On the stability of Voronoi cells

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Abstract

Let T be a given subset of $\mathbb{R}^n$, whose elements are called sites, and let $s \in T$. The Voronoi cell of s with respect to T consists of all points closer to s than to any other site. In many real applications, the position of some elements of T is uncertain due to either random external causes or to measurement errors. In this paper we analyze the effect on the Voronoi cell of small changes in s or in a given non empty set $P \subset T \setminus \{s\}$, but not both. Two types of perturbations of P are considered, one of them not increasing the cardinality of T. More in detail, the paper provides conditions for the corresponding Voronoi cell mappings to be closed, lower and upper semicontinuous. All the involved conditions are expressed in terms of the data.

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1 Introduction

Let $T \subset \mathbb{R}^n$ be a given set containing at least two elements (called *sites*). The *Voronoi cell* of $s \in T$ is the set

$$V_T(s) = \{x \in \mathbb{R}^n : d(x, s) \leq d(x, T \setminus \{s\})\}, \qquad (1.1)$$

where d denotes the Euclidean distance on $\mathbb{R}^n$. In this paper, we assume that some, not all, sites are uncertain. Let P be the set of uncertain elements in $T \setminus \{s\}$. The objective consists of analyzing the effect on $V_T(s)$ of small changes of s or P, but not both at the same time. In formal terms, when s is uncertain and $P = \emptyset$, $V_T(s_1)$ defines a set-valued mapping whose argument s_1 is the result of perturbing s. Alternatively, when s is kept fixed whereas $P \neq \emptyset$ is uncertain, $V_{(T \setminus P) \cup P_1}(s)$ defines a set-valued mapping whose argument P_1 represents the result of perturbing P. We study the continuity properties (lower and upper semicontinuities, closedness) of these multifunctions close to the nominal data s or P. We do not consider simultaneous perturbations of s and $P \neq \emptyset$ because the problem is of high difficulty (the situation is similar to the perturbation of the cost and the right-hand side coefficients in linear optimization: the sensitivity analysis is much more difficult for the simultaneous perturbations of both types of coefficients).

The *Voronoi diagram* of T is $\mathrm{Vor}(T) = \{V_T(t), t \in T\}$. The first works on Voronoi diagrams with T finite are due to Descartes, Dirichlet, and Voronoi (1644, with $n = 2$, 1850, with $n = 3$, and 1908, with $n \in \mathbb{N}$, respectively). Voronoi diagrams w.r.t. finite sets are widely used in operations research, for instance in location problems (Okabe and Suzuki 1997, Blanquero and Carrizosa 2002) and location games (Chawla et al 2006), among a variety of fields: computational geometry, data compression, economics, marketing, geophysics, meteorology, forest management, condensed matter physics, computational chemistry, robot navigation, etc. (see, e.g., Okabe et al 2000). Delaunay (1934) used Voronoi diagrams for infinite sites in the framework of crystallography under conditions guaranteeing that $\mathrm{Vor}(T)$ is formed by polyhedral convex sets. The sets of sites considered by Delaunay were *discrete* in the sense that they have no accumulation point. The Voronoi cells of discrete sets have been studied by Gruber (2007) and Voigt (2008), where it is shown that $\mathrm{Vor}(T)$ is formed by quasipolyhedral sets (i.e., sets whose non empty intersection with polytopes are polytopes). The bounded Voronoi cells w.r.t. arbitrary sets have been characterized by Voigt and Weis (2010).

