JOTA manuscript No.
(will be inserted by the editor)

A Unifying Approach to Robust Convex Infinite Optimization Duality

Nguyen Dinh · Miguel Angel Goberna · Marco Antonio López · Michel Volle

Received: date / Accepted: date

Communicated by Radu Ioan Bot

Abstract This paper considers an uncertain convex optimization problem, posed in a locally convex decision space with an arbitrary number of un-

Nguyen Dinh
International University, Vietnam National University - HCM City
Chi Minh city, Vietnam; ndinh@hcmiu.edu.vn

Miguel Angel Goberna
Department of Mathematics, University of Alicante
Alicante, Spain; mgoberna@ua.es

Marco Antonio López, Corresponding author
Department of Mathematics, University of Alicante
Alicante, Spain; marco.antonio@ua.es; and
CIAO, Federation University, Ballarat, Australia

Michel Volle
Avignon University
LMA EA2151, Avignon, France; michel.volle@univ-avignon.fr

certain constraints. To this problem, where the uncertainty only affects the constraints, we associate a robust (pessimistic) counterpart and several dual problems. The paper provides corresponding dual variational principles for the robust counterpart in terms of the closed convexity of different associated cones.

Keywords Robust convex optimization · Lagrange duality · strong duality · robust strong duality · uniform robust strong duality · robust reverse strong duality.

AMS subject classification 90C25, 46N10, 90C31

1 Introduction

Robust optimization has recently emerged as a useful methodology in the treatment of uncertain optimization problems. In this paper we consider a convex optimization problem posed in a locally convex decision space with an arbitrary number of uncertain constraints. Following the robust approach, we associate to this uncertain problem a deterministic one called robust counterpart ensuring the feasibility of all possible decisions for any conceivable scenario. To this problem we associate five different robust dual problems, two of them already known (the Lagrange dual and the optimistic dual problems), the remaining three dual problems being apparently new in the literature.

The paper provides robust strong duality theorems for the five duality pairs guaranteeing the zero-duality gap with attainment of the dual optimal value which are expressed in terms of the closedness of suitable sets regarding the vertical axis. It also provides corresponding stable robust strong duality theorems guaranteeing robust strong duality for arbitrary linear continuous perturbations of the objective function which are expressed in terms of the closedness and convexity of the above sets. Moreover, the paper gives uniform strong duality theorems guaranteeing the same for the larger class of perturbations of the objective function formed by the proper, lower semicontinuous, and convex functions which are continuous at some robust feasible solution, this time expressed in terms of the closedness and convexity of the so-called robust moment cones. The mentioned duality theorems are specialized in a non-trivial way to uncertain linear optimization problems, obtaining results which are new even for deterministic problems (with singleton uncertainty sets).

We also give reverse strong duality theorems guaranteeing the zero-duality gap together with the solvability of the primal problem which are expressed in terms of the weak-inf-local compactness of certain Lagrange function for particular multipliers, recession conditions on the intersection of sublevel sets of the data functions, and the closedness of certain set associated to the problem regarding the vertical axis.

The mentioned duality theorems are finally applied to a given convex optimization problem with uncertain objective function and (possibly) uncertain constraints by reformulating it as a convex optimization problem with deterministic objective function and uncertain constraints.

2 Background

This paper deals with *uncertain* convex optimization problems of the form

$$(P) \qquad \left\{ \inf_{x\in X} \{f(x) \quad \text{s.t. } g_t(x,u_t) \leq 0, \ \forall t \in T\} \right\}_{(u_t)_{t\in T} \in \prod\limits_{t\in T} U_t}, \quad (1)$$

where X is a locally convex Hausdorff topological space (in brief, lcHtvs), T is a possibly infinite index set, and f and $g_t(\cdot, u_t)$, $t \in T$, $u_t \in U_t$, are convex functions defined on X. We assume that f is deterministic, while the *uncertainty* falls on the constraints in the sense that u_t is not deterministic and belongs to an *uncertainty set* $U_t \subset Z_t$, a lcHtvs depending on t. Additionally, we assume that the functions g_t are real-valued, i.e., $g_t\colon\ X \times U_t \to \mathbb{R}$.

The objective of the paper is to provide duality principles for the *robust counterpart* of (P), which is known to be the problem that *all* the uncertain inequality constraints are satisfied, namely:

$$(RP) \qquad \inf_{x\in X} \{f(x) \quad \text{s.t. } g_t(x,u_t) \leq 0, \ \forall t \in T, \ \forall u_t \in U_t\}. \quad (2)$$

In the formulation of (P) and (RP) we used the notation in [1]. As asserted in [2, p. 472], 'since duality has been shown to play a key role in the tractability of robust optimization, it is natural to ask how duality and robust optimization are connected'. The reaction of the researchers to this question, implicitly posed in 2009 by the seminal paper of Beck and Ben-Tal [3], has been to expand the literature on robust duality in two opposite directions:

1. *Generalization*: getting duality theorems under assumptions, which are as weak as possible for very general robust optimization problems in order to

gain a better understanding of the robust duality phenomena.

2. *Specialization*: getting duality theorems for particular types of uncertain optimization problems in order to ensure the computational tractability of both pessimistic (primal) and optimistic (dual) problems.

Different dual problems can be associated with (RP). Let us denote by (RD) one of these dual problems, which are generically called *robust duals*. *Robust zero duality gap* means the coincidence of their optimal values, i.e., $\inf(RP) = \sup(RD)$. If, additionally, the dual optimal value $\sup(RD)$ is attained, then it is said that *robust strong duality* holds, i.e., $\inf(RP) = \max(RD)$. Analogously, when the primal optimal value $\inf(RP)$ is attained, it is said that *reverse robust strong duality* occurs, i.e., $\min(RP) = \sup(RD)$. Finally, if both problems are solvable and their values coincide, then we say that *robust total duality* holds, i.e., $\min(RP) = \max(RD)$. These desirable properties are said to be *stable* when they are preserved under arbitrary continuous linear perturbations of the objective function f.

With few exceptions (as [4] and [5]), almost any paper on robust duality considers the constraints as a source of uncertainty, in few cases together with the objective function ([3,6]). Most published papers deal with uncertain optimization problems as (1), not necessarily convex.

Defining the *uncertainty set*

$$U := \prod_{t\in T} U_t,$$

and $G : X \times U \longrightarrow \mathbb{R}^T$ such that $G(x,u) := (g_t(x,u_t))_{t\in T} \in \mathbb{R}^T$, for any $x \in X$ and $u \in U$, one can reformulate the robust counterpart of (P) in (2) as the cone constrained problem

$$(\widetilde{RP}) \qquad \inf_{x\in X} f(x) \qquad \text{s.t. } -G(x,u) \in C,\ \forall u \in U,$$

where $C = \mathbb{R}_+^T$ is the positive cone in the product space $\mathbb{R}^T$. Conversely, the robust counterpart of any uncertain cone constrained problem can be reformulated as an uncertain inequality constrained problem of the form

$$(\widehat{RP}) \qquad \inf_{x\in X} f(x) \qquad \text{s.t. } \langle \lambda, G(x,u)\rangle \leq 0,\ \forall \lambda \in C', \forall u \in U,$$

where C' denotes the dual cone to a given closed and convex cone C contained in some lcHtvs.

The works published up to now on robust duality can primarily be classified by the type of constraints of the given uncertain problem, either inequality constraints or conic constraints. Other criteria are the nature of the objective function and the constraints (either ordinary or conic functions).

Table 1 presents a summary of the existing literature on robust optimization problems with uncertain inequality constraints. In almost all references, which are chronologically ordered, the number of variables is finite. We say that a function is *co/co* when it is the quotient of a convex function by a positive concave function, it is *max/co/co* when it is the maximum of finitely many *co/co* functions, it is *sos/co/po* when it is sum-of-squares, and it is *locally* $\mathcal{C}^1$ when it is continuously differentiable on some open set. The codes for the last column, informing about the nature of the duality theorems contained in each paper, are as follows: "zero-gap" stands for the results guaranteeing robust zero duality gap, "strong" means robust strong duality, and "total" for robust zero duality gap with attainment of both problems.

Table 1

Refs.	f	$g_t(\cdot, u_t)$	T	U_t	Dual problem	Ths.
[3]	convex	convex	finite	compact convex	Lagrange	zero-gap
[7]	convex	convex	finite	compact	Lagrange	strong
[8]	co/co	convex	finite	compact convex	Wolfe	total
[9]	locally $\mathcal{C}^1$	locally $\mathcal{C}^1$	finite	compact convex	Lagrange	strong
[10]	linear	affine	infinite	arbitrary	Lagrange	strong
[11]	max/co/co	convex	finite	compact convex	Lagrange	strong
[12]	convex	convex	finite	arbitrary in $\mathbb{R}^q$	Lagrange	zero-gap
[6]	quasiconvex	convex	finite	arbitrary in $\mathbb{R}^q$	surrogate	strong
[13]	linear	affine	finite	compact convex	Dantzig	robust
[14]	sos/co/po	polynomial	finite	finite	Lagrange	total
[15]	$\|\cdot\|^2$	quadratic	finite	ellipsoids	Lagrange	total

The meager literature on robust optimization problems with uncertain cone constraints is compared in Table 2. In all references, the decision space X is a lcHtvs, G is $C-$convex (equivalent to the convexity of g_t for all $t \in T$ in the case of inequality constraints) and the feasible set is the convex set $F = \{x \in S : -G(x, u) \in C, \ \forall u \in \mathcal{U}\}$, where $S \subset X$ is a given convex set. A function is said to be DC when it is the difference of two convex functions. The setting of this paper is intermediate between those of the two types of works reported in Tables 1, and 2 as the decision space X here is infinite dimensional, but we prefer inequality constraints to conic ones as we try to investigate

Table 2

Reference	f	**Dual problem**	**Theorems**
[4]	convex	Lagrange	strong
[16]	convex	Lagrange	total
[17]	co/co	Wolfe	total
[18]	convex	Lagrange	stable zero-gap
[19]	DC, convex	Fenchel, Lagrange	stable strong
[20]	convex	Lagrange	stable strong

the dependence of the duality principles from several cones associated with the constraint functions (more precisely, from the epigraphs of the conjugate functions of $g_t(.,u_t),\ t \in T,\ u_t \in U_t$). Indeed, we associate with the robust counterpart (RP) a convex, infinite optimization parametric problem

$$(RP_{x^*}) \qquad \inf_{x\in X} \left\{ f(x) - \langle x^*, x\rangle \qquad \text{s.t. } g_t(x,u_t) \leq 0,\ \forall t \in T,\ \forall u_t \in U_t \right\},$$

where $\langle x^*, x\rangle$ denotes the duality product of $x \in X$ by $x^* \in X^*$ (the topological dual of X, whose null vector we denote by 0_{X^*}). Obviously, $(RP_{0_{X^*}})$ coincides with (RP), so that we have embedded (RP) into the parametric problem.

Let us give a simple example of robust infinite optimization problem.

Example 2.1 Let X be the Hilbert space $L^2 := L^2([0,1])$. We denote by $\|\cdot\|$ the L^2-norm and consider the unit closed ball $\mathbb{B} := \{x \in L^2 : \|x\| \leq 1\}$. Given $a \in L^2$ and two families of positive numbers, $\{\alpha_t\}_{t\in T}$ and $\{\beta_t\}_{t\in T}$, we consider the uncertain linear problem

$$(P) \left\{ \inf_{x\in L^2} \left\{ \int_0^1 a(s)x(s)\,ds \text{ s.t. } \int_0^1 u_t(s)\,x(s)\,ds \leq \beta_t,\ t \in T \right\} \right\}_{(u_t)_{t\in T} \in \prod\limits_{t\in T} \alpha_t \mathbb{B}}.$$

The feasible set of the robust counterpart (RP) of (P) is

$$\begin{aligned}
&\left\{x \in L^2 : \int_0^1 u_t\left(s\right) x\left(s\right) ds \leq \beta_t,\ \forall t \in T, \forall u_t \in \alpha_t \mathbb{B}\right\} \\
&= \left\{x \in L^2 : \sup_{\|u_t\| \leq \alpha_t} \int_0^1 u_t\left(s\right) x\left(s\right) \leq \beta_t,\ \forall t \in T\right\} \\
&= \left\{x \in L^2 :\ \alpha_t \left\|x\right\| \leq \beta_t,\ \forall t \in T\right\},
\end{aligned}$$

and we have

$$(RP) \qquad \inf_{x \in L^2} \int_0^1 a\left(s\right) x\left(s\right) ds \ \text{ s.t. } \|x\| \leq \inf_{t \in T} \frac{\beta_t}{\alpha_t},$$

equivalently,

$$(RP) \qquad -\sup_{x \in L^2} \int_0^1 a\left(s\right) \left(-x\left(s\right)\right) ds \ \text{ s.t. } \|x\| \leq \inf_{t \in T} \frac{\beta_t}{\alpha_t},$$

and so, $\inf(RP) = -\left\|a\right\| \inf_{t \in T} \frac{\beta_t}{\alpha_t}$.

