(i) *Stable robust duality holds, i.e., $p^* = q$,*

(ii) $\bigcap_{\eta > 0} D^{\varepsilon+\eta}(x) = \overline{D^{\varepsilon}(x)}$ *for all $\varepsilon > 0$ and all $x \in X$.*

Proof. [(i) $\Longrightarrow$ (ii)] follows from Theorem 2.1, Proposition 3.3, and (2.4).

[(ii) $\Longrightarrow$ (i)] Let $(\varepsilon, x) \in \mathbb{R}_+ \times X$. According to Theorem 2.1 one has to check that $\partial^{\varepsilon} p(x) = C^{\varepsilon}(x)$. Assume first that $p(x) = +\infty$. Then, $\partial^{\varepsilon} p(x) = \emptyset$ and, by Lemma 3.3 and (2.4), $C^{\varepsilon}(x) = \emptyset$. Assume now that $p(x) \in \mathbb{R}$. Then, by (ii), Proposition 3.3 and (2.4), one has $C^{\varepsilon}(x) = \overline{D^{\varepsilon}(x)} = \partial^{\varepsilon} p(x)$. ∎

We now provide a convex closedness criterion for stable robust duality:

Corollary 3.2 *Assume that q is convex, $\operatorname{dom} p \neq \emptyset$, (3.6) holds, and*

$$D^{\varepsilon}(x) \;\text{ is } w^* - \text{closed}, \; \forall \varepsilon > 0, \forall x \in X. \tag{3.11}$$

Then $p^* = q$.

Proof. By Proposition 3.3 and (3.11), we have $\partial^{\varepsilon} p(x) = D^{\varepsilon}(x)$ for each $\varepsilon > 0$ and each $x \in X$. It then follows from Theorem 3.1 (with $\delta = 1$) that $p^* = q$. ∎

In fact, the closedness criterion (3.11) can also be used to characterize stable strong robust duality. This is the purpose of the next theorem.

Theorem 3.3 (Stable robust duality via closedness and convexity) *Assume that* $\operatorname{dom} p \neq \emptyset$ *and consider the next statements:*

(i) *Stable strong robust duality holds,*

(ii) $D^{\varepsilon}(x)$ *is* w^**-closed for all* $\varepsilon \geq 0$ *and all* $x \in X$*,*

(iii) $D^{\varepsilon}(x)$ *is* w^**-closed for all* $\varepsilon > 0$ *and all* $x \in X$*.*

One has (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)*.*

If q *is convex,* (3.6) *holds, and* p *is not affine on* $\operatorname{dom} p$*, then the three statements are equivalent.*

Proof. [(i) $\Longrightarrow$ (ii)] follows from Theorem 2.2 and the fact that $\partial^{\varepsilon} p(x)$ is w^*-closed.

[(ii) $\Longrightarrow$ (iii)] is trivial.

[(iii) $\Longrightarrow$ (i)] Let $x^* \in X^*$. If $p^*(x^*) = +\infty$, then $q(x^*) = +\infty$ and (i) holds obviously as $q(x^*) = +\infty = F_u^*(x^*, y_u^*)$ for all $(u, y_u^*) \in \Delta$. Since $\operatorname{dom} p \neq \emptyset$, we have $p^*(x^*) \neq -\infty$, and it only remains to be analyzed the case $p^*(x^*) \in \mathbb{R}$. Since p is not affine on $\operatorname{dom} p$ there exists $x \in \operatorname{dom} p$ such that $\varepsilon := p^*(x^*) + p(x) - \langle x^*, x \rangle > 0$. Then, $x^* \in \partial^{\varepsilon} p(x)$ and, by Proposition 3.3, $x^* \in \overline{D^{\varepsilon}(x)} = D^{\varepsilon}(x)$. By Lemma 3.2, there exist $u \in U$, $y_u^* \in Y_u^*$ such that

$$q(x^*) \leq F_u^*(x^*, y_u^*) \leq \langle x^*, x \rangle - p(x) + \varepsilon = p^*(x^*) \leq q(x^*),$$

and (i) holds. ∎

4 Back to deterministic conjugate duality

Consider $F : X \times Y \to \mathbb{R}_{\infty}$, $p = F(\cdot, 0_Y)$, and $q = \inf_{y^* \in Y^*} F^*(\cdot, y^*)$. This is a deterministic problem, i.e., $F_u = F$. We assume that $\operatorname{dom} p \neq \emptyset$.

