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FUNCTIONAL INEQUALITIES IN THE ABSENCE OF CONVEXITY AND LOWER SEMICONTINUITY WITH APPLICATIONS TO OPTIMIZATION*

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Abstract. In this paper we extend some results in [Dinh, Goberna, López, and Volle, *Set-Valued Var. Anal.*, to appear] to the setting of functional inequalities when the standard assumptions of convexity and lower semicontinuity of the involved mappings are absent. This extension is achieved under certain condition relative to the second conjugate of the involved functions. The main result of this paper, Theorem 1, is applied to derive some subdifferential calculus rules and different generalizations of the Farkas lemma for nonconvex systems, as well as some optimality conditions and duality theory for infinite nonconvex optimization problems. Several examples are given to illustrate the significance of the main results and also to point out the potential of their applications to get various extensions of Farkas-type results and to the study of other classes of problems such as variational inequalities and equilibrium models.

Key words. functional inequalities, Farkas-type lemmas for nonconvex systems, infinite-dimensional nonconvex optimization

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1. Introduction.

Given two convex lower semicontinuous (lsc) extended real-valued functions F and h, defined on locally convex spaces, we provided in [8] a dual transcription of the functional inequality

$$(*)\qquad F(0,\cdot) \geq h(\cdot),$$

in terms of the Legendre–Fenchel conjugates of F and h, and applied this result to convex subdifferential calculus, subgradient-based optimality conditions, Farkas-type results, and, in the optimization field, to linear, convex, semidefinite problems, and to difference of convex functions (DC problems). The main feature of the approach in that paper was the absence of the so-called topological constraint qualifications (CQs) and closedness conditions in the hypotheses.

In many situations the well-known CQs, such as generalized Slater-type/interior-type, Mangasarian–Fromovitz CQs, Robinson-type CQs, and Attouch and Brézis CQs, fail to hold. This is the case in many classes of scalarized forms of (convex) vector optimization problems, in semidefinite programs, and in bilevel programming problems (see, e.g., [5], [9], [36]). Because of that, in the last decades many efforts have been devoted to establishing mathematical tools for such classes of problems (e.g., [2], [3], [8], [9], [12], [24], [27], [32], [33], [35]).

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Nowadays, in science and technology there are a huge number of practical problems that can be modeled as nonconvex optimization problems (see [1], [18], [26], [28], and references therein).

In the present paper, we go a step further than what is done in [8] by relaxing the convexity and the lower semicontinuity on the function F in the left-hand side of $(*)$. In doing so, we use convex tools for nonconvex problems, a tendency whose importance increases nowadays. Even more, we characterize in Theorem 1 the class of functions F for which the dual transcription of $(*)$ obtained in [8] does work. We show that the class of such functions F goes far beyond the usual one of convex and lsc extended real-valued mappings. In fact, this extension is achieved under certain conditions relative to the second Legendre–Fenchel conjugates of the mappings F and $F(0,\cdot)$. A dual geometrical description of this property is given in Proposition 3.

As consequences of Theorem 1, we obtain extensions of the basic convex subdifferential calculus formulas for not necessarily convex functions (Theorem 2 and Proposition 4), Farkas-type results for nonconvex systems (Propositions 5 and 6), optimality conditions for nonconvex optimization problems (Propositions 7, 8, 10, and 11), from which we derive the corresponding recent basic results in the convex setting (Corollaries 1 and 2).

In the same way, we provide duality theorems for nonconvex optimization problems (Proposition 9 and Corollary 3) that cover some recent results in the convex case (Corollary 4).

The results presented in this paper are new, to the knowledge of the authors, and they extend in different directions some relevant results in the literature, such as [6], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], and [24]. The extensions we propose here are such that typical assumptions such as the convexity and/or lower semicontinuity of the involved functions, as well as the closedness-type CQ conditions, are absent. Besides this, Examples 1 and 2 in section 3 also show the potential of Theorem 1 to get further generalizations of Farkas-type theorems and of other results in the field of variational inequalities and equilibrium problems—always in the absence of convexity, of lower semicontinuity, and of any closedness/qualification conditions.

2. Notation and preliminary results.

Let X be a locally convex Hausdorff topological vector space (l.c.H.t.v.s.) whose topological dual is denoted by X^*. The only topology we consider on X^* is the w^*-topology.

Given two nonempty sets A and B in X (or in X^*), we define the *algebraic sum* by

$$A + B := \{a + b \mid a \in A, \quad b \in B\}, \qquad A + \emptyset := \emptyset + A := \emptyset, \tag{2.1}$$

and we set $x + A := \{x\} + A$.

Throughout the paper we adopt the rule $(+\infty) - (+\infty) = +\infty$.

We denote by $\operatorname{co} A$, $\operatorname{cone} A$, and $\operatorname{cl} A$ (or indistinctly by $\overline{A}$), the convex hull, the conical convex hull, and the closure of A, respectively.

Given a function $h \in (\mathbb{R} \cup \{+\infty\})^X$, its *(effective) domain*, *epigraph*, and *level set* are defined, respectively, by

$$\begin{aligned} \operatorname{dom} h &:= \{x \in X : \ h(x) < +\infty\}, \\ \operatorname{epi} h &:= \{(x, \alpha) \in X \times \mathbb{R} : \ h(x) \leq \alpha\}, \\ [h \leq \alpha] &:= \{x \in X : \ h(x) \leq \alpha\}. \end{aligned}$$

The function $h \in (\mathbb{R} \cup \{+\infty\})^X$ is *proper* if $\operatorname{dom} h \neq \emptyset$, it is *convex* if $\operatorname{epi} h$ is convex, and it is *lsc* if $\operatorname{epi} h$ is closed.

The *lsc envelope* of h is the function $\overline{h} \in (\mathbb{R} \cup \{\pm\infty\})^X$ defined by

$$\overline{h}(x) := \inf\{t : (x,t) \in \mathrm{cl}(\mathrm{epi}\, h)\}.$$

Clearly, we have $\mathrm{epi}\, \overline{h} = \overline{\mathrm{epi}\, h}$, which implies that $\overline{h}$ is the greatest lsc function minorizing h, so $\overline{h} \le h$. If h is convex, then $\overline{h}$ is also convex, and then $\overline{h}$ does not take the value $-\infty$ if and only if h admits a continuous affine minorant.

Given $h \in (\mathbb{R} \cup \{+\infty\})^X$, the *lsc convex hull* of h is the convex lsc function $\overline{\mathrm{co}} h \in (\mathbb{R} \cup \{\pm\infty\})^X$ such that

$$\mathrm{epi}(\overline{\mathrm{co}} h) = \overline{\mathrm{co}}(\mathrm{epi}\, h).$$

Obviously, $\overline{\mathrm{co}} h \le \overline{h} \le h$.

We shall denote by $\Gamma(X)$ the class of all the proper lsc convex functions on X. The set $\Gamma(X^*)$ is defined similarly.

Given $h \in (\mathbb{R} \cup \{+\infty\})^X$, the *Legendre–Fenchel conjugate* of h is the function $h^* \in (\mathbb{R} \cup \{\pm\infty\})^{X^*}$ given by

$$h^*(x^*) = \sup\{\langle x^*, x\rangle - h(x) : x \in X\}.$$

The function h^* is convex and lsc. If $\mathrm{dom}\, h = \emptyset$, we have $h^* = \{-\infty\}^{X^*}$ (i.e., $h^*(x^*) = -\infty \ \forall\, x^* \in X^*$). Moreover, $h^* \in \Gamma(X^*)$ if and only if $\mathrm{dom}\, h \neq \emptyset$ and h admits a continuous affine minorant.