Appealing to the standard notation of convex analysis recalled in Section 2, we introduce the following *moment cones*, whose subindexes are the initials of the five robust dual problems, that we introduce in next section:

$$M_O := \bigcup_{u=(u_t)_{t\in T} \in U} \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*}, 1)\} \cup \left(\bigcup_{t \in T} \operatorname{epi}\left(g_t(\cdot, u_t)\right)^*\right)\right\},$$

$$M_C := \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*}, 1)\} \cup \left(\bigcup_{u \in U} \bigcup_{t \in T} \operatorname{epi}\left(g_t(\cdot, u_t)\right)^*\right)\right\},$$

$$M_W := \bigcup_{u=u(t)_{t\in T} \in U} \operatorname{cl}\operatorname{conv}\operatorname{cone}\left\{\bigcup_{t \in T} \operatorname{epi}(g_t(., u_t))^*\right\},$$

$$M_{L_h} := \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*}, 1)\} \cup \left(\bigcup_{t \in T} \operatorname{cl}\operatorname{conv} \bigcup_{u_t \in U_t} \operatorname{epi}(g_t(., u_t))^*\right)\right\},$$

$$M_{L_k} := \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*}, 1)\} \cup \left(\bigcup_{u \in U} \operatorname{cl}\operatorname{conv} \bigcup_{t \in T} \operatorname{epi}(g_t(., u_t))^*\right)\right\}.$$

The cones M_O and M_C are called *robust moment cone* and *robust characteristic cone* in [10], respectively. The above five cones have the same w^*-closed and convex cone hull, $\operatorname{cl} M_C$, which determines the feasible set F of (RP) as, $F \neq \emptyset$ if and only if $(0_{X^*}, 1) \notin \operatorname{cl} M_C$ [21, Theorem 3.1], in which case, by [21, Theorem 4.1] and the separation theorem,

$$F = \left\{ x \in X : \langle v^*, x \rangle \leq \alpha, \ \forall \, (v^*, \alpha) \in \overline{M}_C \right\}. \tag{3}$$

The cone $\operatorname{cl}(M_C)$ can be called *robust reference cone* following the linear semi-infinite programming terminology [22].

The paper is organized as follows. Section 3 introduces the necessary concepts and notations, and yields the basic results to be used later. Section 4 establishes and characterizes various dual variational principles for (RP_{x^*}) in terms of w^*-closed convexity of M_O and M_W, or in terms of w^*-closedness of M_C, M_{L_h}, and M_{L_k}. Section 5 is devoted to uniform robust strong duality (i.e., the fulfilment of robust duality for arbitrary convex objective functions), robust duality for convex problems with linear objective function, the particular case of robust linear optimization and, finally, robust reverse strong duality. The last section is focused on the general uncertain problem, where the constraints and the objective functions are all uncertain. Our approach consists in rewriting such problem in one of the types studied previously, i.e., with deterministic objective function, and applying the results obtained in the first part of the paper.

3 Preliminaries

We start this section with some necessary notation. Given a non-empty subset A of a (real) lcHtvs, we denote by $\operatorname{conv} A$, $\operatorname{cone} A := \mathbb{R}_+ A$, and $\operatorname{cl} A$, the convex hull of A, the cone generated by A, and the closure of A, respectively. Given two subsets A, B of a lcHtvs, A is said to be *closed* (respectively, *closed and convex*) *regarding* B if $(\operatorname{cl} A) \cap B = A \cap B$ (respectively, $(\operatorname{cl} \operatorname{conv} A) \cap B = A \cap B$).

We represent by $\mathbb{R}_+^{(T)}$ the positive cone in $\mathbb{R}^{(T)}$, the so-called *space of generalized finite sequences* $\lambda = (\lambda_t)_{t \in T}$ such that $\lambda_t \in \mathbb{R}$, for each $t \in T$, and with only finitely many λ_t different from zero. The supporting set of $\lambda \in \mathbb{R}^{(T)}$ is $\operatorname{supp} \lambda := \{t \in T \,:\, \lambda_t \neq 0\}$.

Having a function $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we denote by $\operatorname{epi} h$ and h^* the epigraph and the Legendre-Fenchel conjugate of h, respectively. The function h is proper if $\operatorname{epi} h \neq \emptyset$ and never takes the value $-\infty$, it is convex if $\operatorname{epi} h$ is convex, and it is lower semicontinuous (lsc, in brief) if $\operatorname{epi} h$ is closed. We denote by $\Gamma(X)$ the class of all lsc proper convex functions on X. The *indicator function* of $A \subset X$ is represented by i_A (i.e., $i_A(x) = 0$ if $x \in A$, and $i_A(x) = +\infty$ if $x \notin A$). If A is a non-empty, closed and convex set, then $i_A \in \Gamma(X)$. We also denote by $\Upsilon(X) \subset \Gamma(X)$ the class of (real-valued) convex continuous functions on X.

Following [23], we define the *characteristic cone* of a system $\sigma = \{h_t(x) \leq 0,\ t \in T\}$ such that $\{h_t,\ t \in T\} \subset \Gamma(X)$ as

$$K(\sigma) := \operatorname{conv} \operatorname{cone} \left\{ \{(0_{X^*}, 1)\} \cup \bigcup_{t \in T} \operatorname{epi} h_t^* \right\}.$$

Concerning the data in the robust counterpart problem (RP_{x^*}) introduced in (2), we assume that, for each $t \in T$, U_t is an arbitrary subset of the lcHtvs Z_t. Along all the paper we will assume that

$$\begin{cases} f \in \Gamma(X) \\ g_t(\cdot, u_t) \in \Upsilon(X), \ \forall t \in T, \ \forall u_t \in U_t, \\ \exists \bar{x} \in \operatorname{dom} f : g_t(\bar{x}, u_t) \leq 0, \ \forall t \in T, \ \forall u_t \in U_t. \end{cases} \tag{4}$$

If we denote by

$$\mathfrak{U} := \{(t, u_t) : t \in T, \ u_t \in U_t\}$$

the *disjoint union* of the sets U_t, $t \in T$, then the robust counterpart to the uncertain problem (P_{x^*}) can be rewritten as

$$(RP_{x^*}) \qquad \inf_{x \in X} \{f(x) - \langle x^*, x\rangle \qquad \text{s.t. } g_t(x, u_t) \leq 0, \ \forall (t, u_t) \in \mathfrak{U}\},$$

whose feasible set F is represented by the (possibly) infinite convex system of constraints $\sigma := \{g_t(x, u_t) \leq 0, \ (t, u_t) \in \mathfrak{U}\}$.

Throughout the paper we assume that $F \cap \operatorname{dom} f \neq \emptyset$, and so $\inf(RP_{x^*}) < +\infty$. Given $u = (u_t)_{t \in T} \in \prod_{t \in T} U_t = U$, let us consider the convex infinite problem

$$(P_{x^*}^u) \qquad \inf_{x \in X} \{f(x) - \langle x^*, x\rangle\} \qquad \text{s.t. } g_t(x, u_t) \leq 0, \ \forall t \in T,$$

whose feasible set F_u and constraint inequality system σ_u are independent of x^*, and whose characteristic cone $K_u := K(\sigma_u)$ can be expressed as

$$\begin{aligned} K_u &= \operatorname{conv} \operatorname{cone} \left\{ \{(0_{X^*}, 1)\} \cup \left(\bigcup_{t \in T} \operatorname{epi} (g_t(\cdot, u_t))^* \right) \right\} \\ &= \mathbb{R}_+ \{(0_{X^*}, 1)\} \cup \left(\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \sum_{t \in T} \lambda_t \operatorname{epi} (g_t(\cdot, u_t))^* \right). \end{aligned}$$

Since the functions $g_t(\cdot, u_t)$ are convex we have that, by Moreau-Rockafellar formula and from the continuity assumption in (4) (entailing the w^*- closedness of $\sum_{t\in T} \operatorname{epi}(\lambda_t g_t(\cdot, u_t))^*$), we get

$$\operatorname{epi}\left(\sum_{t\in T} \lambda_t g_t(\cdot, u_t)\right)^* = \begin{cases} \sum_{t\in T} \lambda_t \operatorname{epi}(g_t(\cdot, u_t))^*, & \text{if } \lambda = (\lambda_t)_{t\in T} \neq 0_T, \\ \mathbb{R}_+\{(0_{X^*}, 1)\}, & \text{else.} \end{cases}$$

Hence,

$$K_u = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \operatorname{epi}\left(\sum_{t\in T} \lambda_t g_t(\cdot, u_t)\right)^*. \tag{5}$$

By analogy (up to the sign) with [10], on uncertain linear semi-infinite case (the terminology being a bit different in [7] for robust convex programming), we have defined the robust moment cone (non-convex in general) by

$$M_O := \bigcup_{u\in U} K_u, \tag{6}$$

and the *robust characteristic cone* as

$$M_C := \operatorname{conv} M_O.$$

Proposition 3.1 *M_C coincides with the characteristic cone $K(\sigma)$ of the convex system σ.*

Proof Since M_O is a cone, we have

$$\begin{aligned}
M_C &= \operatorname{conv}\operatorname{cone} M_O \\
&= \operatorname{conv}\operatorname{cone}\left(\bigcup_{u\in U}\operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*},1)\}\cup\left(\bigcup_{t\in T}\operatorname{epi}(g_t(.,u_t))^*\right)\right\}\right) \\
&= \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*},1)\}\cup\left(\bigcup_{(u,t)\in U\times T}\operatorname{epi}(g_t(.,u_t))^*\right)\right\} \\
&= \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*},1)\}\cup\left(\bigcup_{(t,u_t)\in\mathfrak{U}}\operatorname{epi}(g_t(.,u_t))^*\right)\right\},
\end{aligned}$$

which is nothing else than the characteristic cone of σ. □

Given $u=(u_t)_{t\in T}\in U$, let us introduce the *Lagrangian dual* of $(P^u_{x^*})$:

$$(D^u_{x^*}) \qquad \sup_{\lambda\in\mathbb{R}^{(T)}_+}\inf_{x\in X}\left\{f(x)-\langle x^*,x\rangle+\sum_{t\in T}\lambda_t g_t(x,u_t)\right\}.$$

Robust duality can be defined via the so-called *optimistic dual problem* of (RP_{x^*}), that is,

$$(RD^O_{x^*}) \qquad \sup_{(u,\lambda)\in U\times\mathbb{R}^{(T)}_+}\inf_{x\in X}\left\{f(x)-\langle x^*,x\rangle+\sum_{t\in T}\lambda_t g_t(x,u_t)\right\}.$$

The optimistic dual problem $(RD^O_{x^*})$ is different from the classical dual $(RD^C_{x^*})$ of the totally explicitly constrained infinite convex problem (RP_{x^*}), i.e.,

$$(RD^C_{x^*}) \qquad \sup_{\left(\lambda_{(t,u_t)}\right)_{(t,u_t)\in\mathfrak{U}}\in\mathbb{R}^{(\mathfrak{U})}_+}\inf_{x\in X}\left\{f(x)-\langle x^*,x\rangle+\sum_{(t,u_t)\in\mathfrak{U}}\lambda_{(t,u_t)}g_t(x,u_t)\right\}.$$

Next we introduce a different type of duality inspired in [7], where only finite index sets are considered. For each $u\in U$ let us consider the Lagrangian $L^u_{x^*}$ associated with the convex infinite problem $(P^u_{x^*})$,

$$L^u_{x^*}(x,\lambda):=f(x)-\langle x^*,x\rangle+\sum_{t\in T}\lambda_t g_t(x,u_t),\ (x,\lambda)\in X\times\mathbb{R}^{(T)}_+,$$

and define the *robust Lagrangian* $L\colon X \times \mathbb{R}_+^{(T)} \to \overline{\mathbb{R}}$ by $L_{x^*} := \sup_{u \in U} L_{x^*}^u$. One has

$$\sup_{\lambda \in \mathbb{R}_+^{(T)}} L_{x^*}(x, \lambda) = f(x) - \langle x^*, x \rangle + i_F(x), \ \forall x \in X,$$

and, consequently, (RP_{x^*}) can be rewritten as

$$(RP_{x^*}) \qquad \inf_{x \in X} \sup_{\lambda \in \mathbb{R}_+^{(T)}} L_{x^*}(x, \lambda).$$

The associated Lagrangian robust *dual* problem is defined by

$$(RD_{x^*}^{L_h}) \qquad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in X} \sup_{u \in U} \left\{ f(x) - \langle x^*, x \rangle + \sum_{t \in T} \lambda_t g_t(x, u_t) \right\}.$$

Let us give a clarifying interpretation of this Lagrangian robust dual. We start by defining, for each $t \in T$, the function

$$h_t := \sup_{u_t \in U_t} g_t(\cdot, u_t),$$

which brings together the uncertain constraints $g_t(x, u_t) \leq 0$, $u_t \in U_t$, giving, for each $t \in T$, the *worse* possible constraint. Observe that $h_t \in \Gamma(X)$, and it is continuous (and belongs to $\Upsilon(X)$) if U_t is a compact subset of Z_t and the function $g_t\colon X \times U_t \to \mathbb{R}$ is continuous for all $t \in T$. Moreover

$$\sup_{u \in U} \sum_{t \in T} \lambda_t g_t(x, u_t) = \sum_{t \in T} \lambda_t h_t(x), \ \forall (x, \lambda) \in X \times \mathbb{R}_+^{(T)},$$

and $(RD_{x^*}^{L_h})$ can be written as

$$(RD_{x^*}^{L_h}) \qquad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in X} \left\{ f(x) - \langle x^*, x \rangle + \sum_{t \in T} \lambda_t h_t(x) \right\};$$

in other words $(RD_{x^*}^{L_h})$ is the classical dual of the partially and explicitly constrained infinite convex problem

$$(RP_{x^*}^{L_h}) \qquad \inf_{x \in X} \{ f(x) - \langle x^*, x \rangle \qquad \text{s.t. } h_t(x) \leq 0, \ \forall t \in T \}.$$

In a similar way, for each $u = (u_t)_{t\in T} \in U$, we can define the function

$$k_u := \sup_{t\in T} g_t(\cdot, u_t),$$

which is a proper function thanks to (4), bringing together the constraints $g_t(x, u_t) \leq 0$, $t \in T$, for each $u \in U$. So, $k_u \in \Gamma(X)$, and it is continuous (and belongs to $\Upsilon(X)$) if T is a compact topological space and the function

$$(x, t) \in X \times T \mapsto g_t(x, u_t) \text{ is continuous, } \forall u = (u_t)_{t\in T} \in U.$$

Then, the problem (RP_{x^*}) can be rewritten as the partially and explicitly constrained infinite convex problem

$$(RP_{x^*}^{L_k}) \qquad \inf_{x\in X} \{f(x) - \langle x^*, x\rangle \qquad \text{s.t. } k_u(x) \leq 0, \ \forall u \in U\},$$

and we consider its classical dual

$$(RD_{x^*}^{L_k}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(U)}} \inf_{x\in X} \left\{ f(x) - \langle x^*, x\rangle + \sum_{u\in U} \lambda_u k_u(x) \right\}.$$

The problem $(RD_{x^*}^{L_k})$ constitutes another kind of Lagrangian robust dual problem of (RP_{x^*}). It is absolutely obvious that

$$\inf(RP_{x^*}) = \inf(RP_{x^*}^{L_h}) = \inf(RP_{x^*}^{L_k}).$$

Let us explore next the relationship among the optimal values of the different duals introduced above.

