The zeros of Riemann zeta partial sums yield solutions to
$f(x)+f(2x)+...+f(nx)=0$

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Abstract. This paper proves that every zero of any n^{th}, $n \geq 2$, partial sum of the Riemann zeta function provides a vector space of basic solutions of the functional equation $f(x)+f(2x)+...+f(nx)=0$, $x \in \mathbb{R}$. The continuity of the solutions depends on the sign of the real part of each zero.

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1 Introduction

The problem of finding continuous solutions of a functional equation is usually more easy than the search of the noncontinuous ones. As an example we have the functional Cauchy equation

$$f(x+y)=f(x)+f(y);\ x,y \in \mathbb{R}, \tag{1.1}$$

whose noncontinuous solutions were found about a century after the continuous ones appeared, because its existence was conditioned to the discovery of the notion of Hamel basis [1, Chap. 2].

In [3], it was found a family of basic solutions of the functional equation

$$F(z)+F(2z)+...+F(nz)=0,\ n \geq 2,$$

on the complex domain $\Omega := \mathbb{C} \setminus (-\infty, 0]$. Then, as these solutions are analytic on Ω, the real and imaginary part of their restrictions on $(0,\infty)$ form a family of continuous solutions, actually of class C^{∞}, of the real functional equation

$$f(x)+f(2x)+...+f(nx)=0,\ x>0. \tag{1.2}$$

Equation (1.2) for small values of n has been used to model some physical processes [3] and, in general, for every integer $n \geq 2$ it represents an important equation as consequence of its connection with the partial sums of the Riemann zeta function, as we will see below. In this paper our aim is the search of vector spaces of solutions of equation (1.2) on $\mathbb{R}$, putting emphasis on the noncontinuous ones. To do it, for the first values of n, we will introduce an algebraic procedure, whereas for arbitrary n, we will link (1.2) to the corresponding n^{th} partial sum of the Riemann zeta function

$$\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z},\ n \geq 2,$$

and we will see then how its zeros produce solutions to (1.2) whose continuity depends on the sign of the real part of each zero. Finally, a special class of real subsets having the property of the n-sum (see Definition 17) will yield the desired vector spaces of noncontinuous solutions of our functional equation.

2 The equation $f(x) + ...+ f(nx) = 0$, $x \in \mathbb{R}$, for small values of n

For $n = 2$ we have the equation

$$f(x) + f(2x) = 0,\ x \in \mathbb{R}, \tag{2.1}$$

whose solutions present a discontinuity at the point $x = 0$ whose meaning will be better understood after Theorem 15 below.

Proposition 1 *Any solution not identically null of the functional equation (2.1) is not continuous at $x = 0$.*

Proof. Let f be a solution not identically null of (2.1). Then, since $f(0) = 0$, there exists a real number $a \neq 0$ such that $f(a) \neq 0$. By considering the sequence $\left(2^{-k}a\right)_{k=1,2,\ldots}$, from (2.1), we have

$$f(2^{-k}a) = (-1)^k f(a) \text{ for all } k \geq 1,$$

which proves that $\lim_{x\to 0} f(x)$ does not exist and then the result follows. ■

The uniqueness of the irreducible expression of a rational number, distinct from 0, as quotient of two integers, allows us to introduce the notion of binary characteristic of a real number.

Definition 2 *The binary characteristic of a real number x, denoted by $[x]_2$, is defined as*

$$[x]_2 = \begin{cases} (-1)^k, & \textit{if } x \in \mathbb{Q}\setminus\{0\},\ \textit{with } x = \dfrac{p}{q} \textit{ (irreducible)} \\ 0, & \textit{otherwise} \end{cases}$$

where k is the power of 2 in the factorial decomposition of p or q.

Noticing the expression $\dfrac{p}{q}$ is irreducible, the binary characteristic is well defined because 2^k can only appear either in the factorial decomposition of p or q. If p and q are both odd, $k = 0$ and then the binary characteristic is 1.

Since the binary characteristic satisfies

$$[x]_2 + [2x]_2 = 0, \text{ for any } x \in \mathbb{R},$$

the next result is immediate.