In many practical applications of Voronoi cells and diagrams, some elements of T are uncertain. Consider, e.g., a bank that operates at present in certain region through branches placed at the points $t_1, ..., t_m$ which is planning to open a new branch at a point to be decided, but close to s. Denote $T = \{s, t_1, ..., t_m\}$. The business area to be assigned to the branch located at s will be $V_T(s)$, but this set depends on the uncertain location of s. Consider the following question: do small perturbations of s provoke small changes in $V_T(s)$? In other words, do $V_T(s_1)$ converge (in some sense) to $V_T(s)$ when s_1 approaches s? Similarly, one can consider the case in which s has been already decided, but new branches could be open at points close to $t_{m+1}, ..., t_k$. Let $T = \{s, t_1, ..., t_k\}$ and $P = \{t_{m+1}, ..., t_k\}$. The question is now: do small perturbations of P provoke small changes in $V_T(s)$? The stability analysis in this paper answers these two questions. Alternative approaches to uncertainty in the sites have been proposed by Jooyandeh et al (2009), where the location of the sites is modelled by means of fuzzy numbers, and by Goberna et al (2010), who consider the robust (or pessimistic) approach, replacing each uncertain site by the (possibly infinite) set of its conceivable positions.

Until recently, the main difficulty in order to get geometric information on $V_T(s)$, or to study its stability, consisted on the lack of a treatable representation of $V_T(s)$ except in some particular cases, as the next example illustrates.

Example 1.1 *Let $c, s \in \mathbb{R}^2$, $\rho > 0$, with $\|s - c\| < \rho$, and $T = \left\{t \in \mathbb{R}^2 : \|t - c\| = \rho\right\} \cup \{s\}$. According to (1.1), $V_T(s)$ is formed by c and by those $x \in \mathbb{R}^n$ such that*

$$\begin{aligned} \|x - s\| &= d(x, s) \leq d(x, T \setminus \{s\}) \\ &= d\left(x, c + \rho \tfrac{x-c}{\|x-c\|}\right) = \rho - \|x - c\|. \end{aligned}$$

So $V_T(s) = \left\{x \in \mathbb{R}^2 : \|x - s\| + \|x - c\| \leq \rho\right\}$ is the region limited by an ellipse with focusses c and s.

Fortunately, as it has been observed by Voigt (2008) and by Voigt and Weis (2010),

$$V_T(s) = \left\{ x \in \mathbb{R}^n : (t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2}, t \in T \right\} \tag{1.2}$$

(where $\|\cdot\|$ denotes the Euclidean norm), which shows that $V_T(s)$ is the solution set of a linear system, i.e., the intersection of a family of closed halfspaces. The linear representation (1.2) also shows that $V_T(s)$ is a polyhedral convex set whenever T is finite.

The theory of linear semi-infinite systems, one of the main research fields of Marco A. López, to whom this volume is dedicated, can be used to get useful information on Voronoi cells of arbitrary sets. The recent work of Goberna et al (2010) exploits this theory to get geometric information, whereas the present paper does the same from the stability perspective.

This paper is organized as follows. Section 2 introduces the notation to be used, reviews the existing theory on Voronoi cells of arbitrary sets, and recalls the basic stability concepts for set-valued mappings. Section 3 introduces the concept of strong Slater condition, which plays a crucial role in the lower semicontinuity property. Sections 4, 5, and 6 study the stability of the given Voronoi cell $V_T(s)$ under small perturbations of s in $\mathbb{R}^n$, global perturbations of $P \subset T \backslash \{s\}$, and individual perturbations of the elements of P, respectively. In all cases, it is shown that the Voronoi cell mapping is closed everywhere, the lower semicontinuity property is characterized, and a sufficient condition for its upper semicontinuity is given. All the involved concepts and conditions can be expressed in terms of the data (T, s, and P).

2 Preliminaries

Throughout the paper we use the following notation. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x'y$, the canonical basis by $\{e_1, ..., e_n\}$, the zero vector by 0_n, and the open unit ball by

B_n. The Chebyshev norm of $x \in \mathbb{R}^n$ is $\|x\|_\infty$. Given $X \subset \mathbb{R}^n$, $\operatorname{int} X$, $\operatorname{cl} X$, and $\operatorname{bd} X$ denote the *interior*, the *closure*, and the *boundary* of X, respectively. Moreover, we denote by $\operatorname{conv} X$, and $\operatorname{cone} X = \mathbb{R}_+ \operatorname{conv} X$, the *convex hull* of X, and the *convex conical hull* of X, respectively.