Proposition 3.2 *For every $x^* \in X^*$ and $i \in \{L_h, L_k\}$, one has*

$$\sup(RD_{x^*}^{O}) \leq \begin{array}{c} \sup_{u\in U} \inf(P_{x^*}^{u}) \\ \\ \sup(RD_{x^*}^{C}) \leq \sup(RD_{x^*}^{i}) \end{array} \leq \inf(RP_{x^*}). \tag{7}$$

Proof • For each $(u,\lambda) \in U \times \mathbb{R}_+^{(T)}$ one has $\sum_{t\in T} \lambda_t g_t(\cdot,u_t) \leq i_{F_u}$ and

$$\inf_X \left\{ f - \langle x^*,\cdot\rangle + \sum_{t\in T} \lambda_t g_t(\cdot,u_t) \right\} \leq \inf_{F_u}(f - \langle x^*,\cdot\rangle) = \inf(P_{x^*}^u).$$

Taking the supremum over $(u,\lambda) \in U\times\mathbb{R}_+^{(T)}$ we get $\sup(RD_{x^*}^O) \leq \sup_{u\in U} \inf(P_{x^*}^u)$.

• If $(\bar{u},\bar{\lambda}) \in U \times \mathbb{R}_+^{(T)}$, let us define

$$\lambda_{(t,u_t)} := \begin{cases} \bar{\lambda}_t, & \text{if } u_t = \bar{u}_t, \\ 0, & \text{if } u_t \neq \bar{u}_t. \end{cases}$$

We have $(\lambda_{(t,u_t)})_{(t,u_t)\in\mathfrak{U}} \in \mathbb{R}_+^{(\mathfrak{U})}$, and $\sum\limits_{t\in T} \bar{\lambda}_t g_t(\cdot,\bar{u}_t) = \sum\limits_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(\cdot,u_t)$. It easily follows that $\sup(RD_{x^*}^O) \leq \sup(RD_{x^*}^C)$.

• Besides, since for each $u \in U$, the feasible set F_u of $(P_{x^*}^u)$ contains the feasible set F of (RP_{x^*}), we have $\sup_{u\in U} \inf(P_{x^*}^u) \leq \inf(RP_{x^*})$.

• We now prove that $\sup(RD_{x^*}^C) \leq \sup(RD_{x^*}^{L_h})$. If $\lambda \in \mathbb{R}_+^{(\mathfrak{U})}$ then we will show that

$$\inf_{x\in X} \left\{ f(x) - \langle x^*,x\rangle + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t) \right\} \leq \sup(RD_{x^*}^{L_h}).$$

For each $t \in T$, define $\overline{\lambda}_t := \sum\limits_{(t,u_t)\in \operatorname{supp}\lambda} \lambda_{(t,u_t)}$. Then, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T}$ belongs to $\mathbb{R}_+^{(T)}$ and we have, for each $x \in X$,

$$\sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t) \leq \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} h_t(x) = \sum_{t\in T} \bar{\lambda}_t h_t(x).$$

It follows that

$$\begin{aligned} \inf_{x\in X}\left\{ f(x) - \langle x^*,x\rangle + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t) \right\} &\leq \inf_{x\in X}\left\{ f(x) - \langle x^*,x\rangle + \sum_{t\in T} \bar{\lambda}_t h_t(x) \right\} \\ &\leq \sup(RD_{x^*}^{L_h}). \end{aligned}$$

• We also have that $\sup(RD^C_{x^*}) \leq \sup(RD^{L_k}_{x^*})$. If $\lambda \in \mathbb{R}^{(\mathfrak{U})}_+$ then we will see that

$$\inf_{x\in X}\left\{f(x) - \langle x^*, x\rangle + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t)\right\} \leq \sup(RD^{L_k}_{x^*}).$$

For each $u = (u_t)_{t\in T} \in U$, define $\overline{\lambda}_u := \sum_{(t,u_t)\in\operatorname{supp}\lambda} \lambda_{(t,u_t)}$. Then, $\bar{\lambda} := (\bar{\lambda}_u)_{u\in U}$ belongs to $\mathbb{R}^{(U)}_+$ and we have, for each $x \in X$,

$$\sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t) \leq \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} k_u(x) = \sum_{u\in U} \bar{\lambda}_u k_u(x).$$

It follows that

$$\begin{aligned}\inf_{x\in X}\left\{f(x) - \langle x^*, x\rangle + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t)\right\} &\leq \inf_{x\in X}\left\{f(x) - \langle x^*, x\rangle + \sum_{u\in U} \bar{\lambda}_u k_u(x)\right\}\\ &\leq \sup(RD^{L_k}_{x^*}).\end{aligned}$$

• It is easy to see that $\sup(LRD^i_{x^*}) \leq \inf(RP_{x^*})$, $i \in \{L_h, L_k\}$, and the proof is complete. □

We illustrate next the case when T is a singleton, U is a compact topological space and $g\colon\ X \times U \to \mathbb{R}$ is such that $g(.,u)$ is continuous and convex for each $u \in U$ and $g(x,.)$ is upper semicontinuous for each $x \in X$. We thus have

$$(RD^O_{x^*}) \qquad \sup_{\lambda\geq 0}\sup_{u\in U}\left[\inf_{x\in X}\{f(x) - \langle x^*, x\rangle + \lambda g(x,u)\}\right],$$

$$(RD^C_{x^*}) = (RD^{L_k}_{x^*}) \qquad \sup_{\lambda\in\mathbb{R}^{(U)}_+}\left[\inf_{x\in X}\left\{f(x) - \langle x^*, x\rangle + \sum_{u\in U}\lambda_u g(x,u)\right\}\right],$$

$$(RD^{L_h}_{x^*}) \qquad \sup_{\lambda\geq 0}\left[\inf_{x\in X}\{f(x) - \langle x^*, x\rangle + \lambda h(x)\}\right],$$

where $h(x) = \max_{u\in U} g(x,u)$.

Now, if Slater condition holds, namely, there exists $\overline{x} \in \operatorname{dom} f$ such that $g(\overline{x}, u) < 0$, for all $u \in U$, then $h(\overline{x}) < 0$ and

$$\inf(RP_{x^*}) = \max(RD_{x^*}^{L_h}). \tag{8}$$

If additionally, U is convex and $g(x, .)$ is (upper semicontinuous) concave for each $x \in X$, then Sion Theorem [24] yields

$$\sup(RD_{x^*}^{O}) = \sup(RD_{x^*}^{L_h}). \tag{9}$$

Consequently, if the two conditions above are satisfied, then (7) - (9) yield

$$\inf(RP_{x^*}) = \sup(RD_{x^*}^{O}) = \max(RD_{x^*}^{L_h}) = \sup(RD_{x^*}^{L_k}) = \sup(RD_{x^*}^{C}).$$

Let us consider the *worst value possible* among the values of the programs $(P_{x^*}^u)$, $u \in U$. That leads to the problem [20]:

$$(RD_{x^*}^{W}) \qquad \sup_{u \in U} \inf(P_{x^*}^u).$$

By Proposition 3.2 we have

$$\sup(RD_{x^*}^{O}) \leq \sup(RD_{x^*}^{W}) \leq \inf(RP_{x^*}).$$

Definition 3.1 Let $i \in \{O, C, W, L_h, L_k\}$.

(a) $\left(RD^i\right)$ *−robust strong duality holds at a given* $x^* \in X^*$ iff

$$\inf(RP_{x^*}) = \max(RD_{x^*}^{i}).$$

(b) The $\left(RD^i\right)$ *−robust strong duality* holds *stably* iff $\left(RD^i\right)-$ robust strong duality holds at each $x^* \in X^*$.

(c) The $\left(RD^i\right)$ *−robust strong duality* holds *uniformly* iff $\left(RD^i\right)$ −robust strong duality holds at $x^* = 0_{X^*}$ for any function f in the family

$$\mathcal{F} := \{f \in \Gamma(X) : f \text{ is continuous at some point of } F\}.$$

(Observe that $f \in \mathcal{F}$ entails $f - \langle x^*, .\rangle \in \mathcal{F}$, $\forall x^* \in X^*$).

From the proof of Proposition 3.2 it is clearly that optimistic robust strong duality entails classical robust strong duality, Lagrangian robust strong duality of both types and worst-value robust strong duality. Robust strong duality of the types defined in (a) and (b) in Definition 3.1 will be studied in next section (Section 4), while the last one and some more complements will be given in Section 5.

We conclude this section by the following note: For the sake of simplicity, in the case when $x^* = 0_{X^*}$ the robust dual problems $(RD^i_{0_{X^*}})$ will be denoted, respectively, by (RD^i), $i \in \{O, C, W, L_h, L_k\}$.

4 The Main Results

In this section, we will study common duality principles between (RP_{x^*}) and the dual problems (RD^i), $i \in \{O, C, W, L_h, L_k\}$. For this, let us associate the mentioned dual problems with the functions $\varphi_i : X^* \longrightarrow \overline{\mathbb{R}}$,

$i \in \{O, C, W, L_h, L_k\}$, defined as

$$\varphi_O := \inf_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(., u_t) \right)^*,$$

$$\varphi_C := \inf_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \left(f + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(., u_t) \right)^*,$$

$$\varphi_W := \inf_{u\in U} (f + i_{F_u})^*,$$

$$\varphi_{L_h} := \inf_{\lambda\in\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t h_t \right)^*, \text{ where } h_t := \sup_{u_t\in U_t} g_t(., u_t),$$

$$\varphi_{L_k} := \inf_{\lambda\in\mathbb{R}_+^{(U)}} \left(f + \sum_{u\in U} \lambda_u k_u \right)^*, \text{ where } k_u := \sup_{t\in T} g_t(., u_t),$$

and with the corresponding sets

$$\mathcal{A}_O := \bigcup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \operatorname{epi} \left(f + \sum_{t\in T} \lambda_t g_t(., u_t) \right)^*,$$

$$\mathcal{A}_C := \bigcup_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \operatorname{epi} \left(f + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(., u_t) \right)^*,$$

$$\mathcal{A}_W := \bigcup_{u\in U} \operatorname{epi} (f + i_{F_u})^*,$$

$$\mathcal{A}_{L_h} := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \operatorname{epi} \left(f + \sum_{t\in T} \lambda_t h_t \right)^*,$$

$$\mathcal{A}_{L_k} := \bigcup_{\lambda\in\mathbb{R}_+^{(U)}} \operatorname{epi} \left(f + \sum_{u\in U} \lambda_u k_u \right)^*.$$

Let us equip the space $X^* \times \mathbb{R}$ with the product topology of the w^*-topology on X^* and the natural topology on $\mathbb{R}$. We denote by $\operatorname{cl} A$ the corresponding w^*-closure of any subset $A \subset X^* \times \mathbb{R}$. Recall that A is said to be w^*-closed (respectively, w^*-closed and convex) regarding a subset $B \subset X^* \times \mathbb{R}$ if $(\operatorname{cl} A) \cap B = A \cap B$ (respectively, $(\operatorname{cl} \operatorname{conv} A) \cap B = A \cap B$). The following facts can easily be checked (the convexity of the sets and functions below can be proved by a similar reasoning to the one followed in Lemma 3.1 in [25]):

$$\begin{cases} \mathcal{A}_i \subset \operatorname{epi}\varphi_i \subset \operatorname{cl}(\mathcal{A}_i), \ i \in \{O, C, W, L_h, L_k\}, \\ \mathcal{A}_C, \mathcal{A}_{L_h}, \text{ and } \mathcal{A}_{L_k} \text{ are convex sets,} \\ \varphi_C, \varphi_{L_h}, \text{ and } \varphi_{L_k} \text{ are convex functions.} \end{cases} \tag{10}$$

Let us give some equivalent expressions of the sets $\mathcal{A}_i$, $i \in \{O, C, W, L_h, L_k\}$, respectively in terms of the robust moment cones with the same indexes and the characteristic cones K_u, $u \in U$.

Proposition 4.1 *(a) One has*

$$\mathcal{A}_O = \operatorname{epi} f^* + M_O, \quad \mathcal{A}_C = \operatorname{epi} f^* + M_C, \quad \mathcal{A}_W = \bigcup_{u\in U} \operatorname{cl}(\operatorname{epi} f^* + K_u).$$

(b) If $f \in \mathcal{F}$, *then* $\mathcal{A}_W = \operatorname{epi} f^* + \bigcup_{u\in U} \operatorname{cl}(K_u) = \operatorname{epi} f^* + M_W$.