The *biconjugate* of h is the function $h^{**} \in (\mathbb{R} \cup \{\pm\infty\})^X$ given by

$$h^{**}(x) := \sup\{\langle x^*, y\rangle - h^*(x^*) : x^* \in X^*\}.$$

We have

$$\{h \in (\mathbb{R} \cup \{+\infty\})^X : h = h^{**}\} = \Gamma(X) \cup \{+\infty\}^X.$$

Moreover, $h^{**} \le \overline{\mathrm{co}} h$, and the equality holds if h admits a continuous affine minorant.

The *indicator* function of $A \subset X$ is defined as

$$i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \in X \setminus A. \end{cases}$$

If $A \neq \emptyset$, the conjugate of i_A is the *support function* of A, $i_A^* : X^* \to \mathbb{R} \cup \{+\infty\}$.

Given $a \in h^{-1}(\mathbb{R})$ and $\varepsilon \ge 0$, the ε-*subdifferential* of h at the point a is defined by

$$\partial_\varepsilon h(a) = \{x^* \in X^* : h(x) - h(a) \ge \langle x^*, x - a\rangle - \varepsilon \ \forall x \in X\}.$$

One has

$$\partial_\varepsilon h(a) = [h^* - \langle \cdot, a\rangle \le \varepsilon - h(a)] = \{x^* \in X^* : h^*(x^*) - \langle x^*, a\rangle \le \varepsilon - h(a)\}.$$

If $a \notin h^{-1}(\mathbb{R})$, set $\partial_\varepsilon h(a) = \emptyset$.

If $h \in (\mathbb{R} \cup \{+\infty\})^X$ is convex and $a \in h^{-1}(\mathbb{R})$, then we have $\partial_\varepsilon h(a) \neq \emptyset \ \forall\, \varepsilon > 0$ if and only if h is lsc at a.

The ε-*normal set* to a nonempty set A at a point $a \in A$ is defined by

$$\mathrm{N}_\varepsilon(A, a) = \partial_\varepsilon i_A(a).$$

The *Young–Fenchel inequality*

$$f^*(x^*) \geq \langle x^*, a \rangle - f(a)$$

always holds. The equality holds if and only if $x^* \in \partial f(a) := \partial_0 f(a)$.

The *limit superior* when $\eta \to 0_+$ of the family $(A_\eta)_{\eta>0}$ of subsets of a topological space is defined (in terms of generalized sequences or nets) by

$$\limsup_{\eta \to 0_+} A_\eta := \left\{ \lim_{i\in I} a_i : a_i \in A_{\eta_i} \ \forall i \in I, \text{ and } \eta_i \to 0_+ \right\},$$

where $\eta_i \to 0_+$ means that $(\eta_i)_{i\in I} \to 0$ and $\eta_i > 0 \ \forall i \in I$.

Let U be another l.c.H.t.v.s. whose topological dual is denoted by U^*, and let us consider $F \in \Gamma(U \times X)$. In [8] we established the following result.

PROPOSITION 1. *Let $F \in \Gamma(U \times X)$ with $\{x \in X : F(0,x) < +\infty\} \neq \emptyset$. For any $h \in \Gamma(X)$, the following statements are equivalent.*

(a) $F(0,x) \geq h(x) \ \forall \ x \in X$.

(b) *For every $x^* \in \operatorname{dom} h^*$, there exists a net $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ such that*

$$F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i \ \forall \ i \in I,$$

and

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

3. Functional inequalities involving not necessarily convex nor lsc mappings. The following theorem constitutes an extension of Proposition 1 to a function F which is neither convex nor lsc, but the theorem is true under certain specific requirements to be satisfied by the second conjugate F^{**}. In fact, it delivers a characterization of that requirement.

THEOREM 1. *Let $F : U \times X \to \mathbb{R} \cup \{+\infty\}$ such that $F(0,\cdot)$ is proper and $\operatorname{dom} F^* \neq \emptyset$. Then the following statements are equivalent.*

(a) $F^{**}(0,\cdot) = (F(0,\cdot))^{**}$.

(b) *For any $h \in \Gamma(X)$,*

$$F(0,x) \geq h(x) \ \forall x \in X \iff \left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} h^*, \textit{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R} \textit{ such that} \\ F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i \ \forall i \in I, \textit{ and} \\ \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

Proof. Assume that (a) holds, and let $h \in \Gamma(X)$, satisfying $F(0,\cdot) \geq h$. Taking biconjugates in both sides, we get $(F(0,\cdot))^{**} \geq h^{**} = h$, and by (a), $F^{**}(0,\cdot) \geq h$. Applying Proposition 1 with $F^{**} \in \Gamma(U \times X)$ playing the role of F (observe that $\{x \in X : F^{**}(0,x) < +\infty\} \supset \operatorname{dom} F(0,\cdot) \neq \emptyset$), and recalling that $F^{***} = F^*$, we get the implication "$\Rightarrow$" in (b).

Assume now that, for a given $h \in \Gamma(X)$, the right-hand side in the equivalence (b) holds. Again, by Proposition 1 applied to F^{**}, we get

$$F(0,x) \geq F^{**}(0,x) \geq h(x) \ \forall x \in X.$$

Thus we have that the converse implication "$\Leftarrow$" in (b) also holds.

Assume now that (b) holds.

Consider any $(x^*,r) \in X^* \times \mathbb{R}$ such that

$$F(0,\cdot) \geq \langle x^*, \cdot\rangle - r. \tag{3.1}$$

Let us apply (b) with $h = \langle x^*, \cdot\rangle - r$ to conclude the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$F^*(u_i^*, x_i^*) \leq h^*(x^*) + \varepsilon_i = r + \varepsilon_i \ \forall i \in I,$$

and

$$\lim_{i\in I} (x_i^*, \varepsilon_i) = (x^*, 0_+).$$

Thus we have, for any $x \in X$,

$$F^{**}(0,x) \geq \langle x_i^*, x\rangle - F^*(u_i^*, x_i^*) \geq \langle x_i^*, x\rangle - r - \varepsilon_i \ \forall i \in I,$$

and, passing to the limit on $i \in I$,

$$F^{**}(0,\cdot) \geq \langle x^*, \cdot\rangle - r. \tag{3.2}$$

Since (3.2) holds whenever (x^*, r) satisfies (3.1), we get

$$\begin{aligned} F^{**}(0,\cdot) &\geq \sup\{\langle x^*, \cdot\rangle - r : \ (x^*,r) \text{ satisfies (3.1)}\} \\ &= (F(0,\cdot))^{**}. \end{aligned}$$

As $\operatorname{dom} F^* \neq \emptyset$ and $F(0,\cdot)$ is proper, one has $F^{**}(0,\cdot) \in \Gamma(X)$. Since $F^{**}(0,\cdot) \leq F(0,\cdot)$, it follows that $F^{**}(0,\cdot) \leq (F(0,\cdot))^{**}$ and, finally, that (a) holds. ∎

Next we provide some geometrical insight on the meaning of condition (a) in Theorem 1. To this aim let us introduce the closed linear spaces $V := \{0\}\times X \subset U\times X$ and $W := V \times \mathbb{R} \subset U \times X \times \mathbb{R}$. Observe that

$$\{0\} \times \operatorname{epi} F(0,\cdot) = W \cap \operatorname{epi} F. \tag{3.3}$$

Since F (and, a fortiori, $F(0,\cdot)$) admits a continuous affine minorant as a consequence of the assumption $\operatorname{dom} F^* \neq \emptyset$, (3.3) yields

$$\{0\} \times \operatorname{epi}(F(0,\cdot))^{**} = \{0\} \times \overline{\operatorname{co}}(\operatorname{epi} F(0,\cdot)) = \overline{\operatorname{co}}(W \cap \operatorname{epi} F),$$

while

$$\operatorname{epi} F^{**}(0,\cdot) = W \cap \overline{\operatorname{co}}(\operatorname{epi} F).$$

Consequently, condition (a) in Theorem 1 may be rewritten as

$$W \cap \overline{\operatorname{co}}(\operatorname{epi} F) = \overline{\operatorname{co}}(W \cap \operatorname{epi} F). \tag{3.4}$$