Since $\operatorname{epi}_s q = \operatorname{proj}_{X^* \times \mathbb{R}} \operatorname{epi}_s F^*$ and $\operatorname{epi}_s F^*$ is convex, the function q is convex, too. By Proposition 3.1 we have:

Corollary 4.1 *Assume that* $p^{**} = F^{**}(\cdot, 0_Y)$*. Then* $p^* = q$ *if and only if* q *is* w^**-lsc.*

Since, in our current setting,

$$D^{\varepsilon}(x) = \operatorname{proj}_{X^*}(\partial^{\varepsilon} F)(x, 0_Y)$$

(see (2.3)), Theorem 2.1, (2.4) and Theorem 3.1 give us:

Corollary 4.2 *The next statements are equivalent:*

(i) $p^* = q$*,*

(ii) $\partial^{\varepsilon} p(x) = \bigcap_{\eta > 0} \operatorname{proj}_{X^*}\big(\partial^{\varepsilon + \eta} F\big)(x, 0_Y)$ *for all* $(\varepsilon, x) \in \mathbb{R}_+ \times X$*,*

(iii) *There exists* $\delta > 0$ *such that*

$$\partial^{\varepsilon} p(x) \subset \operatorname{proj}_{X^*}\big(\partial^{\delta\varepsilon} F\big)(x, 0_Y), \forall \varepsilon > 0, \forall x \in X.$$

By Proposition 3.3 we get:

Corollary 4.3 *Assume that* $p = F^{**}(\cdot, 0_Y)$. *Then for each* $\varepsilon > 0$ *and each* $x \in X$ *we have:*

$$\partial^{\varepsilon} p(x) = \overline{\mathrm{proj}_{X^*}(\partial^{\varepsilon} F)(x, 0_Y)}.$$

From Theorem 3.2 we can state:

Corollary 4.4 *Assume that* $p = F^{**}(\cdot, 0_Y)$. *The next statements are equivalent:*

(i) $p^* = q$,

(ii) $\bigcap_{\eta>0} \mathrm{proj}_{X^*}\big(\partial^{\varepsilon+\eta} F\big)(x, 0_Y) = \overline{\mathrm{proj}_{X^*} \partial^{\varepsilon} F(x, 0_Y)}$ *for all* $\varepsilon > 0$ *and all* $x \in X$.

By Corollary 3.2 we have:

Corollary 4.5 *Assume that* $p = F^{**}(\cdot, 0_Y)$ *and* $\mathrm{proj}_{X^*}\big(\partial^{\varepsilon} F\big)(x, 0_Y)$ *is* w^**-closed for all* $\varepsilon > 0$ *and all* $x \in X$. *Then* $p^* = q$.

From Theorem 3.3 we derive a new characterization of stable strong duality:

Corollary 4.6 *Consider the next statements:*

(i) *Stable strong robust duality holds, i.e.,*

$$\inf_{x \in X} \Big\{ F(x, 0_Y) - \langle x^*, x \rangle \Big\} = - \min_{y^* \in Y^*} F^*(x^*, y^*), \ \forall x^* \in X^*,$$

(ii) $\mathrm{proj}_{X^*}\Big(\partial^{\varepsilon} F\Big)(x, 0_Y)$ *is* w^**-closed for all* $\varepsilon \geq 0$ *and all* $x \in X$,

(iii) $\mathrm{proj}_{X^*}\Big(\partial^{\varepsilon} F\Big)(x, 0_Y)$ *is* w^**-closed for all* $\varepsilon > 0$ *and all* $x \in X$.

Then one has (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). *Moreover, if* $p = F^{**}(\cdot, 0_Y)$ *and* p *is not affine on* $\mathrm{dom}\, p$ *then* (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

We now consider the conjugate duality for the sum of functions and compare our results with those of [1], limiting ourselves to the case of two functions.

Let $f, g \in \mathbb{R}_{\infty}^X$ be two proper functions such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. The classical Fenchel duality for the sum $f + g$ is obtained by considering the perturbation $F : X \times Y \to \mathbb{R}_{\infty}$ given by

$$F(x, y) = f(x + y) + g(x).$$

Here, $Y = X$. We thus have $p = F(\cdot, 0_Y) = f + g$.