(c) If $h_t \in \Upsilon(X)$, $\forall t \in T$, *then* $\mathcal{A}_{L_h} = \operatorname{epi} f^* + M_{L_h}$.

(d) If $k_u \in \Upsilon(X)$, $\forall\, u \in U$, *then* $\mathcal{A}_{L_k} = \operatorname{epi} f^* + M_{L_k}$.

Proof (a) By Moreau-Rockafellar formula (as $\sum_{t\in T} \lambda_t g_t(\cdot, u_t)$ are continuous),

$$\begin{aligned} \mathcal{A}_O &= \bigcup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(\operatorname{epi} f^* + \operatorname{epi}\left(\sum_{t\in T} \lambda_t g_t(\cdot, u_t)\right)^* \right) \\ &= \operatorname{epi} f^* + \bigcup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \operatorname{epi}\left(\sum_{t\in T} \lambda_t g_t(\cdot, u_t)\right)^* \\ &= \operatorname{epi} f^* + \bigcup_{u\in U} K_u = \operatorname{epi} f^* + M_O. \end{aligned}$$

$$\begin{aligned}\mathcal{A}_C &:= \bigcup_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}}\left(\operatorname{epi} f^* + \operatorname{epi}\left(\sum_{(t,u_t)\in\mathfrak{U}}\lambda_{(t,u_t)}g_t(\cdot,u_t)\right)^*\right)\\ &= \operatorname{epi} f^* + \bigcup_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}}\operatorname{epi}\left(\sum_{(t,u_t)\in\mathfrak{U}}\lambda_{(t,u_t)}g_t(\cdot,u_t)\right)^* = \operatorname{epi} f^* + M_C.\end{aligned}$$

In order to have an explicit expression of $\mathcal{A}_W$, let us observe that for all $u=(u_t)_{t\in T}\in U$, $i_{F_u}=\sup_{\lambda\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\lambda_t g_t(\cdot,u_t)$. Since $F_u\neq\emptyset$,

$$\operatorname{epi} i_{F_u}^* = \operatorname{cl}\operatorname{conv}\left(\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\operatorname{epi}\sum_{t\in T}\lambda_t g_t(.,u_t)^*\right) = \operatorname{cl}\operatorname{conv}(K_u) = \operatorname{cl}(K_u)$$

(as K_u is convex), and then

$$\mathcal{A}_W = \bigcup_{u\in U}\operatorname{cl}\left(\operatorname{epi} f^* + \operatorname{epi} i_{F_u}^*\right) = \bigcup_{u\in U}\operatorname{cl}\left(\operatorname{epi} f^* + \operatorname{cl}(K_u)\right) = \bigcup_{u\in U}\operatorname{cl}\left(\operatorname{epi} f^* + K_u\right).$$

(b) Since $f\in\mathcal{F}$, we have

$$\operatorname{epi}(f+i_{F_u})^* = \operatorname{epi} f^* + \operatorname{epi} i_{F_u}^* = \operatorname{epi} f^* + \operatorname{cl}(K_u), \quad \forall u\in U.$$

We now observe that

$$\operatorname{cl}(K_u) = \operatorname{cl}\operatorname{conv}\operatorname{cone}\left(\bigcup_{t\in T}\operatorname{epi}(g_t(.,u_t))^*\right), \quad \forall u\in U. \tag{11}$$

By the very definition of K_u, the inclusion $\supset$ in (11) is obvious. Conversely, let us first check that $(0_{X^*},1)\in\operatorname{cl}\operatorname{cone}(\operatorname{epi}(g_t(.,u_t))^*)$ for any $t\in T$. Pick $(x^*,r)\in\operatorname{epi}(g_t(.,u_t))^*$. For each $n\geq 1$ we have $\frac{1}{n}(x^*,r+n)\in\operatorname{cone}\operatorname{epi}(g_t(.,u_t))^*$, and $(0_{X^*},1)=\lim_{n\to\infty}\frac{1}{n}(x^*,r+n)\in\operatorname{cl}\operatorname{cone}(\operatorname{epi}(g_t(.,u_t))^*)$. Now it is clearly that $(0_{X^*},1)$ belongs to the right-hand side of (11). By definition of K_u it follows that K_u is contained in the right-hand side of (11), which is closed. Finally, we get that (11) holds. We thus have

$$\begin{aligned}\mathcal{A}_W &= \bigcup_{u\in U}(\operatorname{epi} f^* + \operatorname{cl} K_u) = \operatorname{epi} f^* + \bigcup_{u\in U}\operatorname{cl} K_u\\ &= \operatorname{epi} f^* + \bigcup_{u\in U}\left\{\operatorname{cl}\operatorname{conv}\operatorname{cone}\left(\bigcup_{t\in T}\operatorname{epi}(g_t(.,u_t))^*\right)\right\} = \operatorname{epi} f^* + M_W.\end{aligned}$$

(c) For each $\lambda \in \mathbb{R}_+^{(T)}$ one has $\sum_{t\in T}\lambda_t h_t \in \Upsilon(X)$ and, by this,

$$\operatorname{epi}\left(f + \sum_{t\in T}\lambda_t h_t\right)^* = \operatorname{epi} f^* + \operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t\right)^*.$$

We thus have,

$$\mathcal{A}_{L_h} = \operatorname{epi} f^* + \bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t\right)^*.$$

Since $h_t \in \Upsilon(X)$ for all $t \in T$, one has, for each $\lambda \in \mathbb{R}_+^{(T)}$:

$$\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t\right)^* = \begin{cases}\sum_{t\in T}\lambda_t \operatorname{epi} h_t^*, & \text{if } \lambda = (\lambda_t)_{t\in T} \neq 0_T,\\ \{0_{X^*}\}\times\mathbb{R}_+, & \text{else.}\end{cases}$$

Therefore,

$$\begin{aligned}&\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\operatorname{epi}\left(\sum_{t\in T}\lambda_t h_t\right)^*\\ &= \mathbb{R}_+\{(0_{X^*},1)\}\cup\left(\bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\lambda_t\operatorname{epi} h_t^*\right)\\ &= \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*},1)\}\cup\left(\bigcup_{t\in T}\operatorname{epi} h_t^*\right)\right\}\\ &= \operatorname{conv}\operatorname{cone}\left\{\{(0_{X^*},1)\}\cup\left(\bigcup_{t\in T}\operatorname{cl}\operatorname{conv}\bigcup_{u_t\in U_t}\operatorname{epi}(g_t(.,u_t))^*\right)\right\},\end{aligned}$$

and the proof of (c) is complete.

(d) The proof is similar to that of (c). □

Example 4.1 Consider the simple uncertain linear SIP problem

$$(RP) \qquad \inf_{x\in\mathbb{R}^2} \langle c^*, x\rangle$$
$$\text{s.t.} \qquad \langle x_t^*(u_t), x\rangle \leq r_t(u_t),\ \forall t\in[0,1], \forall u_t\in U_t,$$

where $c^* \in \mathbb{R}^2 \diagdown \{(0,0)\}$ is fixed, $U_0 := ([0,2\pi]\cap\mathbb{Q})\times\mathbb{N}\subset\mathbb{Z}_0 = \mathbb{R}^2$ and $U_t = \{t\}\subset\mathbb{Z}_t = \mathbb{R}$, for all $t\in\,]0,1]$, $(x_0^*(u_0), r_0(u_0)) = \left(\cos\alpha_0, \sin\alpha_0, \frac{r_0+1}{r_0}\right)$ for all $u_0 = (\alpha_0, r_0)\in U_0$, and $(x_t^*(u_t), r_t(u_t)) = (0,0,t)$ for all $t\in\,]0,1]$. Here

$$U = \left\{(u_t)_{t\in[0,1]} : u_0\in U_0 \text{ and } u_t = t,\ \forall t\in\,]0,1]\right\}, \text{ and}$$

$$\mathfrak{U} = (\{0\}\times U_0)\cup\{(t,t) : t\in\,]0,1]\}.$$

Moreover, since

$$g_t(x,u_t) = \langle x_t^*(u_t), x\rangle - r_t(u_t), \quad \operatorname{epi}(g_t(\cdot,u_t))^* = \{x_t^*(u_t)\}\times[r_t(u_t), +\infty[$$

and, given $u\in U$ such that $u_0 = (\alpha_0, r_0)$,

$$K_u = \operatorname{conv}\operatorname{cone}\left\{(0,0,1), \left(\cos\alpha_0, \sin\alpha_0, \frac{r_0+1}{r_0}\right)\right\}.$$

Hence, denoting $D := \{(\cos\alpha, \sin\alpha) : \alpha\in[0,2\pi]\cap\mathbb{Q}\}$ (a dense subset in the unit circle $\mathbb{S}^1$), and observing that K_u is closed for all $u\in U$, one has

$$M_O = M_W = \{(0,0,0)\}\cup\left\{x\in\mathbb{R}^3 : (x_1,x_2)\in\mathbb{R}_+D,\ x_3 > \|(x_1,x_2)\|\right\},$$

$$M_C = M_{L_k} = \{(0,0,0)\}\cup\left\{x\in\mathbb{R}^3 : x_3 > \|(x_1,x_2)\|\right\}, \text{and}$$

$$M_{L_h} = \left\{x\in\mathbb{R}^3 : x_3 \geq \|(x_1,x_2)\|\right\}.$$

So, M_{L_h} is a closed and convex cone (actually, the robust reference cone), $M_C = M_{L_k}$ is convex and non-closed and, finally, $M_O = M_W$ is neither closed nor convex.

The functions h_t are continuous for all $t \in [0,1]$ as, given $x \in \mathbb{R}^2$, $h_t(x) = -t$ if $t > 0$, while

$$h_0(x) = \sup\left\{(\cos\alpha_0)\,x_1 + (\sin\alpha_0)\,x_2 - \tfrac{r_0+1}{r_0} : \alpha_0 \in [0,2\pi] \cap \mathbb{Q}, r_0 \in \mathbb{N}\right\}$$
$$= \sup\left\{(\cos\alpha_0)\,x_1 + (\sin\alpha_0)\,x_2 - 1 : \alpha_0 \in [0,2\pi]\right\} = \|x\| - 1.$$

Similarly, the functions k_u are continuous for all $u = (u_t)_{t\in[0,1]} \in U$ as, given $x \in \mathbb{R}^2$,

$$k_u(x) = \sup\nolimits_{t\in T}\left(\langle x_t^*(u_t), x\rangle - r_t(u_t)\right)$$
$$= \max\left\{(\cos\alpha_0)\,x_1 + (\sin\alpha_0)\,x_2 - \tfrac{r_0+1}{r_0}, 0\right\}$$

is the maximum of two affine functions. It is obvious that $F = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$, $\inf(RP) = -\|c^*\|$, and $\operatorname{epi}(\langle c^*, x\rangle)^* = \{c^*\} \times \mathbb{R}_+$. We conclude that $\mathcal{A}_i = M_i + (c_1^*, c_2^*, 0)$, $i \in \{O, C, W, L_h, L_k\}$, which have the same topological and convexity properties as the corresponding moment cones.

Now we proceed by introducing the function $p := f + i_F$. It holds that

$$\inf_{x\in X}\{p - \langle x^*, \cdot\rangle\} = \inf(RP_{x^*}) \quad \text{and} \quad p \in \Gamma(X) \text{ (entailing } p = p^{**}).$$

The two propositions below are mere consequences of Proposition 3.2, as

$$p^*(x^*) = -\inf(RP_{x^*}),$$
$$\varphi_i(x^*) = -\sup(RD^i_{x^*}),\ \forall i \in \{O, C, W, L_h, L_k\}.$$

Proposition 4.2 *For every $x^* \in X^*$ and $i \in \{L_h, L_k\}$ one has*

$$p^*(x^*) \le \begin{matrix} \varphi_W(x^*) \\ \\ \varphi_i(x^*) \le \varphi_C(x^*) \end{matrix} \le \varphi_O(x^*).$$

Proposition 4.3 *(a) Optimistic robust strong duality at* $x^* \in X^*$ *is equivalent to*

$$p^*(x^*) = \min_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(\cdot, u_t) \right)^* (x^*).$$

(b) Classical robust strong duality at $x^* \in X^*$ *is equivalent to*

$$p^*(x^*) = \min_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \left(f + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(\cdot, u_t) \right)^* (x^*).$$

(c) Worst-value robust strong duality at $x^* \in X^*$ *is equivalent to*

$$p^*(x^*) = \min_{u\in U} \left(f + i_{F_u} \right)^* (x^*).$$

(d) Lagrangian robust strong duality of $h-$*type at* $x^* \in X^*$ *is equivalent to*

$$p^*(x^*) = \min_{\lambda\in\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t h_t \right)^* (x^*),$$

where $h_t := \sup_{u_t\in U_t} g_t(., u_t)$, $t \in T$.

(e) Lagrangian robust strong duality of $k-$*type at* $x^* \in X^*$ *is equivalent to*

$$p^*(x^*) = \min_{\lambda\in\mathbb{R}_+^{(U)}} \left(f + \sum_{u\in U} \lambda_u k_u \right)^* (x^*),$$

where $k_u := \sup_{t\in T} g_t(., u_t)$, $u \in U$.

It turns out that the extended real-valued functions φ_i, $i \in \{O, C, W, L_h, L_k\}$ have the same conjugate, namely the function p.

Proposition 4.4 *It holds that* $p = \varphi_i^*$, $i \in \{O, C, W, L_h, L_k\}$.