Observe that (3.4) is a notable weakening of the assumption in Proposition 1, $F \in \Gamma(U \times X)$, which means

$$\operatorname{epi} F = \overline{\operatorname{co}}(\operatorname{epi} F).$$

Since $W \cap \operatorname{epi} F = \operatorname{epi}(F + i_V)$ and $W \cap \overline{\text{co}}(\operatorname{epi} F) = \operatorname{epi}(F^{**} + i_V)$, an analytic reformulation of (3.4) (*alias* condition (a) in Theorem 1) is

$$(F + i_V)^{**} = F^{**} + i_V. \tag{3.5}$$

From Proposition 2(a) below it is easy to observe that (3.5) holds in particular if

$$F(u, x) = F^{**}(u, x) \; \forall \; (u, x) \in V. \tag{3.6}$$

Actually, condition (a) in Theorem 1 may be satisfied while (3.6) fails. This is the case, for instance, when $U = X = \mathbb{R}$ and $F(u, x) = \sqrt{|u|} + \exp(-x^2)$. We actually have $F(0, x) = \exp(-x^2) \neq 0 = F^{**}(0, x) \; \forall x \in \mathbb{R}$, but (a) holds since $(F(0, x))^{**} \equiv 0$.

Proposition 2. (a) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $C \subset X$ be a closed convex set. Assume that* $\operatorname{dom} f^* \neq \emptyset$ *and that*

$$f(x) = f^{**}(x) \; \forall x \in C.$$

Then we have

$$(f + i_C)^{**} = f^{**} + i_C.$$

(b) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper, lsc on the segments, and such that*

$$(f + i_C)^{**} = f^{**} + i_C$$

for every closed segment C such that $C \cap \operatorname{dom} f \neq \emptyset$. Then we have $f \in \Gamma(X)$.

Proof. (a) By assumption, one has $f + i_C = f^{**} + i_C$, and $f^{**} + i_C$ is lsc, convex, and admits a continuous affine minorant. Hence we have

$$(f + i_C)^{**} = (f^{**} + i_C)^{**} = f^{**} + i_C.$$

(b) We first prove

$$f(a) = f^{**}(a) \; \forall a \in \operatorname{dom} f.$$

Let $a \in \operatorname{dom} f$ and take $C = \{a\}$. By assumption, one has

$$f + i_{\{a\}} = (f + i_{\{a\}})^{**} = f^{**} + i_{\{a\}},$$

and so, $f(a) = f^{**}(a)$.

To conclude the proof, we have just to check that $\operatorname{dom} f^{**} \subset \operatorname{dom} f$. Assume the contrary, i.e., the existence of $b \in \operatorname{dom} f^{**}$ such that $f(b) = +\infty$. Pick $a \in \operatorname{dom} f$ and define

$$\Delta := \{\lambda \in [0, 1] : \; (1 - \lambda)a + \lambda b \in \operatorname{dom} f\}.$$

Let us prove that Δ is closed. To this purpose, let $\lambda = \lim_{n\to\infty} \lambda_n$, with $(\lambda_n)_{n\geq 1} \subset \Delta$. Since $(1 - \lambda_n)a + \lambda_n b \in \operatorname{dom} f$, one has, $\forall \; n \in \mathbb{N}$,

$$\begin{aligned} f((1 - \lambda_n)a + \lambda_n b) &= f^{**}((1 - \lambda_n)a + \lambda_n b) \\ &\leq (1 - \lambda_n)f^{**}(a) + \lambda_n f^{**}(b) < +\infty. \end{aligned}$$

Since f is lsc on the segment $[a,b]$, we get

$$f((1-\lambda)a+\lambda b) \leq (1-\lambda)f^{**}(a)+\lambda f^{**}(b) < +\infty,$$

and consequently, $\lambda \in \Delta$. Therefore, Δ is closed, and since $1 \notin \Delta$, there will exist $c \in [a,b[$ such that $[c,b] \cap \operatorname{dom} f = \{c\}$, and so, $f + i_{[c,b]} = f(c) + i_{\{c\}}$. By assumption we thus have

$$f(c) + i_{\{c\}} = (f + i_{[c,b]})^{**} = f^{**} + i_{[c,b]}.$$

Consequently, $f^{**}(b) = +\infty$, which is impossible. So, $\operatorname{dom} f^{**} = \operatorname{dom} f$, and finally, $f = f^{**}$. ∎

Remark 1. When $f : X \to \mathbb{R} \cup \{+\infty\}$ is lsc (or weakly lsc), the equality $f(x) = f^{**}(x)$ must hold at some particular points. More precisely, it is proved in [31, Theorem 2.1] that if x is the Fréchet (or Gâteaux) derivative point of the conjugate function f^*, then $f(x) = f^{**}(x)$.

We now give one more relevant geometrical characterization of condition (a) in Theorem 1.

PROPOSITION 3. *For any $F : U \times X \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent.*

(a) $F^{**}(0,\cdot) = (F(0,\cdot))^{**}$ *and it is proper.*

(b) $\emptyset \neq \operatorname{epi}(F(0,\cdot))^* = \operatorname{cl} \bigcup_{u^* \in U^*} \operatorname{epi} F^*(u^*,\cdot) \neq X^* \times \mathbb{R}$.

Proof. Let us introduce the following marginal dual function:

$$\gamma(x^*) = \inf_{u^* \in U^*} F^*(u^*, x^*), \qquad x^* \in X^*,$$

which is convex [37, Theorem 2.1.3(v)]. Denoting by $\overline{\gamma}$ the w^*-lsc hull of γ, it is well known that

$$\operatorname{epi} \overline{\gamma} = \operatorname{cl} \bigcup_{u^* \in U^*} \operatorname{epi}\ F^*(u^*,\cdot), \tag{3.7}$$

and also that [37, Theorem 2.6.1(i)]

$$\gamma^* = F^{**}(0,\cdot). \tag{3.8}$$

Assume that (a) holds. Then by (3.8) γ^* is proper, and so, $\overline{\gamma} = \gamma^{**}$. Using (3.8) again, we get from (a)

$$\overline{\gamma} = \gamma^{**} = (F(0,\cdot))^{***} = (F(0,\cdot))^*,$$

which yields the properness of $(F(0,\cdot))^*$, and thanks to (3.7), we obtain (b).

Assume now that (b) holds. By (3.7) we conclude that $\overline{\gamma} = (F(0,\cdot))^*$ and $\overline{\gamma}$ is proper. Since $\overline{\gamma} = \gamma^{**}$, we have $\gamma^{**} = (F(0,\cdot))^*$, and hence, $\gamma^* = \gamma^{***} = (F(0,\cdot))^{**}$. Combining this and (3.8), we get $(F(0,\cdot))^{**} = F^{**}(0,\cdot)$ and the properness of this function as well. ∎

Remark 2. It is worth giving here some observations on the assumptions of Proposition 3.

(i) The statement (a) in Proposition 3 is equivalent to

(a′) $F(0,\cdot)$ is proper, $\operatorname{dom} F^* \neq \emptyset$, and $F^{**}(0,\cdot) = (F(0,\cdot))^{**}$.

(ii) The statement (b) in Proposition 3 holds in particular when F is a proper convex and lsc function such that $0 \in P_U(\mathrm{dom} F)$, where P_U denotes the projection of $U \times X$ onto U, since in this case $F^{**}(0,\cdot) = (F(0,\cdot))^{**} = F(0,\cdot)$ and $F(0,\cdot)$ is proper (see [3, Theorem 2]).

As the following examples illustrate, one easily realizes that the class of mappings F satisfying condition (a) of Theorem 1 goes far beyond $\Gamma(U \times X)$. At the same time, these examples show how to check that condition (a) holds in particular problems.