Proposition 3 *The binary characteristic is an everywhere noncontinuous solution of (2.1).*

Since the graph of each noncontinuous solution of Cauchy equation (1.1) is everywhere dense in the plane [1, Theorem 3, p. 14], and the binary characteristic is a bounded noncontinuous solution of (2.1), it follows that this equation is essentially different from Cauchy's equation.

In the next result we prove that two arbitrary numbers a and b, with $ab < 0$, are sufficient to characterize the solutions of (2.1).

Theorem 4 *Given a, b real with $a > 0$, $b < 0$, let $\mathcal{F}_a$, $\mathcal{G}_b$ be the families of all real functions defined on $[a, 2a)$ and $(2b, b]$, respectively. Then, $f(x)$ is solution of (2.1) if and only if*

$$f(x) = \begin{cases} (-1)^{m_x} g_b(2^{-m_x}x), & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ (-1)^{k_x} f_a(2^{-k_x}x), & \text{if } x > 0 \end{cases} \tag{2.2}$$

where $g_b \in \mathcal{G}_b$, $f_a \in \mathcal{F}_a$ and m_x, k_x are the unique integers such that $x \in \left(2^{m_x+1}b, 2^{m_x}b\right]$ if $x < 0$ and $x \in \left[2^{k_x}a, 2^{k_x+1}a\right)$ if $x > 0$.

Proof. The families of disjoint intervals

$$\left\{\left[2^k a, 2^{k+1}a\right) : k \in \mathbb{Z}\right\}, \ \left\{\left(2^{k+1}b, 2^k b\right] : k \in \mathbb{Z}\right\},$$

constitute partitions of $(0, \infty)$ and $(-\infty, 0)$, respectively. Then, given $x > 0$, there is only one integer k_x such that

$$x \in \left[2^{k_x}a, 2^{k_x+1}a\right) \tag{2.3}$$

and, for $x < 0$, there exists only one integer m_x such that

$$x \in \left[2^{m_x+1}b, 2^{m_x}b\right). \tag{2.4}$$

Thus, for each $f_a \in \mathcal{F}_a$, $g_b \in \mathcal{G}_b$, we claim that any function $f(x)$ defined by (2.2) is solution of (2.1). Indeed, since $f(0) = 0$, the case $x = 0$ is trivial. By assuming that $x > 0$, from (2.3), $x \in \left[2^{k_x}a, 2^{k_x+1}a\right)$, so $2x \in \left[2^{k_x+1}a, 2^{k_x+2}a\right)$ and then we get

$$f(x) + f(2x) = (-1)^{k_x} f_a(2^{-k_x}x) + (-1)^{k_x+1} f_a(2^{-k_x-1}2x) =$$

$$= (-1)^{k_x} f_a(2^{-k_x}x) + (-1)^{k_x+1} f_a(2^{-k_x}x) = 0.$$

In a similar way, when $x < 0$, by using (2.4), we also have $f(x) + f(2x) = 0$, which means that the claim is true. Conversely, suppose $f(x)$ is a solution of (2.1). Then, by reiterating (2.1), one has

$$f(x) = (-1)^k f(2^k x), \text{ for all } x \in \mathbb{R} \text{ and every integer } k. \tag{2.5}$$

Now, we define the functions

$$f_a(x) := f(x),\ x \in [a, 2a) \tag{2.6}$$

and

$$g_b(x) := f(x),\ x \in [2b, b)\,. \tag{2.7}$$

Then, given $x > 0$, from (2.3), there exists only one integer k_x such that $2^{-k_x}x \in [a, 2a)$. By putting $k = -k_x$ in (2.5), because (2.6), we get

$$f(x) = (-1)^{-k_x} f(2^{-k_x}x) = (-1)^{k_x} f_a(2^{-k_x}x). \tag{2.8}$$

For $x < 0$, by repeating verbatim the above reasoning and by taking into account (2.7), we obtain

$$f(x) = (-1)^{-m_x} f(2^{-m_x}x) = (-1)^{m_x} g_b(2^{-m_x}x). \tag{2.9}$$

Then, the expressions (2.8) and (2.9) jointly with fact that $f(0) = 0$, show that $f(x)$ can be written of the form (2.2). Hence the theorem follows. ■

In order to study the equation

$$f(x) + f(2x) + f(3x) = 0,\ x \in \mathbb{R}, \tag{2.10}$$

we define the ternary characteristic concept.