Since f and g are proper, the conjugates f^*, g^* do not take value $-\infty$, and we have:

$$-\infty < F^*(x^*, y^*) = f^*(y^*) + g^*(x^* - y^*) \leq +\infty, \ \forall (x^*, y^*) \in X^* \times Y^*. \tag{4.1}$$

The function q coincides with the infimal convolution $f^* \square g^*$ of the conjugate functions f^* and g^*:

$$q(x^*) = \inf_{y^* \in X^*} \Big\{ f^*(y^*) + g^*(x^* - y^*) \Big\}, \ \forall x^* \in X^*.$$

Lemma 4.1 *For each* $(\varepsilon, x) \in \mathbb{R}_+ \times X$ *one has*

$$\mathrm{proj}_{X^*}(\partial^\varepsilon F)(x, 0_X) = \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1\geqslant 0, \varepsilon_2\geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big].$$

Proof. We have $x^* \in \mathrm{proj}_{X^*}\partial^\varepsilon F(x, 0_X)$ if and only if there exists $y^* \in X^*$ such that

$$F^*(x^*, y^*) + F(x, 0_X) - \langle x^*, x\rangle \le \varepsilon,$$

or, equivalently (see (4.1)), $\alpha_1 + \alpha_2 \le \varepsilon$, where

$$\alpha_1 := f^*(y^*) + f(x) - \langle y^*, x\rangle, \quad \alpha_2 := g^*(x^* - y^*) + g(x) - \langle x^* - y^*, x\rangle.$$

Since α_1 and α_2 belong to $\mathbb{R}_+$, this is equivalent to the existence of ε_1 and ε_2 in $\mathbb{R}_+$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$, $\alpha_1 \le \varepsilon_1$, $\alpha_2 \le \varepsilon_2$, which means that $y^* \in \partial^{\varepsilon_1} f(x)$, $x^* - y^* \in \partial^{\varepsilon_2} g(x)$. Therefore, $x^* \in \mathrm{proj}_{X^*}\partial^\varepsilon F(x, 0_X)$ if and only if there exists $y^* \in X^*$ such that $x^* = y^* + x^* - y^* \in \partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)$. ∎

Assuming that

$$\mathrm{dom}\, f^* \neq \emptyset \text{ and } \mathrm{dom}\, g^* \neq \emptyset, \tag{4.2}$$

that means f and g have continuous affine minorants, the biconjugates f^{**} and g^{**} do not take the value $-\infty$, and we have

$$F^{**}(x, y) = f^{**}(x + y) + g^{**}(x), \quad \forall (x, y) \in X \times X. \tag{4.3}$$

So the condition $p^{**} = F^{**}(\cdot, 0_X)$ (used in Corollary 4.1) and the condition $p = F^{**}(\cdot, 0_X)$ (used in Corollaries 4.3-4.6) write respectively

$$(f + g)^{**} = f^{**} + g^{**}, \tag{4.4}$$
$$f + g = f^{**} + g^{**}. \tag{4.5}$$

It is clear that (4.5) $\Longrightarrow$ (4.4).

Condition 4.4 is used in [4, Proposition 3.5] and, for convex functions, in [9, Theorem 13]. Note that if f and g are real-valued then (4.5) amounts to saying that $f = f^{**}$ and $g = g^{**}$. If f and g are just proper functions then (4.5) may hold even if $f \neq f^{**}$ or $g \neq g^{**}$. For instance, if f and g are proper convex functions, and lsc on $\mathrm{dom}\, \bar{f} \cap \mathrm{dom}\, \bar{g}$, which is the hypothesis made in [1, Theorem 3.2], it can be shown that (4.2) and (4.5) hold. The next result completes [1, Theorem 3.2].

Theorem 4.1 *Let* $f, g \in \mathbb{R}_\infty^X$ *be two proper functions such that* $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. *The next statements are equivalent:*

(i) $(f + g)^* = f^* \Box g^*$,

(ii) $\partial^\varepsilon (f + g)(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\eta+\varepsilon \\ \varepsilon_1\geqslant 0, \varepsilon_2\geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big]$ *for all* $(\varepsilon, x) \in \mathbb{R}_+ \times X$,

(iii) *There exists* $\delta > 0$ *such that*

$$\partial^{\varepsilon}(f+g)(x) \subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\delta\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big],\ \forall \varepsilon > 0, \forall x \in X.$$