Example 1. Given a function $f : U \to \mathbb{R} \cup \{+\infty\}$ and a linear continuous map $A : X \to U$, whose adjoint operator is denoted by A^*, let us consider

$$F(u,x) := f(u + Ax), \qquad (u,x) \in U \times X.$$

We thus have

$$F^*(u^*,x^*) = \begin{cases} f^*(u^*) & \text{if } A^*u^* = x^*, \\ +\infty, & \text{otherwise,} \end{cases} \qquad (u^*,x^*) \in U^* \times X^*,$$

and

$$F^{**}(u,x) = f^{**}(u + Ax), \qquad (u,x) \in U \times X.$$

Assuming that $F(0,\cdot) = f \circ A$ is proper, that $(\mathrm{dom}\, f^*) \cap A^*(U^*) \neq \emptyset$, *and that*

$$(F(0,\cdot))^{**} = (f \circ A)^{**} = f^{**} \circ A = F^{**}(0,\cdot),$$

we are in position to apply Theorem 1 with f possibly nonconvex. In such a way we get that for any $h \in \Gamma(X)$,

$$f \circ A \geq h \iff \left\{ \begin{array}{l} \forall x^* \in \mathrm{dom}\, h^*, \text{ there exists a net} \\ (u_i^*, \varepsilon_i)_{i \in I} \subset U^* \times \mathbb{R} \text{ such that} \\ f^*(u_i^*) \leq h^*(x^*) + \varepsilon_i \ \forall i \in I, \\ \text{and } \lim_{i \in I}(A^* u_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

The case when A is an homeomorphism (regular) is of particular interest as the relation $(f \circ A)^{**} = f^{**} \circ A$ holds for any function $f : U \to \mathbb{R} \cup \{+\infty\}$. This is the case when $U = X$ and A is the identity map.

Example 2. Given $f : X \times X \to \mathbb{R} \cup \{+\infty\}$, $a : X \to \mathbb{R} \cup \{+\infty\}$, $b \in \Gamma(X)$, and $K \subset X$, let us consider the following problem, which may be considered an extension of many equilibrium problems:

(P) Find $\overline{x} \in K \cap \mathrm{dom}\, a \cap \mathrm{dom}\, b$ such that $f(\overline{x}, x) + a(x) \geq b(x) + a(\overline{x}) - b(\overline{x})$ $\forall x \in K$.

Problem (P) covers, in particular, the class of generalized equilibrium problems studied in [11].

In order to formulate a dual expression for (P) via Theorem 1, we introduce the following perturbation function associated with $\overline{x} \in K$:

$$F(u,x) := f_{\overline{x}}(x) + (a + i_K)(u + x), \qquad (u,x) \in X \times X,$$

where $f_{\overline{x}} := f(\overline{x}, \cdot)$. One has

$$F^*(u^*,x^*) = (f_{\overline{x}})^*(x^* - u^*) + (a + i_K)^*(u^*), \qquad (u^*,x^*) \in X^* \times X^*,$$

and

$$F^{**}(u,x) = (f_{\overline{x}})^{**}(x) + (a+i_K)^{**}(u+x), \qquad (u,x) \in X \times X.$$

Let us assume that, for every $\overline{x} \in K$, the following conditions hold.

(i) $(\operatorname{dom} f(\overline{x},\cdot)) \cap (\operatorname{dom} a) \cap K \neq \emptyset$; i.e., $F(0,\cdot)$ is proper.

(ii) $\operatorname{dom}(f_{\overline{x}})^* \neq \emptyset$, and $\operatorname{dom}(a+i_K)^* \neq \emptyset$ or, equivalently, $\operatorname{dom} F^* \neq \emptyset$.

(iii) $(f_{\overline{x}})^{**} + (a+i_K)^{**} = (f_{\overline{x}} + a + i_K)^{**}$; i.e., $F^{**}(0,\cdot) = (F(0,\cdot))^{**}$.

Observe that condition (iii) is satisfied in particular when $a \in \Gamma(X)$, K is a closed convex set, and $f(\overline{x},\cdot) \in \Gamma(X)\ \forall\, \overline{x} \in K$, a situation which covers the class of classical variational inequalities.

If we apply Theorem 1 to problem (P), we get that $\overline{x} \in K$ is a solution of (P) if and only if

$$\left\{ \begin{array}{l} \forall x^* \in \operatorname{dom} b^*, \text{ there exists a net} \\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset X^* \times X^* \times \mathbb{R} \text{ such that} \\ (f_{\overline{x}})^*(x_i^* - u_i^*) + (a+i_K)^*(u_i^*) + a(\overline{x}) \leq b^*(x^*) + b(\overline{x}) + \varepsilon_i\ \forall i \in I, \\ \text{and } \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{array} \right\}$$

Example 2 paves the way to apply Theorem 1 to equilibrium problems, and this will be done in a forthcoming paper.

A striking application of Theorem 1 is the following formula of subdifferential calculus that extends [37, Theorem 2.6.3]. Here P_{X^*} denotes the projection of $U^* \times X^*$ onto X^*.

THEOREM 2. *For any* $F : U \times X \to \mathbb{R} \cup \{+\infty\}$ *satisfying*

$$F^{**}(0,.) = (F(0,.))^{**}, \tag{3.9}$$

one has

$$\partial F(0,.)(\overline{x}) = \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0,\overline{x})\ \forall \overline{x} \in X.$$

Proof. We begin with the proof of the inclusion "$\supset$." Let $\overline{x} \in X$ and $x^* \in \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0,\overline{x})$. Then there will exist a net $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0,\overline{x})\ \forall i \in I, \text{ and } \lim_{i\in I}\,(x_i^*, \varepsilon_i) = (x^*, 0_+).$$

We thus have

$$F(u,x) - F(0,\overline{x}) \geq \langle u_i^*, u \rangle + \langle x_i^*, x - \overline{x} \rangle - \varepsilon_i\ \forall (i,u,x) \in I \times U \times X,$$

and, in particular,

$$F(0,x) - F(0,\overline{x}) \geq \langle x_i^*, x - \overline{x} \rangle - \varepsilon_i\ \forall (i,x) \in I \times X.$$

Passing to the limit on i for each fixed $x \in X$, we get

$$F(0,x) - F(0,\overline{x}) \geq \langle x^*, x - \overline{x} \rangle\ \forall x \in X;$$

that is, $x^* \in \partial F(0,.)(\overline{x})$.

We prove now the reverse inclusion "$\subset$." Let $\overline{x} \in X$ and $x^* \in \partial F(0,.)(\overline{x})$. This entails $F(0, \overline{x}) \in \mathbb{R}$, $F(0,.)$ is proper, and (3.9), together with [37, Theorem 2.4.1(ii)], yields

$$F^{**}(0, \overline{x}) = (F(0,.))^{**}(\overline{x}) = F(0,.)(\overline{x}) \equiv F(0, \overline{x}) \in \mathbb{R},$$

which entails that F^* is proper, and so, $\operatorname{dom} F^* \neq \emptyset$ (otherwise, $F^* \equiv +\infty$ and $F^{**} = -\infty$). The inclusion now readily follows from Theorem 1 with $h \in \Gamma(X)$ being the affine continuous mapping defined as follows:

$$h(x) := \langle x^*, x - \overline{x} \rangle + F(0, \overline{x}) \ \forall x \in X.$$

Indeed, since $x^* \in \partial F(0,.)(\overline{x})$, we have

$$F(0,.) \geq h,$$

and, by Theorem 1, there exists a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset U^* \times X^* \times \mathbb{R}$ such that

$$F^*(u_i^*, x_i^*) \leq \langle x^*, \overline{x} \rangle - F(0, \overline{x}) + \varepsilon_i \ \forall i \in I,$$

and $(x_i^*, \varepsilon_i) \to (x^*, 0_+)$. According to this,

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0, \overline{x}), \text{ and } (x_i^*, \varepsilon_i) \to (x^*, 0_+),$$

which means

$$x^* \in \limsup_{\varepsilon \to 0_+} P_{X^*} \partial_\varepsilon F(0, \overline{x}). \qquad \square$$

From Theorem 2 we obtain the following extension of the Hiriart–Urruty and Phelps formula [17, Corollary 2.1] and of Theorem 13 in [14]. See also [25, Theorem 4] for another approach of this result.