Definition 5 *The ternary characteristic of a real number x, denoted by $[x]_3$, is defined as*

$$[x]_3 := \begin{cases} (-1)^m, \text{ if } x \in \mathbb{Q}\setminus\{0\} \text{ with } x = \dfrac{p}{q} \text{ (irreducible)} \\ 0, \quad \text{otherwise} \end{cases}$$

where m is the power of 3 in the factorial decomposition of p or q.

It is immediate to check the validity of the following property of the ternary characteristic.

Lemma 6 *The property*

$$[2x]_3 = [x]_3\,,\ \ [3x]_3 = -[x]_3$$

holds for any $x \in \mathbb{R}$.

By using the above lemma and the solutions of Cauchy functional equation (1.1), the next result allows us to obtain families of noncontinuous solutions of (2.10).

Theorem 7 *Let φ be a solution, not identically null on the rationals, of Cauchy functional equation. Then*

$$F_{3,\varphi}(x) := \varphi\left(x\,[x]_3\right)$$

is a non-continuous solution of (2.10).

Proof. Noticing the linearity of φ, for every $x \in \mathbb{R}$ we have

$$F_{3,\varphi}(x) + F_{3,\varphi}(2x) + F_{3,\varphi}(3x) = \varphi\left(x\,[x]_3 + 2x\,[2x]_3 + 3x\,[3x]_3\right),$$

but according to Lemma 6, we get

$$x\,[x]_3 + 2x\,[2x]_3 + 3x\,[3x]_3 = 0$$

and then

$$F_{3,\varphi}(x) + F_{3,\varphi}(2x) + F_{3,\varphi}(3x) = \varphi(0) = 0,$$

which proves that $F_{3,\varphi}(x)$ is a solution of (2.10). It only remains to prove that $F_{3,\varphi}(x)$ is noncontinuous. Indeed, since φ is not identically null on the rationals and $\varphi(q) = q\varphi(1)$ for all $q \in \mathbb{Q}$, it follows that $\varphi(1) \neq 0$ and consequently $\varphi(q) \neq 0$ for all $q \neq 0$. Then for a fixed rational $a \neq 0$, given any sequence of irrational numbers (s_n) such that $s_n \to a$, by virtue of the definition of ternary characteristic, we have

$$F_{3,\varphi}(s_n) = 0.$$

On the other hand, since $[a]_3 = \pm 1$, we get

$$F_{3,\varphi}(a) = \varphi(\pm a) = \pm\varphi(a) \neq 0,$$

which proves that $F_{3,\varphi}$ is not continuous at a. Now the proof is completed. ■

A large class of solutions of (2.10) is obtained by using the notion of Hamel basis [1, Chap. 2].

Theorem 8 *Let $\mathcal{B} = \{x_i : i \in I\}$ be a Hamel basis of $\mathbb{R}$ on $\mathbb{Q}$, h an arbitrary real function on $\mathcal{B}$ and φ a solution of Cauchy functional equation. Then*

$$F_{3,\varphi,h}(x) := \sum_{i \in A_x} \varphi(q_i\,[q_i]_3) h(x_i)$$

is a solution of (2.10), where A_x is the finite subset of the index set I which expresses $x = \sum_{i \in A_x} q_i x_i$ under a unique form.