Proof By Proposition 4.2 it suffices to check that $p = \varphi_O^*$. Now

$$\varphi_O^* = \left(\inf_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(.,u_t) \right)^* \right)^* = \sup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(.,u_t) \right)^{**}$$

$$= \sup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(.,u_t) \right) = f + \sup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(\sum_{t\in T} \lambda_t g_t(.,u_t) \right)$$

$$= f + \sup_{u\in U} \sup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\sum_{t\in T} \lambda_t g_t(.,u_t) \right) = f + \sup_{u\in U} i_{F_u} = f + i_F = p.$$

□

Proposition 4.5 *One has*

$$\operatorname{epi} p^* = \operatorname{cl}\operatorname{conv}(\operatorname{epi} f^* + M_O) = \operatorname{cl}(\operatorname{epi} f^* + M_C)$$
$$= \operatorname{cl}\operatorname{conv}\left(\bigcup_{u\in U} \operatorname{cl}(\operatorname{epi} f^* + K_u) \right) = \operatorname{cl}\operatorname{conv}\left(\operatorname{epi} f^* + \bigcup_{u\in U} K_u \right),$$

or, equivalently,

$$\operatorname{epi} p^* = \operatorname{cl}\operatorname{conv}(\mathcal{A}_O) = \operatorname{cl}(\mathcal{A}_C) = \operatorname{cl}\operatorname{conv}(\mathcal{A}_W).$$

If $h_t \in \Upsilon(X)$ *for all* $t \in T$ *(resp.* $k_u \in \Upsilon(X)$ *for all* $u \in U$*), then we have additionally*

$$\operatorname{epi} p^* = \operatorname{cl}(\mathcal{A}_{L_h}) \quad (\text{respectively, } \operatorname{epi} p^* = \operatorname{cl}(\mathcal{A}_{L_k})).$$

Proof By (10) we have $\operatorname{cl}\operatorname{conv}(\mathcal{A}_i) = \operatorname{cl}\operatorname{conv}(\operatorname{epi}\varphi_i)$, $i \in \{O, W\}$, and $\operatorname{cl}(\mathcal{A}_i) = \operatorname{cl}(\operatorname{epi}\varphi_i)$, $i \in \{C, L_h, L_k\}$. Since $\operatorname{dom} p \neq \emptyset$, Proposition 4.4 and Proposition 4.1 give

$$\operatorname{epi} p^* = \operatorname{epi}\varphi_i^{**} = \operatorname{cl}\operatorname{conv}(\operatorname{epi}\varphi_i) = \operatorname{cl}\operatorname{conv}(\mathcal{A}_i),\ i \in \{O, W\}.$$

$$\operatorname{epi} p^* = \operatorname{epi}\varphi_C^{**} = \operatorname{cl}(\operatorname{epi}\varphi_C) = \operatorname{cl}(\mathcal{A}_C).$$

The rest of equations in the first statement follow from Proposition 4.1.

If $h_t \in \Upsilon(X)$ for all $t \in T$, then

$$\operatorname{epi} p^* = \operatorname{epi} \varphi_{L_h}^{**} = \operatorname{cl}(\operatorname{epi} \varphi_{L_h}) = \operatorname{cl}(\mathcal{A}_{L_h}).$$

The last statement, involving $\mathcal{A}_{L_k}$, holds similarly. □

We are now in a position to establish the duality principles corresponding to the dual problems $(RD^i_{x^*})$, $i \in \{O, C, W, L_h, L_k\}$. In the special cases, these duality principles lead to characterizations of robust and robust stable strong duality between (RP) and $(RD^i_{x^*})$, $i \in \{O, C, W, L_h, L_k\}$. The theorems below extend, complete and unify some results in the literature; in particular, some results concerning optimistic robust duality in [4, 6–10, 13–15, 19, 20, 26], etc., classical robust duality in [10], Lagrangian robust duality of type L_h in [7], etc.

Theorem 4.1 (Robust strong duality at a point) *Let $i \in \{O, C, W, L_h, L_k\}$ be such that $h_t \in \Upsilon(X)$ for all $t \in T$ when $i = L_h$, and that $k_u \in \Upsilon(X)$ for all $u \in U$ when $i = L_k$. Then, given $x^* \in X^*$, (RD^i) −robust strong duality holds at x^* if and only if $\mathcal{A}_i$ is w^*-closed and convex regarding $\{x^*\} \times \mathbb{R}$ for $i \in \{O, W\}$, and w^*-closed regarding $\{x^*\} \times \mathbb{R}$ for $i \in \{C, L_h, L_k\}$.*

Proof We discuss the five possible cases.

• Case $i = O$: By Proposition 4.5 we have

$$(\{x^*\} \times \mathbb{R}) \cap \operatorname{cl} \operatorname{conv}(\mathcal{A}_O) = (\{x^*\} \times \mathbb{R}) \cap \operatorname{epi} p^* = \{x^*\} \times \{r \in \mathbb{R} : p^*(x^*) \leq r\}. \tag{12}$$

Let us begin with the case that $p^*(x^*) = +\infty$. By (12) we have

$(\{x^*\}\times\mathbb{R})\cap \operatorname{cl}\operatorname{conv}(\mathcal{A}_O) = \emptyset$ and, hence, $\mathcal{A}_O$ is w^*-closed and convex regarding $\{x^*\} \times \mathbb{R}$. By Proposition 4.2 we have $\varphi_O(x^*) = +\infty$, and

$$\begin{aligned}\inf_{x\in F}\{f(x) - \langle x^*, x\rangle\} &= -p^*(x^*) = -\infty = -\varphi_O(x^*) = \\ &= \max_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \inf_{x\in X} \left\{ f(x) - \langle x^*, x\rangle + \sum_{t\in T} \lambda_t g_t(x, u_t) \right\},\end{aligned}$$

the maximum being attained at each $(u, \lambda) \in U \times \mathbb{R}_+^{(T)}$. So, if $p^*(x^*) = +\infty$, then both statements in Theorem 4.1 are true.

Consider now that $p^*(x^*) \neq +\infty$. Since $\operatorname{dom} p \neq \emptyset$, we get $p^*(x^*) \in \mathbb{R}$. By Proposition 4.5 we have $(x^*, p^*(x^*)) \in \operatorname{cl}\operatorname{conv}(\mathcal{A}_O)$.

Assume first that $\mathcal{A}_O$ is w^*-closed and convex regarding $\{x^*\}\times\mathbb{R}$. We then have $(x^*, p^*(x^*)) \in \mathcal{A}_O$, and by the definition of $\mathcal{A}_O$, there exists $(\bar{u}, \bar{\lambda}) \in U \times \mathbb{R}_+^{(T)}$ such that

$$(x^*, p^*(x^*)) \in \operatorname{epi}\left(f + \sum_{t\in T} \overline{\lambda}_t g_t(., \overline{u}_t)\right)^* . \tag{13}$$

Now, combining the definition of φ_O with (13) and Proposition 4.2, one gets

$$\varphi_O(x^*) \leq \left(f + \sum_{t\in T} \overline{\lambda}_t g_t(., \overline{u}_t)\right)^* (x^*) \leq p^*(x^*) \leq \varphi_O(x^*).$$

It follows that

$$p^*(x^*) = \left(f + \sum_{t\in T} \overline{\lambda}_t g_t(., \overline{u}_t)\right)^* (x^*) = \min_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \left(f + \sum_{t\in T} \lambda_t g_t(., u_t)\right)^* (x^*),$$

and, taking Proposition 4.3(a) into account, optimistic robust strong duality holds at x^*.

Conversely, assume that optimistic robust strong duality holds at x^*, and let $(x^*, r) \in \operatorname{cl}\operatorname{conv}(\mathcal{A}_O) = \operatorname{epi} p^*$. One has to check that $(x^*, r) \in \mathcal{A}_O$. As the optimistic robust strong duality holds at x^*, there exists $(\overline{u},\overline{\lambda}) \in U \times \mathbb{R}_+^{(T)}$ such that $\left(f+\sum_{t\in T}\overline{\lambda}_t g_t(.,\overline{u}_t)\right)^*(x^*) = p^*(x^*) \leq r$ (see Proposition 4.3(a)), showing that $(x^*,r) \in \operatorname{epi}\left(f+\sum_{t\in T}\overline{\lambda}_t g_t(.,\overline{u}_t)\right)^* \subset \mathcal{A}_O$.

• Case $i = C$: As in Case $i = O$, we first observe that both statements are true when $p^*(x^*) = +\infty$. Assume now that $p^*(x^*) \in \mathbb{R}$, and note that $(x^*, p^*(x^*)) \in \operatorname{epi} p^* = \operatorname{cl}(\mathcal{A}_C)$ by Proposition 4.5. If $\mathcal{A}_C$ is w^*-closed regarding $\{x^*\} \times \mathbb{R}$, then we get $(x^*, p^*(x^*)) \in \mathcal{A}_C$, and there exists $\overline{\lambda} \in \mathbb{R}_+^{(\mathfrak{U})}$ such that

$$(x^*, p^*(x^*)) \in \operatorname{epi}\left(f+\sum_{(t,u_t)\in\mathfrak{U}}\overline{\lambda}_{(t,u_t)}g_t(.,u_t)\right)^*, \text{ or}$$

$$\left(f+\sum_{(t,u_t)\in\mathfrak{U}}\overline{\lambda}_{(t,u_t)}g_t(.,u_t)\right)^*(x^*) \leq p^*(x^*).$$

The last inequality, together with Proposition 4.2 and the definition of φ_C, gives rise to

$$p^*(x^*) \leq \varphi_C(x^*) \leq \left(f+\sum_{(t,u_t)\in\mathfrak{U}}\overline{\lambda}_{(t,u_t)}g_t(.,u_t)\right)^*(x^*) \leq p^*(x^*),$$

or, equivalently, there exists $\overline{\lambda} \in \mathbb{R}_+^{(\mathfrak{U})}$ such that

$$p^*(x^*) = \left(f+\sum_{(t,u_t)\in\mathfrak{U}}\overline{\lambda}_{(t,u_t)}g_t(.,u_t)\right)^*(x^*) = \min_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}}\left(f+\sum_{(t,u_t)\in\mathfrak{U}}\lambda_{(t,u_t)}g_t(.,u_t)\right)^*(x^*),$$

which (taking Proposition 4.3(b) into account) means that classic robust strong duality holds at x^*.

Conversely, assume that classic robust strong duality holds at x^* and let $(x^*,r) \in \operatorname{cl}(\mathcal{A}_C) = \operatorname{epi} p^*$ (see Proposition 4.5). One has to prove that $(x^*,r) \in \mathcal{A}_C$. Note that, by Proposition 4.3(b), classical robust strong duality at x^* means that

$$p^*(x^*) = \min_{\lambda \in \mathbb{R}_+^{(\mathfrak{U})}} \left(f + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(.,u_t) \right)^* (x^*).$$

So, there exists $\bar{\lambda} = (\bar{\lambda}_{(t,u_t)}) \in \mathbb{R}_+^{(\mathfrak{U})}$ such that $\left(f + \sum_{(t,u_t)\in\mathfrak{U}} \bar{\lambda}_{(t,u_t)} g_t(.,u_t) \right)^* (x^*) = p^*(x^*) \leq r$, entailing

$$(x^*, r) \in \operatorname{epi} \left(f + \sum_{(t,u_t)\in\mathfrak{U}} \bar{\lambda}_{(t,u_t)} g_t(.,u_t) \right)^* \subset \mathcal{A}_C.$$

• Case $i = W$: We firstly recall that, by Proposition 4.3(c),

$$\inf_{x\in F} \{f(x) - \langle x^*, x\rangle\} = \max_{u\in U} \inf_{x\in F_u} \{f(x) - \langle x^*, x\rangle\} \tag{14}$$

is equivalent to

$$p^*(x^*) = \min_{u\in U} (f + i_{F_u})^* (x^*). \tag{15}$$

Similar to the proofs of Cases $i = O$ and $i = C$, both statements hold if $p^*(x^*) = +\infty$. Assume now $p^*(x^*) \in \mathbb{R}$. Then, $(x^*, p^*(x^*)) \in \operatorname{epi} p^* = \overline{\operatorname{co}}\, \mathcal{A}_W$ by Proposition 4.5. If $\mathcal{A}_W$ is w^*-closed and convex regarding $\{x^*\} \times \mathbb{R}$, then $(x^*, p^*(x^*)) \in \mathcal{A}_W$, and there is $\bar{u} \in U$ such that $(x^*, p^*(x^*)) \in \operatorname{epi}(f + i_{F_{\bar{u}}})^*$, and so $(f + i_{F_{\bar{u}}})^*(x^*) \leq p^*(x^*)$. Applying Proposition 4.2, we get

$$p^*(x^*) \leq \varphi_W(x^*) \leq (f + i_{F_{\bar{u}}})^*(x^*) \leq p^*(x^*),$$

which means that (15) holds, and so, (14) does, too.

Conversely, assume that (14) holds and let $(x^*, r) \in \operatorname{cl}\operatorname{conv}(\mathcal{A}_W) = \operatorname{epi} p^*$. We now show that $(x^*, r) \in \mathcal{A}_W$. As (14) holds, (15) does, too, and there exists $\bar{u} \in U$ such that $(f + i_{F_{\bar{u}}})^*(x^*) = p^*(x^*) \leq r$, which yields

$(x^*, r) \in \operatorname{epi}(f + i_{F_{\bar{u}}})^* \subset \mathcal{A}_W$.

• The proofs for cases $i = L_h$, assuming $h_t \in \Upsilon(X)$ for all $t \in T$, and $i = L_k$, assuming $k_u \in \Upsilon(X)$ for all $u \in U$, are completely similar to the proof for the case $i = C$ (they also can be derived from [27, Theorem 1] with f being replaced by $f - x^*$). □

Next very important results are consequences of Theorem 4.1.