Proposition 4 (subdifferential of the sum). *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be a couple of functions satisfying*

$$(f+g)^{**} = f^{**} + g^{**}. \tag{3.10}$$

Then, for any $\overline{x} \in X$,

$$\partial(f+g)(\overline{x}) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})\right).$$

Proof. The inclusion "$\supset$" always holds, and it is not difficult to be proved. So, we only have to prove the reverse inclusion "$\subset$". Let $\overline{x} \in X$ and $x^* \in \partial(f+g)(\overline{x})$. Setting

$$F(u, x) := f(u + x) + g(x), \qquad (u, x) \in X^2,$$

we get

$$F(0,.) = f + g. \tag{3.11}$$

Since $\partial(f+g)(\overline{x}) \neq \emptyset$, one has by (3.10)

$$f^{**}(\overline{x}) + g^{**}(\overline{x}) = (f+g)^{**}(\overline{x}) = f(\overline{x}) + g(\overline{x}) \in \mathbb{R}.$$

It follows easily that all the functions f^*, g^*, f^{**}, and g^{**} are proper. We have then, straightforwardly,

$$F^*(u^*, x^*) = f^*(u^*) + g^*(x^* - u^*), \qquad (u^*, x^*) \in (X^*)^2, \tag{3.12}$$

$$F^{**}(u, x) = f^{**}(u + x) + g^{**}(x), \qquad (u, x) \in X^2, \tag{3.13}$$

and so, by (3.10), (3.11), and (3.13), we have $F^{**}(0, .) = (F(0, .))^{**}$. Since $x^* \in \partial F(0, .)(\overline{x})$, we can thus apply Theorem 2 to conclude the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset (X^*)^2 \times \mathbb{R}$ such that

$$(u_i^*, x_i^*) \in \partial_{\varepsilon_i} F(0, \overline{x}), \text{ and } (x_i^*, \varepsilon_i) \to (x^*, 0_+). \tag{3.14}$$

By (3.12) and (3.14), one has

$$[f^*(u_i^*) + f(\overline{x}) - \langle u_i^*, \overline{x}\rangle] + [g^*(x_i^* - u_i^*) + g(\overline{x}) - \langle x_i^* - u_i^*, \overline{x}\rangle] \leq \varepsilon_i \ \forall i \in I.$$

Since both expressions in the brackets are nonnegative (by the Fenchel inequality), each of them is less or equal to ε_i. We thus have $u_i^* \in \partial_{\varepsilon_i} f(\overline{x})$, and $x_i^* - u_i^* \in \partial_{\varepsilon_i} g(\overline{x}) \ \forall \ i \in I$; so,

$$x^* = \lim_{i\in I}(u_i^* + x_i^* - u_i^*) \in \limsup_{\varepsilon\to 0_+} (\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})) = \bigcap_{\varepsilon>0} \operatorname{cl}(\partial_\varepsilon f(\overline{x}) + \partial_\varepsilon g(\overline{x})). \qquad \square$$

Remark 3. It is worth observing that if $f, g \in \Gamma(X)$, then

$$(f + g)^{**} = f + g = f^{**} + g^{**}.$$

Thus Proposition 4 is a nonconvex version of [17, Corollary 2.1].

4. The generalized Farkas lemma for nonconvex systems. Farkas-type results (in asymptotic or nonasymptotic form) have been used extensively as one of the main tools for establishing results on optimality, duality, primal and dual stability, etc., for many classes of problems such as cone-constrained convex problems, convex semidefinite problems, convex semi-infinite and infinite problems, DC problems, variational inequalities, second-order cone programming, equilibrium problems, and bilevel convex problems, as well as some models arising from the relaxation of the convexity and lower semicontinuity of the involved functions (see, e.g., [4], [7], [10], [11], [12], [20], [22], and references therein).

This section addresses asymptotic versions of the Farkas lemma for systems without convexity and lower semicontinuity. Here, not only some generalized Farkas lemmas are established, but necessary and sufficient conditions for them are proposed as well. This additional feature, to the best of authors' knowledge, is new, even for convex simplex cases (see [22]).

Given $H : \operatorname{dom} H \subset X \to U$ and $g : U \to \mathbb{R} \cup \{+\infty\}$, we set

$$(g \circ H)(x) = \begin{cases} g(H(x)) & \text{if } x \in \operatorname{dom} H, \\ +\infty & \text{if } x \in X \setminus \operatorname{dom} H. \end{cases}$$

We consider a cone $S \subset U$ (i.e., $u \in S$ and $\alpha > 0$ imply $\alpha u \in S$) whose nonnegative polar cone is defined by S^+:

$$S^+ := \{u^* \in U^* : \ \langle u^*, u\rangle \geq 0 \ \forall u \in S\}.$$

In contrast with [8], neither lower semicontinuity nor convexity are required for the mapping $u^* \circ H$, with $u^* \in S^+$.

As a consequence of Theorem 1, we get the following versions of the Farkas lemma for nonconvex systems.

PROPOSITION 5 (the Farkas lemma for nonconvex systems I). *Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $H : \operatorname{dom} H \subset X \to U$, and S a cone in U. Assume that the two following conditions hold:*

$$(\operatorname{dom} f) \cap C \cap H^{-1}(-S) \neq \emptyset, \tag{4.1}$$

$$\begin{aligned}&\text{there exists } (u_0^*, x_0^*, \eta_0) \in S^+ \times X^* \times \mathbb{R} \text{ such that}\\ &f(x) + (u_0^* \circ H)(x) \geq \langle x_0^*, x\rangle - \eta_0 \;\forall x \in C.\end{aligned} \tag{4.2}$$

Then the following statements are equivalent.

(a) $(f + i_C + i_{-S} \circ H)^{**} = \sup_{u^* \in S^+} (f + i_C + u^* \circ H)^{**}$.

(b) *For any $h \in \Gamma(X)$, we have $(\alpha) \Leftrightarrow (\beta)$, where*

$$(\alpha) \qquad C \cap H^{-1}(-S) \subset [f - h \geq 0],$$

and

$$(\beta) \quad \left\{\begin{array}{l}\forall x^* \in \operatorname{dom} h^*, \text{ there exists a net}\\ (u_i^*, x_i^*, \varepsilon_i)_{i\in I} \subset S^+ \times X^* \times \mathbb{R}\\ \text{such that } \left\{\begin{array}{l}(f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) + \varepsilon_i \;\forall i \in I,\\ \text{and } \lim_{i\in I}(x_i^*, \varepsilon_i) = (x^*, 0_+).\end{array}\right.\end{array}\right.$$

Proof. Define $g = f + i_C$ and

$$F(u, x) := g(x) + i_{-S}(H(x) + u),\; (u, x) \in U \times X.$$

(According to our convention, if $x \notin \operatorname{dom} H$, $i_{-S}(H(x) + u) = +\infty \;\forall u \in U$.)

Observe that $F(0, .) = g + i_{-S} \circ H$. Since S is a cone, we get easily

$$F^*(u^*, x^*) = \begin{cases}(g + u^* \circ H)^*(x^*) & \text{if } u^* \in S^+,\\ +\infty & \text{otherwise,}\end{cases} \tag{4.3}$$

and so,

$$F^{**}(0, \cdot) = \sup_{u^* \in S^+} (g + u^* \circ H)^{**}.$$

By (4.1) $F(0, .)$ is proper. By (4.2) and (4.3) one has $\operatorname{dom} F^* \neq \emptyset$. Thus the equivalence between (a) and (b) follows directly from Theorem 1. ▯

Let us now specify a standard situation in which the condition (a) in Proposition 5 is satisfied. To this end one needs the following lemma whose proof can be obtained by standard arguments in convex analysis and, hence, will be omitted.