Proof. By expressing

$$x = \sum_{i \in A_x} q_i x_i,$$

it follows

$$2x = \sum_{i \in A_x} 2q_i x_i, \quad 3x = \sum_{i \in A_x} 3q_i x_i.$$

From Theorem 7, $F_{3,\varphi}(x) := \varphi\left(x\,[x]_3\right)$ is a solution of (2.10); then, for every $x \in \mathbb{R}$ we have

$$F_{3,\varphi,h}(x) + F_{3,\varphi,h}(2x) + F_{3,\varphi,h}(3x) =$$

$$= \sum_{i \in A_x} F_{3,\varphi}(q_i)h(x_i) + \sum_{i \in A_x} F_{3,\varphi}(2q_i)h(x_i) + \sum_{i \in A_x} F_{3,\varphi}(3q_i)h(x_i) =$$

$$= \sum_{i \in A_x} \left[F_{3,\varphi}(q_i) + F_{3,\varphi}(2q_i) + F_{3,\varphi}(3q_i) \right] h(x_i) = 0,$$

which proves that $F_{3,\varphi,h}(x)$ is a solution of (2.10). ■

For the equation

$$f(x) + f(2x) + f(3x) + f(4x) = 0,\ x \in \mathbb{R}, \tag{2.11}$$

we can also obtain basic families of noncontinuous solutions by defining the quaternary characteristic concept.

Definition 9 *The quaternary characteristic of a real number x, denoted by $[x]_4$, is defined as*

$$[x]_4 := \begin{cases} (-1)^{k+m}, & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ with } x = \dfrac{p}{q} \text{ (irreducible)} \\ 0, & \text{otherwise} \end{cases}$$

where k, m are the powers of 2 and 3, respectively, in the factorial decomposition of p and q.

From the above definition, for any real x we have:

$$[2x]_4 = [3x]_4 = -[x]_4\,; \quad [4x]_4 = [x]_4\,. \tag{2.12}$$

Property (2.12) allows us to prove similar results to the preceding cases $n = 2$, 3, as follows.

Proposition 10 *The quaternary characteristic function is a noncontinuous solution of the functional equation (2.11).*

Theorem 11 *Let φ be any solution of Cauchy functional equation not identically null on the rationals. Then the function*

$$F_{4,\varphi}(x) := \varphi(x\,[x]_4)$$

is a noncontinuous solution of (2.11).

Theorem 12 *Let $\mathcal{B} = \{x_i : i \in I\}$ be a Hamel basis of $\mathbb{R}$ on $\mathbb{Q}$, h an arbitrary real function on $\mathcal{B}$ and φ a solution of Cauchy functional equation. Then*

$$F_{4,\varphi,h}(x) := \sum_{i \in A_x} \varphi(q_i\,[q_i]_4) h(x_i)$$

is a solution of (2.11), where A_x is the finite subset of the index set I that expresses $x = \sum_{i \in A} q_i x_i$ under a unique form.

3 The equation $f(x) + ...+ f(nx) = 0$, $x \in \mathbb{R}$, for any $n \geq 2$

As far as we know, by means of the procedure exhibited in the cases $n = 2, 3, 4$, it is still possible to construct, for each particular value of $n \geq 2$, the n^{th} characteristic of a real number to obtain basic noncontinuous solutions of

$$f(x) + f(2x) + ... + f(nx) = 0,\ x \in \mathbb{R}, \tag{3.1}$$

nevertheless, it is not necessary to resort to the n^{th} characteristic concept to find noncontinuous solutions of (3.1), as we will see. Indeed, as an alternative method we propose, firstly, to consider the continuous solutions of equation (1.2)

$$f(x) + f(2x) + ... + f(nx) = 0,\ x > 0,$$

generated by the zeros of the Riemann zeta partial sums

$$\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z},\ n \geq 2,$$

and, secondly, to define the concept of real subset having the n-sum property (Definition 17).

The next result proves that each zero of $\zeta_n(z)$ generates a vector space of continuous solutions of (1.2).

Theorem 13 *Fixed any integer $n \geq 2$, every zero of the n^{th} partial sum $\zeta_n(z)$ of the Riemann zeta function generates a vector space of real continuous solutions of (2.1).*