Theorem 4.2 (Robust strong duality) *Let $i \in \{O, C, W, L_h, L_k\}$ be such that $h_t \in \Upsilon(X)$ for all $t \in T$ when $i = L_h$, and that $k_u \in \Upsilon(X)$ for all $u \in U$ when $i = L_k$. Then, $(RD^i)-$ robust strong duality holds for the robust counterpart (RP), i.e., $\inf(RP) = \max(RD^i)$, if and only if $\mathcal{A}_i$ is w^*-closed and convex regarding $\{0_{x^*}\} \times \mathbb{R}$ for $i \in \{O, W\}$, and w^*-closed regarding $\{0_{x^*}\} \times \mathbb{R}$ for $i \in \{C, L_h, L_k\}$.*

Theorem 4.3 (Stable strong duality) *Let $i \in \{O, C, W, L_h, L_k\}$ be such that $h_t \in \Upsilon(X)$ for all $t \in T$ when $i = L_h$, and that $k_u \in \Upsilon(X)$ for all $u \in U$ when $i = L_k$. Then, $(RD^i)-$ robust strong duality holds stably if and only if $\mathcal{A}_i$ is w^*-closed and convex for $i \in \{O, W\}$, and w^*-closed for $i \in \{C, L_h, L_k\}$.*

In Example 4.1, $\mathcal{A}_i$ is closed and convex regarding $\{(0,0)\} \times \mathbb{R}$ if and only if M_i is closed and convex regarding $\{-c^*\} \times \mathbb{R}$, which is true for $i = L_h$ and false for $i \in \{C, W, L_k\}$, as well as for $i = O$ when $\mathcal{A}_i \cap (\{(0,0)\} \times \mathbb{R}) \neq \emptyset$. Thus, only (RD^{L_h}) is solvable, independently of the objective function. Since $\mathcal{A}_{L_h}$ is w^*-closed and convex, (RD^{L_h}) enjoys stable robust strong duality.

Concerning optimistic robust strong duality (see Theorem 4.1, Case $i = O$ and the corresponding corollaries) the following question is of particular interest: when is M_O (hence, $\mathcal{A}_O = \text{epi}\, f^* + M_O$) convex? Next result provides an answer for that, including the "convexity condition" introduced in [10] for robust linear semi-infinite problems, and extending [7, Proposition 2.3] to robust infinite convex programs.

Proposition 4.6 *Assume that, for each $t \in T$, U_t is a convex subset of Z_t and that, for each $x \in X$, the function $u_t \in U_t \mapsto g_t(x, u_t)$ is concave. Then, the robust moment cone M_O is convex.*

Proof Let $(x_1^*, r_1), (x_2^*, r_2) \in M_O$. Since M_O is a cone, it suffices to check that $(x_1^* + x_2^*, r_1 + r_2) \in M_O$. Taking into account (5) and (6), there exist $(u^1, \lambda^1), (u^2, \lambda^2) \in U \times \mathbb{R}_+^{(T)}$ such that

$$\langle x_1^*, x\rangle - \sum_{t\in T} \lambda_t^1 g_t(x, u_t^1) \le r_1, \ \forall x \in X,$$

$$\langle x_2^*, x\rangle - \sum_{t\in T} \lambda_t^2 g_t(x, u_t^2) \le r_2, \ \forall x \in X.$$

Define $\lambda^0 := \lambda^1 + \lambda^2 \in \mathbb{R}_+^{(T)}$, and $u^0 \in \prod_{t\in T} Z_t$ such that

$$u_t^0 := \begin{cases} u_t^1, & \text{if } \lambda_t^0 = 0 \text{ (i.e., } \lambda_t^1 = \lambda_t^2 = 0), \\ \frac{\lambda_t^1}{\lambda_t^0} u_t^1 + \frac{\lambda_t^2}{\lambda_t^0} u_t^2, & \text{else (i.e., if } \lambda_t^0 > 0). \end{cases}$$

Since, for each $t \in T$, U_t is convex, we have $u^0 \in \prod_{t\in T} U_t = U$. Let us check that $(x_1^* + x_2^*, r_1 + r_2) \in \text{epi} \left(\sum_{t\in T} \lambda_t^0 g_t(., u_t^0)\right)^*$, and this will conclude the proof. For each $x \in X$ we have, since $g_t(x, .)$ is concave,

$$\sum_{t\in T} \lambda_t^0 g_t(x, u_t^0) \ge \sum_{t\in T} \left(\lambda_t^1 g_t(x, u_t^1) + \lambda_t^2 g_t(x, u_t^2)\right),$$

and for any $x \in X$,

$$\begin{aligned}\langle x_1^* + x_2^*, x\rangle - \sum_{t\in T} \lambda_t^0 g_t(x, u_t^0) &\leq \langle x_1^*, x\rangle - \sum_{t\in T} \lambda_t^1 g_t(x, u_t^1) + \langle x_2^*, x\rangle - \sum_{t\in T} \lambda_t^2 g_t(x, u_t^2) \\ &\leq r_1 + r_2.\end{aligned}$$

□

The last result in this section provides a sufficient condition for M_W to be convex. Recall that, since $\operatorname{epi} i_{F_u}^* = \operatorname{cl}(K_u)$, for all $u \in U$, $M_W = \bigcup_{u\in U} \operatorname{epi} i_{F_u}^*$, where $F_u = \{x \in X : g_t(x, u_t) \leq 0, t \in T\}$. Recall also that $\mathfrak{U}$ is the *disjoint union* of the sets U_t, $t \in T$, and that $U = \prod_{t\in T} U_t$.

Definition 4.1 The family of functions $(g_t(\cdot, u_t))_{(t,u_t)\in\mathfrak{U}}$ is *filtering* iff for any two elements $u, v \in U$ there exists a third one $w \in U$ such that

$$\max\{g_t(\cdot, u_t), g_t(\cdot, v_t)\} \leq g_t(\cdot, w_t),\ \forall t \in T. \tag{16}$$

Proposition 4.7 *If* $(g_t(\cdot, u_t))_{(t,u_t)\in\mathfrak{U}}$ *is filtering, then* M_W *is convex.*

Proof Since M_W is a cone, we have to check that it is stable for the sum. Let $(x^*, r), (y^*, s) \in M_W$. Then, there exist $u, v \in U$ such that $i_{F_u}^*(x^*) \leq r$ and $i_{F_v}^*(y^*) \leq s$. From the filtering assumption, we can take $w \in U$ such that (16) holds and we get $F_w \subset F_u \cap F_v$. We then have

$$\begin{aligned} i_{F_w}^*(x^* + y^*) &\leq i_{F_w}^*(x^*) + i_{F_w}^*(y^*) \leq i_{F_u}^*(x^*) + i_{F_v}^*(y^*) \\ &\leq r + s, \end{aligned}$$

and so $(x^*, r) + (y^*, s) \in \operatorname{epi} i_{F_w}^* \subset M_W$. □

5 Uniformly Robust Strong Duality and Complements

5.1 Uniformly Robust Strong Duality

Recall that, according to Definition 3.1, if F is the feasible set of (RP_{x^*}), then the $\left(RD^i\right)-$*robust strong duality* holds *uniformly* if $\left(RD^i\right)-$*robust strong duality holds at* $x^* = 0_{X^*}$ for any function f in the family

$$\mathcal{F} = \{f \in \Gamma(X) : f \text{ is continuous at a point of } F\}.$$

Applying Theorem 4.3 we can easily prove the following results, which extend [7, Theorems 3.1, 3.2] and [10, Theorems 1, 2] for $i = O$, [10, Proposition 4] for $i = C$, and [7, Theorem 5.3] for $i = L_h$.

Theorem 5.1 (Uniform robust strong duality) *Let* $i \in \{O, C, W, L_h, L_k\}$ *be such that* $h_t \in \Upsilon(X)$ *for all* $t \in T$ *when* $i = L_h$*, and that* $k_u \in \Upsilon(X)$ *for all* $u \in U$ *when* $i = L_k$*. Then, the following statements are equivalent:*

(i) *The robust moment cone* M_i *is* w^**-closed and convex for* $i \in \{O, W\}$*, and* w^**-closed for* $i \in \{C, L_h, L_k\}$*,*

$$(ii)\begin{cases} \inf\limits_{x\in F} \langle x^*, x\rangle = \max\limits_{(u,\lambda)\in U\times R_+^{(T)}} \inf\limits_{x\in X} \left\{\langle x^*, x\rangle + \sum_{t\in T} \lambda_t g_t(x,u_t)\right\}, \ \forall x^*\in X^*, \textit{ if } i = O, \\ \inf\limits_{x\in F} \langle x^*, x\rangle = \max\limits_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \inf\limits_{x\in X} \left\{\langle x^*, x\rangle + \sum\limits_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t)\right\}, \ \forall x^*\in X^*, \textit{ if } i = C, \\ \inf\limits_{x\in F} \langle x^*, x\rangle = \max\limits_{u\in U} \inf\limits_{x\in X} \left\{\langle x^*, x\rangle : g_t(x,u_t) \le 0, \ \forall t\in T\right\}, \ \forall x^*\in X^*, \quad \textit{if } i = W, \\ \inf\limits_{x\in F} \langle x^*, x\rangle = \max\limits_{\lambda\in\mathbb{R}_+^{(T)}} \inf\limits_{x\in X} \sup\limits_{u\in U} \left\{\langle x^*, x\rangle + \sum\limits_{t\in T} \lambda_t g_t(x,u_t)\right\}, \ \forall x^*\in X^*, \quad \textit{if } i = L_h, \\ \inf\limits_{x\in F} \langle x^*, x\rangle = \max\limits_{\lambda\in\mathbb{R}_+^{(U)}} \inf\limits_{x\in X} \left\{\langle x^*, x\rangle + \sum\limits_{u\in U} \lambda_u \sup\limits_{t\in T} g_t(x,u_t)\right\}, \ \forall x^*\in X^*, \quad \textit{if } i = L_k. \end{cases}$$

(iii) $\left(RD^i\right)-$ *strong duality holds uniformly.*

Proof The proofs are very similar for $i \in \{O, C, W, L_h, L_k\}$, so we may assume $i = O$. Applying Proposition 4.1 for $f \equiv 0$ we get that

$$\mathcal{A}_O = \operatorname{epi} f^* + M_O = \{0_{X^*}\} \times \mathbb{R}_+ + M_O = M_O,$$

and the equivalence $[(i) \Leftrightarrow (ii)]$ holds.

$[(ii) \Leftrightarrow (iii)]$ Note that $\inf_{x \in F} f(x) = -(f + i_F)^*(0_{X^*})$. By Moreau-Rockafellar theorem, there exists $\bar{x}^* \in X^*$ such that $-\inf_{x \in F} f(x) = f^*(\bar{x}^*) + i_F^*(-\bar{x}^*)$. If (ii) holds, then there will exist $(\bar{u}, \bar{\lambda}) \in U \times \mathbb{R}_+^{(T)}$, $\bar{\lambda} = (\bar{\lambda}_t) \in \mathbb{R}_+^{(T)}$, such that

$$\begin{aligned} i_F^*(-\bar{x}^*) &= -\inf_{x \in F} \langle \bar{x}^*, x \rangle = -\inf_{x \in X} \left\{ \langle \bar{x}^*, x \rangle + \sum_{t \in T} \bar{\lambda}_t g_t(x, \bar{u}_t) \right\} \\ &= \left(\sum_{t \in T} \bar{\lambda}_t g_t(., \bar{u}_t) \right)^* (-\bar{x}^*), \end{aligned}$$

and we have

$$\begin{aligned} f^*(\bar{x}^*) + i_F^*(-\bar{x}^*) &= f^*(\bar{x}^*) + \left(\sum_{t \in T} \bar{\lambda}_t g_t(., \bar{u}_t) \right)^* (-\bar{x}^*) \\ &\geq \left(f + \sum_{t \in T} \bar{\lambda}_t g_t(., \bar{u}_t) \right)^* (0_{X^*}). \end{aligned}$$

Consequently, by weak-duality, we get

$$\begin{aligned} &\sup_{(u,\lambda) \in U \times \mathbb{R}_+^{(T)}} \inf_{x \in X} \left\{ f(x) + \sum_{t \in T} \lambda_t g_t(x, u_t) \right\} \leq \inf_{x \in F} f(x) = -\{ f^*(\bar{x}^*) + i_F^*(-\bar{x}^*) \} \\ &\leq -\left(f + \sum_{t \in T} \bar{\lambda}_t g_t(., \bar{u}_t) \right)^* (0_{X^*}) = \inf_{x \in X} \left\{ f(x) + \sum_{t \in T} \bar{\lambda}_t g_t(x, \bar{u}_t) \right\} \\ &\leq \sup_{(u,\lambda) \in U \times \mathbb{R}_+^{(T)}} \inf_{x \in X} \left\{ f(x) + \sum_{t \in T} \lambda_t g_t(x, u_t) \right\}, \end{aligned}$$

and (iii) holds.

Since $[(iii) \Rightarrow (ii)]$ is obvious, the proof is complete. □

As a consequence of the above results, $\left(RD^{L_k}\right)-$ strong duality uniformly holds for the problem in Example 4.1.