LEMMA 1. *Assume that the cone $S \subset U$ is closed and convex. Then for any map $H : \operatorname{dom} H \subset X \to U$, one has*

$$i_{-S} \circ H = \sup_{u^* \in S^+} u^* \circ H.$$

Remark 4. From Lemma 1, it easily follows that the condition (a) in Proposition 5 is, in particular, satisfied whenever S is a closed convex cone and

$$(f + i_C + u^* \circ H) \in \Gamma(X) \;\forall u^* \in S^+.$$

PROPOSITION 6 (the Farkas lemma for nonconvex systems II). *Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $H : \operatorname{dom} H \subset X \to Z$, and S is a cone in Z. Assume that* (4.1) *holds together with*

$$\begin{aligned} &\text{there exists } (u_0^*, y_0^*, t_0^*, x_0^*, \eta_0) \in S^+ \times (X^*)^3 \times \mathbb{R} \text{ such that} \\ &f(y) + (u_0^* \circ H)(x) \geq \langle y_0^*, y\rangle + \langle t_0^*, t\rangle + \langle x_0^* - y_0^* - t_0^*, x\rangle - \eta_0 \\ &\qquad \forall (y,t,x) \in X \times C \times \operatorname{dom} H. \end{aligned} \tag{4.4}$$

Then the following statements are equivalent.

(c) $(f + i_C + i_{-S} \circ H)^{**} = f^{**} + i_{\overline{\text{co}}C} + \sup_{u^* \in S^+} (u^* \circ H)^{**}$.

(d) *For any $h \in \Gamma(X)$, one has $(\gamma) \Leftrightarrow (\delta)$, where*

(γ) $\quad C \cap H^{-1}(-S) \subset [f - h \geq 0]$,

and

$$(\delta) \begin{cases} \forall x^* \in \operatorname{dom} h^*, \textit{ there exists a net} \\ (u_i^*, y_i^*, t_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times (X^*)^3 \times \mathbb{R} \textit{ such that} \\ \begin{cases} f^*(y_i^*) + i_C^*(t_i^*) + (u_i^* \circ H)^*(x_i^* - y_i^* - t_i^*) \leq h^*(x^*) + \varepsilon_i \ \forall i \in I, \\ \textit{and } \lim_{i \in I}(x_i^*, \varepsilon_i) = (x^*, 0_+). \end{cases} \end{cases}$$

Proof. Define now $F : U \times X \to \mathbb{R} \cup \{+\infty\}$, with $U = Z \times X^2$ and

$$F(u, y, t, x) := f(x + y) + i_C(x + t) + i_{-S}(H(x) + u), \qquad (u, y, t, x) \in U \times X^3.$$

(According to our convention, if $x \notin \operatorname{dom} H$, $F(u, y, t, x) = +\infty$.)

Observe that

$$F(0, 0, 0, \cdot) = f + i_C + i_{-S} \circ H.$$

Since S is a cone, a straightforward computation leads us to

$$F^*(u^*, y^*, t^*, x^*) = \begin{cases} f^*(y^*) + i_C^*(t^*) + (u^* \circ H)^*(x^* - y^* - t^*) \\ \text{if } (u^*, y^*, t^*, x^*) \in S^+ \times (X^*)^3, \\ +\infty, \ \text{otherwise}, \end{cases} \tag{4.5}$$

and so,

$$F^{**}(0, 0, 0, \cdot) = f^{**} + i_{\overline{\text{co}}C} + \sup_{u^* \in S^+} (u^* \circ H)^{**}.$$

By (4.1) $F(0, 0, 0, \cdot)$ is proper. By (4.4) and (4.5) one has $\operatorname{dom} F^* \neq \emptyset$. Thus the equivalence between (c) and (d) follows directly from Theorem 1. ∎

Remark 5. Propositions 5 and 6 establish necessary and sufficient conditions for the Farkas lemma in asymptotic forms, and they are new (even for convex data), to the knowledge of the authors. These types of conditions for the nonasymptotic form and for the convex, lsc systems without set constraint (i.e., where $h \equiv 0$, $C = X$) were proposed recently in [22].

COROLLARY 1 (see [8, Theorem 3]). *Let $f, h \in \Gamma(X)$, C be a closed convex set in X, S be a closed convex cone in Z, and $H : X \to Z$ be a mapping. Assume that* (4.1) *holds together with*

$$u^* \circ H \in \Gamma(X) \ \forall u^* \in S^+. \tag{4.6}$$

Then the statements (γ) *and* (δ) *in Proposition* 6 *are again equivalent.*

Proof. By Lemma 1 one has

$$i_{-S} \circ H = \sup_{u^* \in S^+} u^* \circ H.$$

By (4.6) we get $i_{-S} \circ H \in \Gamma(X)$ (recall that $H^{-1}(-S) \neq \emptyset$). Since $f \in \Gamma(X)$ and C is closed and convex, condition (4.4) holds. To see this, we can simply take $u_0^* = t_0^* = 0$, $y_0^* \in \operatorname{dom} f^*$, $x_0^* = y_0^*$, and $\eta_0 = f^*(y_0^*)$. It is easy to see that the condition (c) in Proposition 6 holds, too. Consequently, the statement (d) in Proposition 6 is true, and this is precisely what Corollary 1 says. ∎

Remark 6. When H is S-convex; i.e., when

$$H(\lambda x + (1-\lambda)y) - \lambda H(x) - (1-\lambda)H(y) \in -S \ \forall x, y \in X \ \forall \lambda \in [0,1],$$

the condition (4.6) is satisfied if H is lsc in the following sense (see [29]):

$$\forall x \in X \text{ and } \forall V \in \mathcal{N}(H(x)) \text{ there exists } W \in \mathcal{N}(x) \text{ subject to } H(W) \subset V + S,$$

where $\mathcal{N}(y)$ denotes a neighborhood basis of y.

5. Nonconvex optimization problems. Optimality and duality. We consider the nonconvex optimization problem

$$\text{(P)} \qquad \text{minimize } [f(x) - h(x)] \text{ subject to } x \in C \text{ and } H(x) \in -S,$$

where $f, h : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, S is a cone in U, and $H : \operatorname{dom} H \subset X \to U$.

PROPOSITION 7 (optimality condition for (P)). *Consider* $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, $H : \operatorname{dom} H \subset X \to U$, *and* S *is a cone in* U. *Assume that* (4.2) *holds together with*

$$(f + i_C + i_{-S} \circ H)^{**} = \sup_{u^* \in S^+} (f + i_C + u^* \circ H)^{**}. \tag{5.1}$$

Then for each $h \in \Gamma(X)$ *and any* $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$, *the following statements are equivalent.*

(a) *a is a global optimal solution of* (P).

(b) $\forall x^* \in \operatorname{dom} h^*$, *there exists a net* $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times X^* \times \mathbb{R}$ *such that*

$$(f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i \ \forall i \in I,$$

and

$$\lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+).$$

Proof. This is a straightforward consequence of Proposition 5. Indeed, $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$ is a global optimal solution of (P) if and only if

$$x \in C, \qquad H(x) \in -S \implies f(x) - [h(x) + f(a) - h(a)] \geq 0,$$

and this happens if and only if the statement (α) in Proposition 5 holds with $\tilde{h}$, defined as $\tilde{h}(x) := h(x) + f(a) - h(a)$, instead of h. The conclusion follows from Proposition 5, taking into account the fact that $\tilde{h}^*(x^*) = h^*(x^*) - f(a) + h(a)$. ∎

The following optimality condition is a consequence of Proposition 6. The proof follows the same line as that of Proposition 7 and, therefore, it will be omitted.