Proof. We fix $n \geq 2$; let β_n be an arbitrary zero of the n^{th} partial sum $\zeta_n(z)$. By taking the principal logarithm, we define the function

$$g_n(z) := z^{-\beta_n}$$

on the complex domain $\Omega := \mathbb{C} \backslash (\infty, 0]$, and we claim that the family of functions on the interval $(0, \infty)$ of the form

$$\left\{ \lambda \operatorname{Re}(x^{-\beta_n}) + \mu \operatorname{Im}(x^{-\beta_n}) :\ \lambda, \mu \in \mathbb{R} \right\} \tag{3.2}$$

is a vector space of continuous solutions of (1.2). Indeed, the structure of the functional equation (1.2) is linear in the following sense: if $f(x)$ and $g(x)$ are solutions of (1.2), then the function

$$h(x) := \lambda f(x) + \mu g(x),$$

for arbitrary reals λ and μ, is also a solution of (1.2). A similar argument proves that if $f(z)$ is a solution of the complex functional equation

$$F(z) + F(2z) + ... + F(nz) = 0,\ z \in \Omega, \tag{3.3}$$

then the function

$$\lambda \operatorname{Re} f(x) + \mu \operatorname{Im} f(x)$$

is a solution of (1.2) on $(0, \infty)$, and therefore the claim is true. Now, it only remains to check that the function $g_n(z) = z^{-\beta_n}$, $z \in \Omega$, is a solution of (3.3). Indeed, by taking into account that β_n is a zero of $\zeta_n(z)$, we have

$$g_n(z) + g_n(2z) + ... + g_n(nz) = z^{-\beta_n} + (2z)^{-\beta_n} + ... + (nz)^{-\beta_n} =$$

$$= z^{-\beta_n}(1 + 2^{-\beta_n} + ... + n^{-\beta_n}) = z^{-\beta_n}\zeta_n(\beta_n) = 0,$$

which definitively proves the theorem. ∎

An easy consequence of the previous theorem is the following:

every function $f(x)$ of the form (3.2) defines a function

$$F_n(x) = \begin{cases} f(x), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ f(-x), & \text{if } x < 0 \end{cases} \tag{3.4}$$

which is a continuous solution of (3.1), except possibly at the point 0.

Since all the zeros of $\zeta_2(z)$ are on the imaginary axis, the next result must be necessarily formulated for $n > 2$.

Theorem 14 *For each $n > 2$, every zero of the n^{th} partial sum $\zeta_n(z)$ of the Riemann zeta function situated in the left half-plane generates a vector space of real continuous solutions of (3.1).*

Proof. Let $\beta_n = a_n + ib_n$ be a zero of $\zeta_n(z)$ with $a_n < 0$. Noticing the preceding theorem, it is enough to prove that

$$\lim_{x \to 0^+} F_n(x) = 0,$$

where $F_n(x)$ is defined by (3.4) and $f(x)$ belongs to the family (3.2). Indeed, for $x > 0$ we have

$$\operatorname{Re}(x^{-\beta_n}) = e^{-a_n \log x} \cos(b_n \log x), \ \operatorname{Im}(x^{-\beta_n}) = e^{-a_n \log x} \sin(b_n \log x). \tag{3.5}$$

Then, by taking into account that $a_n \log x > 0$ for $0 < x < 1$, we get

$$\lim_{x \to 0^+} \operatorname{Re}(x^{-\beta_n}) = \lim_{x \to 0^+} \operatorname{Im}(x^{-\beta_n}) = 0$$

and consequently

$$\lim_{x \to 0^+} F_n(x) = 0,$$

as we claimed. The proof is now completed. ∎

We have just seen that if $\beta_n = a_n + ib_n$ is a zero of $\zeta_n(z)$, $n \geq 2$, then (3.5) are basic solutions of equation (1.2). Now, the next result easily follows.

Theorem 15 *For each $n \geq 2$, every zero β_n, with $\operatorname{Re}\beta_n \geq 0$, of the n^{th} partial sum $\zeta_n(z)$ of the Riemann zeta function generates a vector space of real continuous solutions of (3.1) except at the point $x = 0$.*

Proof. It is enough to note that, since $a_n = \operatorname{Re}\beta_n \geq 0$, the limits

$$\lim_{x\to 0} e^{-a_n \log x}\cos(b_n \log x),\ \lim_{x\to 0} e^{-a_n \log x}\sin(b_n \log x)$$

do not exist. ■

For all $n > 2$, the existence of zeros of every $\zeta_n(z)$ outside the imaginary axis is guaranteed by means of the following result.