The characterization of the bounded Voronoi cells mentioned in Section 1 is as follows: $V_T(s)$ is bounded if and only if $s \in \operatorname{int} \operatorname{conv} T$ (Voigt and Weis 2010). The proofs of the remaining results on Voronoi cells that we shall use can be found in Goberna et al (2010): if $T \subset T_1 \subset \mathbb{R}^n$, we have $V_{T_1}(s) \subset V_T(s)$, and $V_{T_1}(s) = V_T(s)$ whenever $T \subset T_1 \subset \operatorname{cl} T$; $s \in \operatorname{int} V_T(s)$ if and only if s is an isolated point of T. Moreover, $V_T(s) = \{s\}$ if and only if the closure of the *characteristic cone* of T at s,

$$K_T(s) := \operatorname{cone}\left\{\left(t - s, \frac{\|t\|^2 - \|s\|^2}{2}\right), t \in T; (0_n, 1)\right\},$$

is a halfspace. In that case, s is an accumulation point of T. If $K_T(s)$ is closed, then s is isolated in T; the converse statement holds when T is closed. If $T, T_1 \subset \mathbb{R}^n$, then $V_{T_1}(s) \subset V_T(s)$ if and only if $\operatorname{cl} K_T(s) \subset \operatorname{cl} K_{T_1}(s)$. Actually, $\operatorname{cl} K_T(s)$ captures all the relevant information on $V_T(s)$ (see Goberna and López 1998, and references therein).

For the sake of completeness, we recall now the stability concepts for set-valued mappings introduced by Bouligand and Kuratowski that we shall consider in this paper. Let $\mathcal{V} : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, where Ω is a pseudometric space whose elements are called *parameters*.

$\mathcal{V}$ is *lower semicontinuous* at $\omega_0 \in \Omega$ (the *nominal parameter*) *in the Berge sense* (lsc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $W \cap \mathcal{V}(\omega_0) \neq \emptyset$, there exists an open set $V \subset \Omega$, containing ω_0, such that $W \cap \mathcal{V}(\omega) \neq \emptyset$ for each $\omega \in V$.

$\mathcal{V}$ is *upper semicontinuous* at $\omega_0 \in \Omega$ *in the Berge sense* (usc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $\mathcal{V}(\omega_0) \subset W$, there exists an open set $V \subset \Omega$, containing ω_0, such that $\mathcal{V}(\omega) \subset W$ for each $\omega \in V$.

$\mathcal{V}$ is *closed* at $\omega_0 \in \operatorname{dom} \mathcal{V}$ if for all sequences $\{\omega_r\} \subset \Omega$ and $\{x_r\} \subset \mathbb{R}^n$ satisfying $x_r \in \mathcal{V}(\omega_r)$ for all $r \in \mathbb{N}$, $\omega_r \to \omega_0$ and $x_r \to x_0$, one has $x_0 \in \mathcal{V}(\omega_0)$.

It is well known that $\mathcal{V}$ is usc at $\omega_0 \in \operatorname{dom} \mathcal{V}$ whenever $\mathcal{V}$ is closed and *locally bounded* at ω_0 (i.e., there exists a bounded set $C \subset \mathbb{R}^n$ such that $\mathcal{V}(\omega_1) \subset C$ for any ω_1 sufficiently close to ω_0).

We also consider $\liminf_{\omega \to \omega_0} \mathcal{V}(\omega)$, which is the set of points x such that for every sequence $\{\omega_k\} \subset \Omega$, $\omega_k \to \omega_0$, it is possible to find a sequence $\{x_k\} \subset \mathbb{R}^n$, $x_k \in \mathcal{V}(\omega_k)$ for all k, with $x_k \to x$; whereas $\limsup_{\omega \to \omega_0} \mathcal{V}(\omega)$ is the set of points x for which there exist sequences $\{\omega_k\} \subset \Omega$ and $\{x_k\} \subset \mathbb{R}^n$, such that $x_k \in \mathcal{V}(\omega_k)$, $\omega_k \to \omega_0$, and $x_k \to x$. According to Rockafellar and Wets (1998), $\mathcal{V}$ closed and lsc at ω_0 is equivalent to

$$\liminf_{\omega \to \omega_0} \mathcal{V}(\omega) = \limsup_{\omega \to \omega_0} \mathcal{V}(\omega) = \mathcal{V}(\omega_0). \tag{2.1}$$