5.2 Robust Duality for Convex Problems with Linear Objective Function

The robust counterpart with linear objective $f(x) = \langle c^*, x\rangle$, $c^* \in X^*$, has the form

$$(ROLP) \qquad \inf_{x\in X} \langle c^*, x\rangle \qquad \text{s.t. } g_t(x, u_t) \le 0,\ \forall (t, u_t) \in \mathfrak{U},$$

with corresponding robust dual problems

$$(ROLD^O) \qquad \sup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \inf_{x\in X} \left\{ \langle c^*, x\rangle + \sum_{t\in T} \lambda_t g_t(x, u_t) \right\},$$

$$(ROLD^C) \qquad \sup_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \inf_{x\in X} \left\{ \langle c^*, x\rangle + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x, u_t) \right\},$$

$$(ROLD^W) \qquad \sup_{u\in U} \inf_{x\in X} \left\{ \langle c^*, x\rangle : g_t(x, u_t) \le 0,\ \forall t \in T \right\},$$

$$(ROLD^{L_h}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in X} \sup_{u\in U} \left\{ \langle c^*, x\rangle + \sum_{t\in T} \lambda_t g_t(x, u_t) \right\},$$

$$(ROLD^{L_k}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(U)}} \inf_{x\in X} \sup_{t\in T} \left\{ \langle c^*, x\rangle + \sum_{u\in U} \lambda_u k_u(x) \right\}.$$

We now give a geometric interpretation of the optimal values of the above five dual problems in terms of the corresponding moment cones.

Proposition 5.1 (Robust duality and moment cones)

Let $i \in \{O, C, W, L_h, L_k\}$ be such that $h_t \in \Upsilon(X)$ for all $t \in T$ when $i = L_h$, and that $k_u \in \Upsilon(X)$ for all $u \in U$ when $i = L_k$. Then,

$$\sup(ROLD^i) = \sup\left\{ r \in \mathbb{R} : -(c^*, r) \in M_i \right\}. \tag{17}$$

Proof Given $i \in \{O, C, W, L_h, L_k\}$, from the definitions of φ_i and $\mathcal{A}_i$ one gets

$$\operatorname{epi}_s \varphi_i \subset \mathcal{A}_i \subset \operatorname{epi} \varphi_i,$$

where $\operatorname{epi}_s \varphi_i$ denotes the strict epigraph of φ_i. Consequently,

$$\inf\{r \in \mathbb{R} : (x^*, r) \in \mathcal{A}_i\} = \varphi_i(x^*) = -\sup(RD^i_{x^*}),$$

or, in other words,

$$\sup(RD^i_{x^*}) = \sup\{r \in \mathbb{R} : (x^*, -r) \in \mathcal{A}_i\}. \tag{18}$$

Concerning the moment cones, from Proposition 4.1 we get the identity

$$\mathcal{A}_i = \{c^*\} \times \mathbb{R}_+ + M_i, \tag{19}$$

for $i \in \{O, C, W\}$ by statements (a) and (b), and for $i \in \{L_h, L_k\}$ by statements (c) and (d). Combining (18) and (19), and recalling that $\sup(ROLD^i) = \sup(RD^i_{0_{X^*}})$, one gets

$$\begin{aligned}\sup(ROLD^i) &= \sup\{r \in \mathbb{R} : (0^*_{X^*}, -r) \in \{c^*\} \times \mathbb{R}_+ + M_i\} \\ &= \sup\{r \in \mathbb{R} : \exists \delta \geq 0 \text{ such that } -(c^*, r + \delta) \in M_i\} \\ &= \sup\{r \in \mathbb{R} : -(c^*, r) \in M_i\},\end{aligned}$$

i.e., (17) holds. □

5.3 Robust Duality for Linear Programs

Let us consider the important particular case of *robust linear programs*, which have been already studied in [10] when $X = \mathbb{R}^n$. Putting $f(x) = \langle c^*, x\rangle$, $c^* \in X^*$, and for each $(t, u_t) \in \mathfrak{U}$, $g_t(x, u_t) = \langle x_t^*(u_t), x\rangle - r_t(u_t)$, $(x_t^*(u_t), r_t(u_t)) \in X^* \times \mathbb{R}$, the robust linear infinite problem can be expressed as

$$(RLP) \qquad \inf_{x \in X} \langle c^*, x\rangle \qquad \text{s.t. } \langle x_t^*(u_t), x\rangle \leq r_t(u_t),\ \forall (t, u_t) \in \mathfrak{U}.$$

It is easy to see that the corresponding moment cones are

$$M_O := \bigcup_{u=(u_t)_{t\in T}\in U} \operatorname{conv}\operatorname{cone}\{(x_t^*(u_t), r_t(u_t)),\ t\in T;\ (0_{X^*},1)\},$$

$$M_C := \operatorname{conv}\operatorname{cone}\{(x_t^*(u_t), r_t(u_t)),\ (t,u_t)\in\mathfrak{U};\ (0_{X^*},1)\},$$

$$M_W := \bigcup_{u=(u_t)_{t\in T}\in U} \operatorname{cl}\operatorname{conv}\operatorname{cone}\{(x_t^*(u_t), r_t(u_t)),\ t\in T\},$$

$$M_{L_h} := \operatorname{conv}\operatorname{cone} \bigcup_{t\in T}\operatorname{cl}\operatorname{conv}\left(\bigcup_{u_t\in U_t}\{(x_t^*(u_t), r_t(u_t)) + \mathbb{R}_+(0_{X^*},1)\};\ (0_{X^*},1)\right),$$

$$M_{L_k} := \operatorname{conv}\operatorname{cone} \bigcup_{u\in U}\operatorname{cl}\operatorname{conv}\left(\bigcup_{t\in T}\{(x_t^*(u_t), r_t(u_t)) + \mathbb{R}_+(0_{X^*},1)\};\ (0_{X^*},1)\right).$$

From (3),

$$\begin{aligned}\inf(RLP) &= \sup\{r\in\mathbb{R} : \langle c^*, x\rangle \geq r, \forall x\in F\} \\ &= \sup\left\{r\in\mathbb{R} : -(c^*, r)\in \overline{M}_C\right\}.\end{aligned}$$

We also associate with problem (RLP) the corresponding dual problems:

$$(RLD^O) \qquad \sup_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}} \inf_{x\in X} \left\{\langle c^*, x\rangle + \textstyle\sum_{t\in T}\lambda_t(\langle x_t^*(u_t), x\rangle - r_t(u_t))\right\},$$

$$(RLD^C) \qquad \sup_{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}} \inf_{x\in X} \left\{\langle c^*, x\rangle + \sum_{(t,u_t)\in\mathfrak{U}}\lambda_{(t,u_t)}(\langle x_t^*(u_t), x\rangle - r_t(u_t))\right\},$$

$$(RLD^W) \qquad \sup_{u\in U}\inf_{x\in X}\{\langle c^*, x\rangle : \langle x_t^*(u_t), x\rangle \leq r_t(u_t),\ \forall t\in T\},$$

$$(RLD^{L_h}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in X}\sup_{u\in U} \left\{\langle c^*, x\rangle + \sum_{t\in T}\lambda_t(\langle x_t^*(u_t), x\rangle - r_t(u_t))\right\},$$

$$(RLD^{L_k}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(U)}} \inf_{x\in X} \left\{\langle c^*, x\rangle + \sum_{u\in U}\lambda_u \sup_{t\in T}(\langle x_t^*(u_t), x\rangle - r_t(u_t))\right\}.$$

Proposition 5.2 *Let $i\in\{O, C, W, L_h, L_k\}$ be such that $h_t\in\Upsilon(X)$ for all $t\in T$ when $i = L_h$, and that $k_u\in\Upsilon(X)$ for all $u\in U$ when $i = L_k$. Then, the following statements hold:*

(i) $\sup(RLD^i) = \sup\{r \in \mathbb{R} : -(c^*, r) \in M_i\}$.

(ii) $(RLD^i)-$ *strong duality holds uniformly for any* $c^* \in X^*$ *if and only if the robust moment cone* M_i *is* w^**-closed and convex for* $i \in \{O, W\}$, *and* w^**-closed for* $i \in \{C, L_h, L_k\}$.

Proof (i) It follows immediately from Proposition 5.1.

(ii) It is a consequence of the equivalence $[(i) \Leftrightarrow (ii)]$ in Theorem 5.1. □

An alternative proof of Proposition 5.2(i) for $i \in \{C, L_h, L_k\}$ can be obtained from [26, Lemma 4.3], by taking into account that (RLD^C), (RLD^{L_h}) and (RLD^{L_h}) are nothing else than the Lagrange dual problems corresponding to linear representations of the feasible set of (RLP) with characteristic cones M_C, M_{L_h}, and M_{L_k}, respectively. Proposition 5.2(ii) generalizes [22, Theorem 8.4], where $i = C$ and $X = \mathbb{R}^n$.

Let us revisit Example 4.1. According to (17), $\sup(RLD^i) = -\|c^*\|$, $i \in \{C, L_h, L_k\}$. We now check the fulfilment of (17) for $i \in \{O, W\}$. Given $(u, \lambda) \in U \times \mathbb{R}_+^{([0,1])}$,

$$\begin{aligned}
&\inf_{x\in\mathbb{R}^2}\left\{\langle c^*, x\rangle + \sum_{t\in[0,1]} \lambda_t(\langle x_t^*(u_t), x\rangle - r_t(u_t))\right\}\\
&= \sum_{t\in]0,1]} \lambda_t(-t) - \lambda_0\left(\tfrac{r_0+1}{r_0}\right) + \inf_{x\in\mathbb{R}^2}\{\langle c^*, x\rangle + \lambda_0\left((\cos\alpha_0)\,x_1 + (\sin\alpha_0)\,x_2\right)\}\\
&= \sum_{t\in]0,1]} \lambda_t(-t) - \lambda_0\left(\tfrac{r_0+1}{r_0}\right) + \inf_{x\in\mathbb{R}^2}\langle c^* + \lambda_0(\cos\alpha_0, \sin\alpha_0), x\rangle,
\end{aligned}$$

and so

$$\inf_{x\in\mathbb{R}^2}\left\{\langle c^*,x\rangle+\sum_{t\in[0,1]}\lambda_t(\langle x_t^*(u_t),x\rangle-r_t(u_t))\right\}$$
$$=\begin{cases}\sum\limits_{t\in]0,1]}\lambda_t(-t)-\lambda_0\left(\frac{r_0+1}{r_0}\right), \text{ if } c^*=-\lambda_0(\cos\alpha_0,\sin\alpha_0)\\ -\infty, \text{ otherwise.}\end{cases}$$

Therefore,

$$\sup(RLD^O)=\begin{cases}\sup\limits_{(u,\lambda)\in U\times\mathbb{R}_+^{(T)}}\left\{\sum\limits_{t\in]0,1]}\lambda_t(-t)-\lambda_0\left(\frac{r_0+1}{r_0}\right)\right\}, \text{ if } c^*\in-\lambda_0 D,\\ -\infty, \text{ otherwise,}\end{cases}$$
$$=\begin{cases}\sup_{\lambda\in\mathbb{R}_+^{(T)}}\left\{\sum\limits_{t\in]0,1]}\lambda_t(-t)-\|c^*\|\right\}, \text{ if } -c^*\in\mathbb{R}_{++}D,\\ -\infty, \text{ otherwise,}\end{cases}$$
$$=\begin{cases}-\|c^*\|, \text{ if } -c^*\in\mathbb{R}_{++}D,\\ -\infty, \quad \text{else.}\end{cases}$$

(observe that $c^*\in-\lambda_0 D$ with $c^*\neq(0,0)$ entails that $\lambda_0>0$, and so the supremum of (RLD^O) is not attained).

We now take $u\in U$. Eliminating redundant constraints, (P^u) can be expressed as

$$(P^u)\qquad \inf_{x\in\mathbb{R}^2}\langle c^*,x\rangle \qquad \text{s.t. } (\cos\alpha_0)\,x_1+(\sin\alpha_0)\,x_2\leq\frac{r_0+1}{r_0},$$

with

$$\inf(P^u)=\begin{cases}-\left(\frac{r_0+1}{r_0}\right)\|c^*\|, \text{ if } (\cos\alpha_0,\sin\alpha_0)=-\frac{c^*}{\|c^*\|},\\ -\infty, \qquad \text{else.}\end{cases}$$

Thus, for $i\in\{O,W\}$, if $-c^*\notin\mathbb{R}_{++}D$, we have an infinite robust duality gap; otherwise,

$$\sup(RLD^i)=\min(RLP),$$

i.e., we have $(RLD^i)-$robust zero-gap (but not strong) duality. In fact, $\mathcal{A}_i$ is closed and convex regarding $\{0_{x^*}\} \times \mathbb{R}$ if and only if M_i is closed and convex regarding $\{-c^*\} \times \mathbb{R}$, which is not the case. Observe also that

$$\sup\{r \in \mathbb{R} : -(c^*, r) \in M_i\} = \begin{cases} -\|c^*\|, \text{ if } -c^* \in \mathbb{R}_{++}D, \\ -\infty, \text{ else}, \end{cases}$$

so that (17) holds for $i \in \{O, W\}$.

Finally, we observe that only $\left(RLD^{L_h}\right)$ enjoys uniform robust strong duality.

5.4 Reverse Robust Strong Duality

In addition to the main results on robust strong duality provided in the previous section, some results on reverse robust strong duality can be derived from convex infinite duality, recently revisited in [27]. In fact, Theorem 5.3 below is a slight adaptation of [27, Theorem 2] to robust case. Recall that a function $h \in \Gamma(X)$ is *weakly-inf-locally compact* when for each $r \in \mathbb{R}$, the *sublevel set* $[h \leq r]$ is weakly-locally compact (i.e., locally compact for the weak-topology in X). We also denote by h_∞ the recession function of h (whose epigraph is the recession cone of $\operatorname{epi} h$).