PROPOSITION 8 (optimality condition for (P)). *Consider $f : X \to \mathbb{R} \cup \{+\infty\}$, $C \subset X$, S is a cone in U, and $H : \operatorname{dom} H \subset X \to U$. Assume that* (4.4) *holds together with*

$$(f + i_C + i_{-S} \circ H)^{**} = f^{**} + i_{\overline{\text{co}}C} + \sup_{u^* \in S^+} (u^* \circ H)^{**}. \tag{5.2}$$

Then for each $h \in \Gamma(X)$ and $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent.

(a) *a is a global optimal solution of* (P).

(b) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(u_i^*, y_i^*, t_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times (X^*)^3 \times \mathbb{R}$ such that*

$$f^*(y_i^*) + i_C^*(t_i^*) + (u_i^* \circ H)^*(x_i^* - y_i^* - t_i^*) \leq h^*(x^*) + h(a) - f(a) + \varepsilon_i \ \forall i \in I,$$

and

$$\lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+).$$

COROLLARY 2 (see [8, Proposition 2]). *Let $f, h \in \Gamma(X)$, C be a closed convex set in X, S be a closed convex cone in U, and $H : X \to U$ be a mapping. Assume additionally that* (4.6) *holds. Then for each $a \in C \cap H^{-1}(-S) \cap \operatorname{dom} f \cap \operatorname{dom} h$, the statements* (a) *and* (b) *in Proposition* 8 *are equivalent.*

Proof. By Lemma 1 and (4.6) one has

$$i_{-S} \circ H = \sup_{u^* \in S^+} u^* \circ H \in \Gamma(X)$$

(recall that $H^{-1}(-S) \neq \emptyset$ as $a \in H^{-1}(-S)$). Since $f \in \Gamma(X)$ and C is closed and convex, conditions (4.4) in Proposition 6 and (5.2) in Proposition 8 hold (see the proof of Corollary 1). Therefore, statements (a) and (b) in Proposition 8 are equivalent. ∎

PROPOSITION 9 (duality theorem for (P)). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, $h \in \Gamma(X)$, $C \subset X$, $S \subset U$, and $H : \operatorname{dom} H \subset X \to U$ be as in Proposition* 7 *(i.e., satisfying* (4.2) *and* (5.1)*). Moreover, assume that $\alpha := \inf(\text{P}) \in \mathbb{R}$. Then it holds that*

$$\inf(\text{P}) = \inf_{x^* \in \operatorname{dom} h^*} \sup_{\substack{(u_i^*)_{i \in I} \subset S^+ \\ (x_i^*)_{i \in I} \subset X^* \\ x_i^* \to x^*}} \left[h^*(x^*) - \limsup_{i \in I} (f + i_C + u_i^* \circ H)^*(x_i^*) \right]. \tag{5.3}$$

Proof. We begin with the inequality [$\leq$]. Take $x^* \in \operatorname{dom} h^*$ and observe that

$$x \in C, \qquad H(x) \in -S \ \Rightarrow \ f(x) - [h(x) + \alpha] \geq 0.$$

By Proposition 5, with $\tilde{h}(x) := h(x) + \alpha$ playing the role of h, the previous inequality implies the existence of a net $(u_i^*, x_i^*, \varepsilon_i)_{i \in I} \subset S^+ \times X^* \times \mathbb{R}$ such that

$$(f + i_C + u_i^* \circ H)^*(x_i^*) \leq h^*(x^*) - \alpha + \varepsilon_i \ \forall i \in I,$$

and

$$\lim_{i \in I} (x_i^*, \varepsilon_i) = (x^*, 0_+),$$

which in fact entails

$$\limsup_{i\in I}(f+i_C+u_i^*\circ H)^*(x_i^*)\leq h^*(x^*)-\alpha,$$

and thus

$$\inf(\mathrm{P})\leq \sup_{\substack{(u_i^*,x_i^*)_{i\in I}\subset S^+\times X^*\\ x_i^*\to x^*}}\left\{h^*(x^*)-\limsup_{i\in I}(f+i_C+u_i^*\circ H)^*(x_i^*)\right\}$$

$\forall\, x^*\in \operatorname{dom} h^*$, so the inequality [$\leq$] in (5.3) holds.

We now prove the inequality [$\geq$] in (5.3). If $x^*\in\operatorname{dom} h^*$ for any net $(u_i^*,x_i^*)_{i\in I}\subset S^+\times X^*$ such that $x_i^*\to x^*$, one has

$$(f+i_C+u_i^*\circ H)^*(x_i^*)\geq \langle x_i^*,x\rangle - f(x)-\langle u_i^*,H(x)\rangle \ \ \forall i\in I,\forall x\in C\cap \operatorname{dom} H,$$

and since $(u_i^*)_{i\in I}\subset S^+$,

$$(f+i_C+u_i^*\circ H)^*(x_i^*)\geq \langle x_i^*,x\rangle - f(x)\ \forall i\in I,\forall x\in C\cap H^{-1}(-S).$$

It follows then that $\forall i\in I$ and $\forall x\in C\cap H^{-1}(-S)$,

$$h^*(x^*)-\limsup_{i\in I}(f+i_C+u_i^*\circ H)^*(x_i^*)\leq h^*(x^*)-\langle x^*,x\rangle+f(x),$$

and so,

$$\begin{aligned}&\sup_{\substack{(u_i^*,x_i^*)_{i\in I}\subset S^+\times X^*\\ x_i^*\to x^*}}\left\{h^*(x^*)-\limsup_{i\in I}(f+i_C+u_i^*\circ H)^*(x_i^*)\right\}\\ &\quad\leq h^*(x^*)-\langle x^*,x\rangle+f(x)\ \forall i\in I,\forall x\in C\cap H^{-1}(-S).\end{aligned}$$

Now, since x^* is an arbitrary element of $\operatorname{dom} h^*$, we get by taking the infimum on $x^*\in\operatorname{dom} h^*$ in the last inequality, that the right-hand side of (5.3) is less or equal to

$$f(x)-h^{**}(x)=f(x)-h(x)\ \forall x\in C\cap H^{-1}(-S)$$

so that, finally, the inequality [$\geq$] in (5.3) holds. ∎

Now we derive from (5.3) another duality formula for (P) in which we denote by

$$L(u^*,x):=f(x)+(u^*\circ H)(x),\qquad (u^*,x)\in S^+\times X$$

the *Lagrange function* associated with f and H.

COROLLARY 3. *With the same assumptions as in Proposition* 9, *one also has*

$$\inf(\mathrm{P})=\inf_{x^*\in\operatorname{dom} h^*}\ \sup_{(u_i^*)_{i\in I}\subset S^+}\ \inf_{x\in C}\left\{h^*(x^*)-\langle x^*,x\rangle+\liminf_{i\in I} L(u_i^*,x)\right\}.$$

Proof. By (5.3) one easily gets

$$\inf(\mathrm{P})\leq\inf_{x^*\in\operatorname{dom} h^*}\ \sup_{\substack{(u_i^*,x_i^*)_{i\in I}\subset S^+\times X^*\\ x_i^*\to x^*}}\ \inf_{x\in C}\left\{h^*(x^*)+\liminf_{i\in I}(L(u_i^*,x)-\langle x_i^*,x\rangle)\right\}.$$

Since $x_i^* \to x^*$, one has

$$\liminf_{i\in I}(L(u_i^*, x) - \langle x_i^*, x\rangle) = \left(\liminf_{i\in I} L(u_i^*, x)\right) - \langle x^*, x\rangle\,,$$

and so,

$$\inf(\mathrm{P}) \leq \inf_{x^*\in \operatorname{dom} h^*} \sup_{(u_i^*)_{i\in I}\subset S^+} \inf_{x\in C} \left\{h^*(x^*) - \langle x_i^*, x\rangle + \liminf_{i\in I} L(u_i^*, x)\right\} =: \beta.$$