Theorem 16 *For $n > 2$, every partial sum $\zeta_n(z)$ of the Riemann zeta function possesses infinitely many zeros in at least one half-plane.*

Proof. By defining

$$G_n(z) := \zeta_n(-z),\ n \geq 2,$$

the sets of zeros of the functions $G_n(z)$ and $\zeta_n(z)$ satisfy the relation

$$Z_{G_n(z)} = -Z_{\zeta_n(z)}. \tag{3.6}$$

About the zeros of $G_n(z)$ we note that in [3, Propositions 1,2,3] it was proved:

a) Every $G_n(z)$, $n \geq 2$, is an entire function of order 1 and it has infinitely many zeros.

b) There exist real numbers r_n, s_n, such that all the zeros of $G_n(z)$ are in the vertical strip $\{z \in \mathbb{C} : r_n \leq \operatorname{Re} z \leq s_n\}$.

c) The functions $G_n(z)$ do not have all the zeros on the imaginary axis, except for $n = 2$.

Now, assume by *reductio ad absurdum* that there exists an integer $m > 2$ such that all the zeros of $G_m(z)$ are:

$$(\alpha_m^{(l)})_{l=1,2,\dots p},\ p \geq 1, \text{ with } \operatorname{Re}\alpha_m^{(l)} \neq 0;\ (\alpha_{m,k})_{k=1,2,\dots} \text{ with } \operatorname{Re}\alpha_{m,k} = 0. \tag{3.7}$$

Taking into account that, for any $n \geq 2$, $G_n(\overline{z}) = \overline{G_n(z)}$ for all $z \in \mathbb{C}$, the zeros of $G_m(z)$ are conjugate. Then, p is even, so $p = 2q \geq 2$, and $\alpha_{m,k} = \pm i y_k$ with $y_k > 0$ for all $k \geq 1$. Hence, the polynomial defined by the zeros $(\alpha_m^{(l)})_{l=1,2,\dots p}$, say $P_m(z)$, will be

$$P_m(z) = (z^2 - 2a^{(1)}z + \left|\alpha_m^{(1)}\right|^2)\cdots(z^2 - 2a^{(q)}z + \left|\alpha_m^{(q)}\right|^2), \tag{3.8}$$

where $a^{(l)} = \operatorname{Re}\alpha_m^{(l)}$, $l = 1, \dots, q$. Now, the function

$$H_m(z) := \frac{G_m(z)}{P_m(z)}$$

is entire of order 1 with zeros $(\pm iy_k)_{k=1,2,\ldots}$ and then, from Hadamard's factorization theorem [2, Theorem 4.4.3], we have

$$H_m(z) = e^{Az+B} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{y_k^2}\right), \tag{3.9}$$

where the constants A, B, because (3.8), are given by

$$e^B = H_m(0) = \frac{m}{\left|\alpha_m^{(1)} \ldots \alpha_m^{(q)}\right|^2}$$

and

$$A = \frac{H'_m(0)}{e^B} = \frac{\log(m!)}{m} + \frac{a^{(1)}}{\left|\alpha_m^{(1)}\right|^2} + \ldots + \frac{a^{(q)}}{\left|\alpha_m^{(q)}\right|^2}.$$

Noticing (3.9), the function $H_m(z)e^{-Az}$ is even and then

$$H_m(z)e^{-Az} = H_m(-z)e^{Az},$$

which means that

$$\frac{P_m(-z)}{P_m(z)} = \frac{G_m(-z)e^{2Az}}{G_m(z)}. \tag{3.10}$$

Now, assume all the zeros of $P_m(z)$ are zeros of $P_m(-z)$, then necessarily $P_m(z) \equiv P_m(-z)$ and hence the zeros are conjugate and opposite. In consequence q is even and then

$$\frac{a^{(1)}}{\left|\alpha_m^{(1)}\right|^2} + \ldots + \frac{a^{(q)}}{\left|\alpha_m^{(q)}\right|^2} = 0,$$

which implies that

$$A = \frac{\log(m!)}{m}.$$

Then, noticing $(m!)^2 > m^m$ for all $m > 2$, by taking the limit in (3.10) when $z = x \to +\infty$ we are led to a contradiction because the left side of (3.10) is identically equal to 1 whereas the right side tends to $+\infty$.