Proposition 5.3 *Assume that* $\sup(RD^C) \neq +\infty$, *and additionally, the following conditions are fulfilled:*

(a) $\exists \overline{\lambda} \in \mathbb{R}_+^{(\mathfrak{U})} : f + \sum_{(t,u_t)\in\mathfrak{U}} \overline{\lambda}_{(t,u_t)} g_t(., u_t)$ *is weakly-inf-locally compact,*

(b) *the recession cone of* (RP), *namely*

$$[f_\infty \leq 0] \bigcap \left(\bigcap_{(t,u_t)\in\mathfrak{U}} [(g_t(., u_t))_\infty \leq 0] \right),$$

is a linear space.

Then, $\min\inf(RP) = \sup(RD^C)$ *and the optimal set of* (RP) *is the sum of a non-empty, weakly-compact and convex set and a finite dimensional linear space.*

In the same way, Theorem 5.4 below is a simple adaptation of [27, Theorem 3]. The topology on $\mathbb{R}^{(\mathfrak{U})} \times \mathbb{R}$ is the product topology.

Proposition 5.4 *Assume* $\sup(RD^C) \neq -\infty$*. Then, the following statements are equivalent:*

(i) $\min(RP) = \sup(RD^C)$.

(ii) $\bigcup_{x \in \operatorname{dom} f} \left(\left((g_t(x, u_t) \right)_{(t,u_t) \in \mathfrak{U}}, f(x) \right) + \left(\mathbb{R}_+^{\mathfrak{U}} \times \mathbb{R}_+ \right)$ *is closed regarding* $\{0_{\mathbb{R}^{\mathfrak{U}}}\} \times \mathbb{R}$.

6 The General Uncertain Problem

We consider in this last section the general uncertain problem

$$(Q) \qquad \left\{ \inf_{x \in X} \{ f(x, v) \qquad \text{s.t. } g_t(x, u_t) \leq 0, \ \forall t \in T \} \right\}_{(u_t)_{t \in T} \in U, \ v \in V}, \tag{20}$$

where $U = \prod_{t \in T} U_t$, V is another uncertainty set, which is a subset of some lcHtvs, and $f(., v) \in \Gamma(X)$, for all $v \in V$. This problem admits the following *pessimistic* reformulation as an uncertain problem with deterministic objective function (of the type studied in the previous sections):

$$(P) \left\{ \inf_{(x,r) \in X \times \mathbb{R}} \{ F(x, r) \text{ s.t. } G_t(x, r, u_t) \leq 0, \ t \in T, \ H(x, r, v) \leq 0 \} \right\}_{(u_t)_{t \in T} \in U, \ v \in V},$$

where

$$F(x, r) := r, \ G_t(x, r, u_t) = g_t(x, u_t) \text{ and } H(x, r, v) = f(x, v) - r.$$

The totally explicit robust counterpart of (P) is the problem

$$(RPC) \qquad \inf_{(x,r)\in X\times\mathbb{R}} \left\{ F(x,r) \text{ s.t. } \begin{array}{c} G_t(x,r,u_t) \leq 0,\ \forall (t,u_t)\in\mathfrak{U}, \\ \text{and } H(x,r,v)\leq 0,\ \forall v\in V, \end{array} \right\},$$

where $\mathfrak{U} := \{(t,u_t) : t\in T,\ u_t\in U_t\}$.

The classical dual of the convex infinite problem (RPC) is

$$(RDC) \qquad \sup_{\substack{\lambda\in\mathbb{R}_+^{(\mathfrak{U})} \\ \mu\in\mathbb{R}_+^{(V)}}} \left[\inf_{(x,r)\in X\times\mathbb{R}} \left\{ \begin{array}{c} F(x,r) + \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} G_t(x,r,u_t) \\ + \sum_{v\in V} \mu_v H(x,r,v) \end{array} \right\} \right].$$

According to the definition of the functions F, G_t and H, we easily obtain

$$(RDC) \qquad \sup_{\substack{\lambda\in\mathbb{R}_+^{(\mathfrak{U})}, \\ \mu\in\mathbb{R}_+^{(V)}, \\ \sum_{v\in V}\mu_v=1}} \left[\inf_{x\in X} \left\{ \sum_{(t,u_t)\in\mathfrak{U}} \lambda_{(t,u_t)} g_t(x,u_t) + \sum_{v\in V} \mu_v f(x,v) \right\} \right].$$

We have

$$(G_t(.,.,u_t))^*(x^*,s) = \begin{cases} g_t(.,u_t))^*(x^*), \text{ if } s=0, \\ +\infty,\ \text{else}, \end{cases}$$

$$\operatorname{epi}(G_t(.,.,u_t))^* = \left\{(x^*,0,\theta)\in X^*\times\mathbb{R}\times\mathbb{R} :\ g_t(.,u_t))^*(x^*)\leq\theta\right\},$$

$$(H(.,.,v))^*(x^*,s) = \begin{cases} f(.,v))^*(x^*), \text{ if } s=-1, \\ +\infty,\ \text{else}, \end{cases}$$

$$\operatorname{epi}(H(.,.,v))^* = \left\{(x^*,-1,\theta)\in X^*\times\mathbb{R}\times\mathbb{R} :\ f(.,v))^*(x^*)\leq\theta\right\}.$$

Thus, the moment cone associated with (RPC) is

$$M'_C = \operatorname{conv\,cone} \left\{ \begin{array}{c} \{(0_{X^*},0,1)\} \cup \bigcup_{(t,u_t)\in\mathfrak{U}} \{(x^*,0,\theta) :\ g_t(.,u_t))^*(x^*)\leq\theta\} \\ \cup \bigcup_{v\in V} \{(x^*,-1,\theta) :\ f(.,v))^*(x^*)\leq\theta\} \end{array} \right\}.$$

Since

$$F^*(x^*, s) = \begin{cases} 0, \text{ if } x^* = 0_{X^*} \text{ and } s = 1, \\ \\ +\infty, \text{ else}, \end{cases}$$

we get $\operatorname{epi} F^* = \{(0_{X^*}, 1)\} \times \mathbb{R}_+$. Assuming that $\inf(RPC) \neq +\infty$, and applying Theorem 4.1 and Proposition 4.1, we conclude that

$\inf(RPC) = \max(RDC)$ if and only if

$$\{(0_{X^*}, 1)\} \times \mathbb{R}_+ + M'_C \text{ is closed regarding } (0_{X^*}, 0)\} \times \mathbb{R}.$$

7 Conclusions

In most applications of convex optimization, the data defining the nominal problem are uncertain, so that the decision maker must choose among different uncertainty models. Parametric models (stability and sensitivity analyses) are based on embedding the nominal problem into a suitable topological space of admissible perturbed problems, the so-called space of parameters. Sensitivity analysis provides estimations of the impact of a given perturbation of the nominal problem on the optimal value while stability analysis provides conditions under which sufficiently small perturbations of the nominal problem provoke only small changes in the optimal value, the optimal set and the feasible set, as well as approximate distances, in the space of parameters, from the nominal problem to important families of problems. Stochastic optimization, in turn, assumes that the uncertain data are random variables with a known probability distribution and provides either the probability distribution of the optimal value under strong assumptions or its empirical distribution via simulation. Both approaches to uncertain convex optimization, the parametric and the stochastic ones, are considered unrealistic by many practitioners for which it is preferable to describe the uncertainty via sets. Indeed, robust models assume that all instances of the data belong to prescribed sets (the so-called uncertainty sets), and select an "optimal decision" among those which are feasible under any conceivable data. Assuming that the optimal value function f is deterministic, the robust decision makers agree in minimizing f on the set of robust feasible solutions. In contrast with the existing unanimity of the robust optimization community in solving this (pessimistic) primal problem, there exists a variety of possible choices of its (optimistic) dual counterpart. We have chosen in Sections 5 the so-called min-max robust counterpart, which consists of minimizing the worst case for the objective function on the robust feasible set.

This paper examines five different dual pairs in robust convex optimization (two of them already known), each one based on a corresponding moment cone. In particular, we characterize:

– Robust strong (or inf-max) duality in terms of the closedness regarding the vertical axis of the corresponding moment cones.

– Uniform robust strong duality (i.e., the fulfilment of robust strong duality for arbitrary convex objective functions) in terms of the closedness regarding the whole space and the convexity of the moment cones.

– Robust reverse strong (or min-sup) duality in terms of the linearity of the recession cone of the robust primal problem and the closedness of certain set regarding the vertical axis.

Moreover, we analyze robust duality for convex problems with linear objective function $x \mapsto \langle c^*, x\rangle$ and the particular case of robust linear optimization, for which we provide results which are new even in the deterministic setting (when the uncertainty sets are singleton), e.g., the characterization of the optimal value of the five dual problems in terms of the intersection of a vertical line through the point $(-c^*, 0)$ with the corresponding moment cone.

Acknowledgements The authors are grateful to the referees for their constructive comments and helpful suggestions which have contributed to the final preparation of the paper.

This research was supported by the National Foundation for Science & Technology Development (NAFOSTED) of Vietnam, Project 101.01-2015.27, *Generalizations of Farkas lemma with applications to optimization*, by the Ministry of Economy and Competitiveness of Spain and the European Regional Development Fund (ERDF) of the European Commission, ProjectMTM2014-59179-C2-1-P, and by the Australian Research Council, Project DP160100854. Parts of the work of the first author were developed during his visit to the Department of Mathematics, University of Alicante in July 2016, for which he would like to express his sincere thanks to the support and the hospitality he received.

References

1. Ben-Tal A., El Ghaoui L., Nemirovski A., Robust Optimization, Princeton U.P., Princeton (2009)
2. Gabrel V., Murat C., Thiele A., Recent advances in robust optimization: an overview. European J. Oper. Res. 235, 471-483 (2014)

3. Beck A., Ben-Tal A., Duality in robust optimization: Primal worst equals dual best, Oper. Res. Letters 37, 1-6 (2009)

4. Li G.Y., Jeyakumar V., Lee G.M., Robust conjugate duality for convex optimization under uncertainty with application to data classification, Nonlinear Analysis 74, 2327-2341 (2011)

5. Wang W., Fang D., Chen Z., Strong and total Fenchel dualities for robust convex optimization problems, J. Inequal. Appl. 70, 21 (2015)

6. Suzuki S., Kuroiwa D., Lee G.M., Surrogate duality for robust optimization, European J. Oper. Res. 231, 257- 262 (2013)

7. Jeyakumar V., Li G.Y., Strong duality in robust convex programming: complete characterizations, SIAM J. Optim. 20, 3384-3407 (2010)

8. Jeyakumar V., Li G.Y., Robust duality for fractional programming problems with constraint-wise data uncertainty, J. Optim. Theory Appl. 151, 292 - 303 (2011)

9. Jeyakumar V., Li G.Y., Lee G.M., Robust duality for generalized convex programming problems under data uncertainty, Nonlinear Anal. 75, 1362-1373 (2012)

10. Goberna M.A., Jeyakumar V., Li G., López M.A., Robust linear semi-infinite programming duality under uncertainty, Math. Programming 139B, 185-203,(2013)

11. Jeyakumar V., Li G.Y., Srisatkunarajah S., Strong duality for robust minimax fractional programming problems, European J. Oper. Res. 228, 331-336 (2013)

12. Jeyakumar V., Li G.Y., Wang J.H., Some robust convex programs without a duality gap, J. Convex Anal. 20, 377-394 (2013)

13. Gorissen B.L., Blanc H., den Hertog D., Ben-Tal A., Technical note - deriving robust and globalized robust solutions of uncertain linear programs with general convex uncertainty sets, Oper. Res. 62, 672-679 (2014)

14. Jeyakumar V., Lee G.M., Lee J.H., Generalized SOS-convexity and strong duality with SDP dual programs in polynomial optimization, J. Convex Anal. 22, 999-1023 (2015)

15. Wang Y., Shi R., Shi J., Duality and robust duality for special nonconvex homogeneous quadratic programming under certainty and uncertainty environment, J. Global Optim. 62, 643-659 (2015)

16. Boţ R.I., Jeyakumar V., Li G.Y., Robust duality in parametric convex optimization, Set-Valued Var. Anal. 21, 177-189 (2013)

17. Sun X.-K., Chai Y., On robust duality for fractional programming with uncertainty data, Positivity 18, 9-28 (2014)

18. Fang D., Li C., Yao J.-C, Stable Lagrange dualities for robust conical programming, J. Nonlinear Convex Anal. 16, 2141-2158 (2015)

19. Dinh N., Mo T.H., Vallet G., Volle M., A unified approach to robust Farkas-type results with applications to robust optimization problems, SIAM J. Optim. 27, 1075-1101 (2017)

20. Barro M., Ouédraogo A., Traoré S., On uncertain conical convex optimization problems, Pacific J. Optim. 13, 29-42 (2017)

21. Dinh N., Goberna M.A., López M.A., From linear to convex systems: Consistency, Farkas Lemma and applications. J. Convex Anal. 13, 279-290 (2006)

22. Goberna M.A., López M.A., Linear Semi-infinite Optimization, Wiley, Chichester (1998)

23. Dinh N., Goberna M.A., López M.A., Son T.Q., New Frakas-Minkowski constraint qualifications in convex infinite programming, ESAIM: COCV 13, 580-597 (2007)

24. Sion M., On general minimax theorems, Pacific Journal of Mathematics. 8, 171–176 (1958)

25. Goberna M.A., López M.A., Volle M., Primal attainment in convex infinite optimization duality, J. Convex Anal. 21, 1043-1064 (2014)

26. Goberna M.A., Jeyakumar V., López M.A., Necessary and sufficient constraint qualifications for systems of infinite convex inequalities, Nonlinear Analysis 68, 1184-1194 (2008)

27. Goberna M.A., López M.A., Volle M., New glimpses on convex infinite optimization duality, Rev. R. Acad. Cien. (Serie A. Mat.) 109, 431-450 (2015)