In order to prove the opposite inequality, we have to check that for every $\overline{x} \in C \cap H^{-1}(-S)$,

$$\begin{aligned} f(\overline{x}) - h(\overline{x}) &= f(\overline{x}) - h^{**}(\overline{x}) \\ &= \inf_{x^*\in \operatorname{dom} h^*} \{f(\overline{x}) + h^*(x^*) - \langle x^*, \overline{x}\rangle\} \\ &\geq \beta, \end{aligned}$$

and this happens if, for every $\overline{x} \in C \cap H^{-1}(-S)$ and every $\overline{x}^* \in \operatorname{dom} h^*$, we have

$$f(\overline{x}) + h^*(\overline{x}^*) - \langle \overline{x}^*, \overline{x}\rangle \geq \beta.$$

In fact, we have

$$\begin{aligned} \beta &\leq \sup_{(u_i^*)_{i\in I}\subset S^+} \inf_{x\in C} \left\{h^*(\overline{x}^*) - \langle \overline{x}^*, x\rangle + \liminf_{i\in I} L(u_i^*, x)\right\} \\ &\leq \sup_{(u_i^*)_{i\in I}\subset S^+} \left\{h^*(\overline{x}^*) - \langle \overline{x}^*, \overline{x}\rangle + \liminf_{i\in I} L(u_i^*, \overline{x})\right\}, \end{aligned}$$

and since $(u_i^*, \overline{x})_{i\in I} \subset S^+ \times H^{-1}(-S)$, one has

$$L(u_i^*, \overline{x}) = f(\overline{x}) + (u_i^* \circ H)(\overline{x}) \leq f(\overline{x})$$

so that we are done. ☐

Corollary 4 (see [8, Proposition 7], [9]). *Assume that $f \in \Gamma(X)$, C is a closed convex set in X, S is a closed convex cone in U, $H : X \to U$ satisfies* (4.6)*, and* $(\operatorname{dom} f) \cap C \cap H^{-1}(-S) \neq \emptyset$. *Then*

$$\begin{aligned} \inf_{x\in C\cap H^{-1}(-S)} f(x) &= \sup_{(u_i^*)_{i\in I}\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(u_i^*, x) \\ &= \inf_{x\in C} \sup_{(u_i^*)_{i\in I}\subset S^+} \liminf_{i\in I} L(u_i^*, x). \end{aligned}$$

Proof. Since $L(u_i^*, x) := f(x) + (u_i^* \circ H)(x) \leq f(x)$, for any $(u_i^*, x)_{i\in I} \subset S^+ \times H^{-1}(-S)$, it is easy to see that

$$\inf_{x\in C} \sup_{(u_i^*)_{i\in I}\subset S^+} \liminf_{i\in I} L(u_i^*, x) \leq \inf_{x\in C\cap H^{-1}(-S)} f(x).$$

Observe also that

$$\alpha := \inf_{x\in C\cap H^{-1}(-S)} f(x) \leq \sup_{(u_i^*)_{i\in I}\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(u_i^*, x).$$

This is obvious if $\alpha = -\infty$. Note that the assumptions of the corollary imply that (4.2) and (5.1) hold, and so, if $\alpha \in \mathbb{R}$, the last inequality comes from Corollary 3 (applied with $h = 0$) and from the fact that $\alpha < +\infty$.

On the other hand, since

$$\sup_{(u_i^*)_{i\in I}\subset S^+} \inf_{x\in C} \liminf_{i\in I} L(u_i^*, x) \leq \inf_{x\in C} \sup_{(u_i^*)_{i\in I}\subset S^+} \liminf_{i\in I} L(u_i^*, x),$$

we are done. □

By taking $H \equiv 0$ in (P), we get the problem

$$(\mathrm{P}_1) \qquad \text{minimize } [f(x) - h(x)] \text{ subject to } x \in C.$$

So, it is not surprising that the previous results cover, as a special case, the well-known duality for DC problems [34] (see also, [30]). For instance, from Corollary 3 with $H = 0$ and $C = X$, we straightforwardly get that, for any $h \in \Gamma(X)$ and any $f : X \to \mathbb{R} \cup \{+\infty\}$, with f^* proper, one has

$$\inf_{x\in X}\{f(x) - h(x)\} = \inf_{x^*\in X^*}\{h^*(x^*) - f^*(x^*)\}, \tag{5.4}$$

which still holds when f^* is not proper.

According to Proposition 8, we provide next a characterization of the optimal solution set for the problem (P_1).

Proposition 10. *Let $h \in \Gamma(X)$, $C \subset X$, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be such that f^* proper and*

$$(f + i_C)^{**} = f^{**} + i_{\overline{\mathrm{co}}C}. \tag{5.5}$$

Then for any $a \in C \cap \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent.

(a) *a is a global minimum of (P_1).*

(b) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(x_i^*, y_i^*, \varepsilon_i)_{i\in I} \subset (X^*)^2 \times \mathbb{R}$ such that*

$$f^*(y_i^*) + i_C^*(x_i^* - y_i^*) + f(a) \leq h^*(x^*) + h(a) + \varepsilon_i \ \forall i \in I,$$

and

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

Proof. The proof follows from Proposition 8 by taking $H \equiv 0$. □

Remark 7. Condition (5.5) is, in particular, satisfied in the following two important cases.

(i) C is closed and convex, and $f(x) = f^{**}(x)\, \forall x \in C$ (see Proposition 2).

(ii) $C = X$.

We may give examples of a nonconvex function f for which (5.5) holds for every closed and convex set C meeting $\operatorname{dom} f$. This is, for instance, the case of the indicator functions of the rational numbers $\mathbb{Q}$ in the real line $\mathbb{R}$, i.e., when $f = i_{\mathbb{Q}}$ and $C \cap \mathbb{Q} \neq \emptyset$, since $f^* = i_{\{0\}}$, $f^{**} \equiv 0$, and

$$(f + i_C)^{**} = (i_{C\cap\mathbb{Q}})^{**} = i_{\overline{\mathrm{co}}(C\cap\mathbb{Q})} = i_C = f^{**} + i_C.$$

If C is a convex and closed set such that $C \cap \operatorname{dom} f = \emptyset$, it may happen that (5.5) fails. This is the case for $f = i_{\mathbb{Q}}$ and $C := \{c\}$, with $c \in \mathbb{R} \setminus \mathbb{Q}$, since

$$(f + i_C)^{**} \equiv +\infty \neq i_C = f^{**} + i_C.$$

Relative to the case (ii) above, we have the following.

PROPOSITION 11. *Let $h \in \Gamma(X)$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ such that f^* is proper. Then for any $a \in \operatorname{dom} f \cap \operatorname{dom} h$, the following statements are equivalent.*

(a) *a is a global minimum of $f - h$ on X.*

(b) *$\forall x^* \in \operatorname{dom} h^*$,*

$$f^*(x^*) + f(a) \leq h^*(x^*) + h(a).$$

(c) *$\forall x^* \in \operatorname{dom} h^*$, there exists a net $(x_i^*, \varepsilon_i)_{i \in I} \subset X^* \times \mathbb{R}$ such that*

$$f^*(x_i^*) + f(a) \leq h^*(x^*) + h(a) + \varepsilon_i \ \forall i \in I,$$

and

$$(x_i^*, \varepsilon_i) \to (x^*, 0_+).$$

Proof. $[(a) \Rightarrow (b)]$: Let $x^* \in \operatorname{dom} h^*$. For any $x \in X$, it holds that

$$h^*(x^*) + h(a) \geq \langle x^*, x\rangle - h(x) + h(a) \geq \langle x^*, x\rangle - f(x) + f(a),$$

and we get (b) by taking the supremum over $x \in X$.

$[(b) \Rightarrow (c)]$: Take $x_i^* = x^*$, $\varepsilon_i = 0 \ \forall \ i \in I$ (an arbitrary directed set).

$[(c) \Rightarrow (a)]$: Apply Proposition 10 with $C = X$. ∎

Remark 8. The equivalence of (a) and (b) in Proposition 11 also follows from (5.4).

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