On the other hand, the right side of (3.10) is a quotient of exponential polynomials whose number of poles is finite and then, Shields's theorem [4] implies that this quotient is an exponential polynomial. Hence if we suppose that at least one zero of $P_m(z)$ is not a zero of $P_m(-z)$, we are led to a contradiction consisting on the left side of (3.10) is a meromorphic function with at least a pole whereas the right side one is an exponential polynomial. As consequence, there is no function $G_n(z)$, $n > 2$, having its zeros like in (3.7). It means that every function $G_n(z)$, $n > 2$, has infinitely many zeros in at least one half-plane and consequently, from (3.6), the result follows. ■

Now, by using the family (3.2) of continuous solutions of (1.2) we can construct large families of noncontinuous solutions of (3.1). To do it, we introduce a new class of real subsets.

Definition 17 *Given an integer $n \geq 2$, a proper subset S of $\mathbb{R}$ is said to have the n-sum property if for each $x \in S$ one has*

$$2x, 3x, ..., nx \in S,$$

and for each $x \in \mathbb{R}\backslash S$ one has

$$2x, 3x, ..., nx \in \mathbb{R}\backslash S.$$

Observe that if $S \subset \mathbb{R}$ has the n-sum property, then $\mathbb{R}\backslash S$ also has the same property. For each integer $n \geq 2$, a non-trivial example of a real subset S having the n-sum property is the multiplicative group Π_n generated by the set $\{p_1, p_2, ..., p_{k_n}\}$ of all prime numbers less than or equal to n,

$$\Pi_n := \left\{x = p_1^{\alpha_1} p_2^{\alpha_2} ... p_{k_n}^{\alpha_{k_n}} : \alpha_1, \alpha_2, ... \alpha_{k_n} \in \mathbb{Z}\right\}. \tag{3.11}$$

At this point we obtain the following result.

Theorem 18 *For every integer $n > 2$, let Π_n be the multiplicative group defined in (3.11) and φ any solution of Cauchy functional equation not identically null on the rationals. Then for any $F_n(x)$ of the form (3.4), not identically zero, the function*

$$F_{n,\varphi}(x) := \begin{cases} \varphi(F_n(x)), & \text{if } x \in \mathbb{R} \setminus \Pi_n \\ 0, & \text{if } x \in \Pi_n \end{cases} \tag{3.12}$$

is a noncontinuous solution of (3.1).

Proof. Firstly, since φ is not identically null on the rationals, necessarily

$$\varphi(q) = q\varphi(1) \neq 0, \text{ for all } q \in \mathbb{Q},\ q \neq 0.$$

On the other hand, from the linearity of φ, the n-sum property of the subsets Π_n, $\mathbb{R} \setminus \Pi_n$, and the fact of $F_n(x)$ is of the form (3.4), it follows that every function defined by (3.12) is a solution of (3.1). Now we observe that, for each $x \in \Pi_n$, there exists

$$\min \{|y - x| : y \in \Pi_n,\ y \neq x\} > 0$$

and hence there is some $y_x \in \Pi_n$ with $y_x \neq x$ such that either (y_x, x) or (x, y_x) is contained in $\mathbb{R}\backslash\Pi_n$. Then, as $F_n(x)$ is a function of the form (3.4) not identically zero, it is continuous and not constant on $(0, \infty)$. Therefore, either the image $F_n((y_x, x))$ or $F_n((x, y_x))$ is a connected set of $\mathbb{R}$ not reduced to a point and consequently either $F_n((y_x, x))$ or $F_n((x, y_x))$ contains at least a rational, say q_x, distinct from 0. Then, there exists $z_x \in \mathbb{R} \setminus \Pi_n$ such that $F_n(z_x) = q_x \neq 0$ for which we have

$$\varphi(F_n(z_x)) = \varphi(q_x) \neq 0 \tag{3.13}$$

and it means that the function defined in (3.12) is not identically 0.