If (4.2) *holds and* $(f+g)^{**} = f^{**} + g^{**}$ *we can add:*

(iv) $f^* \Box g^*$ *is* w^**-lsc.*

If (4.2) *holds and* $f + g = f^{**} + g^{**}$ *we can add:*

(v) *For each* $\varepsilon > 0$ *and each* $x \in X$ *one has*

$$\bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\eta+\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big] = \overline{\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big]}.$$

(vi) *There exists* $\delta > 0$ *such that*

$$\overline{\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big]} \subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\delta\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big], \forall \varepsilon > 0, \forall x \in X.$$

(vii) *There exists* $\delta > 0$ *such that*

$$\overline{\partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x)} \subset \partial^{\delta\varepsilon} f(x) + \partial^{\delta\varepsilon} g(x), \forall \varepsilon > 0, \forall x \in X.$$

Proof. [(i) ⇔ (ii) ⇔ (iii)] by Corollary 4.2 and Lemma 4.1.

[(i) ⇔ (iv)] by Corollary 4.1.

[(i) ⇔ (v)] by Corollary 4.4 and Lemma 4.1.

[(i) ⇔ (vi)] by Corollaries 4.2-4.3 and Lemma 4.1.

[(vi) ⇒ (vii)] Let $\varepsilon > 0$ and $x \in \operatorname{dom} f \cap \operatorname{dom} g$. Then

$$\partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x) \subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=2\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big],$$

and, by (vi), there exists $\delta' > 0$ such that

$$\overline{\partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x)} \subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=2\delta'\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big] \subset \partial^{2\delta'\varepsilon} f(x) + \partial^{2\delta'\varepsilon} g(x),$$

and (vii) holds with $\delta = 2\delta'$.

[(vii) ⇒ (vi)] Let $\varepsilon > 0, x \in \operatorname{dom} f \cap \operatorname{dom} g$. We obviously have

$$\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0}} \Big[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\Big] \subset \partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x),$$

and, by (vii), there exists $\delta' > 0$ such that

$$\overline{\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}} \left[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\right]} \subset \overline{\partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x)} \subset \partial^{\delta'\varepsilon} f(x) + \partial^{\delta'\varepsilon} g(x)$$

$$\subset \bigcup_{\substack{\varepsilon_1+\varepsilon_2=2\delta'\varepsilon\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}} \left[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\right],$$

and (vi) holds with $\delta = 2\delta'$. ∎

The following closedness criteria are immediate consequences of Theorem 4.1 (see statements (vi), (vii) with $\delta = 1$).

Proposition 4.1 *Let $f, g \in \mathbb{R}_\infty^X$ be two proper functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, (4.2) *holds, $f + g = f^{**} + g^{**}$, and either*

$$\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}} \left[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\right] \ \textit{is } w^* - \text{closed}, \ \forall \varepsilon > 0, \forall x \in X,$$

or the Bertsekas constraint qualification holds, i.e.,

$$\partial^{\varepsilon} f(x) + \partial^{\varepsilon} g(x) \text{ is } w^* - \text{closed}, \ \forall \varepsilon > 0, \forall x \in X.$$

Then $(f + g)^* = f^* \Box g^*$.

From Corollary 4.6 and Lemma 4.1 we derive a new characterization for the stable strong duality for the sum of functions.

Proposition 4.2 *Let $f, g \in \mathbb{R}_\infty^X$ be two proper functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Consider the next statements:*

(i) $(f + g)^*(x^*) = \min_{y^* \in X^*} \left\{ f^*(y^*) + g^*(x^* - y^*) \right\}$ *for all $x^* \in X^*$,*

(ii) $\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}} \left[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\right]$ is w^* − closed *for all $\varepsilon \geq 0$ and all $x \in X$,*

(iii) $\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}} \left[\partial^{\varepsilon_1} f(x) + \partial^{\varepsilon_2} g(x)\right]$ is w^* − closed *for all $\varepsilon > 0$ and all $x \in X$,*

Then one has (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

If, additionally, (4.2) *holds, $f + g = f^{**} + g^{**}$, and $f + g$ is not affine on* $\operatorname{dom} f \cap \operatorname{dom} g$, *then* (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Acknowledgements This research was supported by the Ministry of Economy and Competitiveness of Spain and the European Regional Development Fund (ERDF) of the European Commission, Project MTM2014-59179-C2-1-P, by the Australian Research Council, Project DP160100854, and by Vietnam National University -HCM city, Vietnam, project "Generalized scalar and vector Farkas-type results with applications to optimization theory".

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