Let us denote by Z_n the set of the positive zeros of $F_n(x)$, then we claim that Π_n is not contained in Z_n. Indeed, for $x > 0$, we have

$$F_n(x) = \lambda e^{-a_n \log x} \cos(b_n \log x) + \mu e^{-a_n \log x} \sin(b_n \log x),\ \lambda,\ \mu \in \mathbb{R}. \tag{3.14}$$

Then, if $\mu = 0$, necessarily, $\lambda \neq 0$ and therefore the claim is true by taking $x = 1$. In the case $\lambda = 0$, it must be $\mu \neq 0$ and, since $2 \in \Pi_n$, by admitting that $2 \notin Z_n$, the claim is true. By supposing $2 \in Z_n$, we have

$$b_n \log 2 = k\pi, \text{ with } k \neq 0,\ k \in \mathbb{Z}$$

and, necessarily, $x = 3$ is a point of Π_n which is not in Z_n and then the claim follows. Otherwise, if $x = 3 \in Z_n$, then

$$b_n \log 3 = l\pi, \text{ with } l \neq 0,\ l \in \mathbb{Z},$$

implying that $\log 2$ and $\log 3$ are linearly dependent on the rationals, which is false. Finally, when $\lambda,\ \mu \neq 0$, clearly $x = 1 \in \Pi_n$ but $1 \notin Z_n$ and then the claim is true. Therefore, let x_0 be a point of Π_n such that $F_n(x_0) \neq 0$; by continuity of $F_n(x)$, for m large enough we get

$$F_n(x) \neq 0 \text{ for all } x \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\right).$$

Thus, by using (3.13), we can determine a sequence $(z_m)_m \in \mathbb{R} \setminus \Pi_n$ such that $z_m \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\right)$ with $F_n(z_m) = q_m \in \mathbb{Q}$. Now, taking into account that $z_m \to x_0$, we get

$$\lim_{m\to\infty} F_n(z_m) = F_n(x_0)$$

and then

$$\lim_{m\to\infty} q_m = F_n(x_0).$$

Hence

$$\lim_{m\to\infty} F_{n,\varphi}(z_m) = \lim_{m\to\infty} \varphi(F_n(z_m)) = \lim_{m\to\infty} \varphi(q_m) = \lim_{m\to\infty} q_m \varphi(1) =$$

$$= F_n(x_0)\varphi(1) \neq 0. \tag{3.15}$$

On the other hand, as $x_0 \in \Pi_n$, we have $F_{n,\varphi}(x_0) = 0$ and then, from (3.15), it follows that $F_{n,\varphi}(x)$ is not continuous at x_0. This completes the proof. ∎

Observe that if $n = 2$, by taking a function $F_2(x)$ of the form (3.14) with $\lambda = 0$, $\mu \neq 0$, we have that the group $\Pi_2 \subset Z_2$, where Z_2 is the set of the positive zeros of $F_2(x)$. Therefore, the case $n = 2$ cannot be treated as the preceding one. Nevertheless, when $n = 2$, noncontinuous solutions of (3.1) can be obtained as follows.

Proposition 19 *Let φ be a noncontinuous solution of Cauchy functional equation. Then, for any $F_2(x)$ of the form (3.4) with $n = 2$, not identically null, the function*

$$F_{2,\varphi}(x) := \varphi(F_2(x)),\ x \in \mathbb{R}$$

is a noncontinuous solution of (3.1).

Proof. From the linearity of φ and the fact of $F_2(x)$ is a solution of (3.1) for $n = 2$, it follows that $F_{2,\varphi}(x)$ is a solution of (3.1). We pick an interval $[a, b]$ with $0 < a < b$ and, because the continuity of $F_2(x)$ on $(0, \infty)$, $F_2([a, b])$ is a compact and connected set of $\mathbb{R}$ not reduced to a point. Hence $F_2([a, b]) = [A, B]$ with $A < B$. Now, noticing the image $\varphi([A, B])$ is dense in $\mathbb{R}$ [1, Corollary 4, p. 15], we get that $F_{2,\varphi}(x)$ is a noncontinuous function. This completes the proof. ■

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