In this case we say that $\lim_{\omega \to \omega_0} \mathcal{V}(\omega) = \mathcal{V}(\omega_0)$ and call it the Painlevé-Kuratowski limit (see also Exercise 6.5 in Goberna and López 1998).

The next result is useful in order to determine those properties of $V_T(s)$ which are preserved by any (sufficiently small) perturbation of the set of sites. We omit the proof, which follows from the continuity properties of the determinant function.

Lemma 2.1 *Let $\{a_s, s \in S\} \subset \mathbb{R}^n$ and $a \in \operatorname{int} \operatorname{conv} \{a_s, s \in S\}$. Then there exists some $\varepsilon > 0$ such that $c \in \operatorname{int} \operatorname{conv} \{c_s, s \in S\}$ if $\sup_{s \in S} \|c_s - a_s\| < \varepsilon$ and $\|c - a\| < \varepsilon$.*

3 The strong Slater condition

Given a set Q with $\emptyset \neq Q \subset T \backslash \{s\}$, an element $\bar{x}$ of the set

$$X_Q^s := \left\{ x \in \mathbb{R}^n : (t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2},\ t \in T \backslash Q \right\}$$

is called a *strong Slater point* for the pair (Q, s) with associated scalar $\delta > 0$ whenever $(t-s)' \overline{x} \leq \frac{\|t\|^2 - \|s\|^2}{2} - \delta$ for all $t \in Q$. Observe that $X_{T \backslash \{s\}}^s = \mathbb{R}^n$ for all $s \in T$. When there exists a strong Slater point for (Q, s) we say that Q satisfies the *strong Slater condition (SSC) for* s.

Proposition 3.1 *Given a set Q such that $\emptyset \neq Q \subset T \backslash \{s\}$, the following statements are equivalent:*
(i) Q satisfies SSC for (Q, s).
(ii) $s \notin \operatorname{cl} Q$, i.e., $d(s, Q) > 0$.
(iii) s is a strong Slater point for (Q, s).

Proof. [(i) $\Rightarrow$ (ii)] Suppose that $\bar{x}$ is a strong Slater point for (Q, s) with associated scalar $\delta > 0$. If $s \in \operatorname{cl} Q$, then we would get $0 \leq -\delta < 0$. So $s \notin \operatorname{cl} Q$.

[(ii) $\Rightarrow$ (iii)] Assume that $s \notin \operatorname{cl} Q$. Let $\rho > 0$ be such that $s + \rho B_n \subset \mathbb{R}^n \backslash \operatorname{cl} Q$. Hence $\|t - s\| \geq \rho$ for any $t \in Q$, which gives

$$(t-s)' s - \tfrac{\|t\|^2 - \|s\|^2}{2} = -\tfrac{\|t-s\|^2}{2} \leq -\frac{\rho^2}{2}.$$

Since $s \in X_Q^s$, it follows that s is a strong Slater point for (Q, s).

[(iii) $\Rightarrow$ (i)] It is trivial. $\square$

So, any closed set Q not containing s satisfies SSC for s.

Proposition 3.2 *Assume that Q is bounded. If $\bar{x}$ is a strong Slater point for (Q, s), then there exists $\varepsilon > 0$ such that $\bar{x}$ is a strong Slater point for any pair (Q_1, s) such that $Q_1 \subset Q + \varepsilon B_n$.*

Proof. Suppose that $Q \subset \mu B_n$ and that $\bar{x}$ is a strong Slater point for (Q, s) with associated scalar $\delta > 0$. Let ε be such that $0 < \varepsilon < \min\left\{1, \frac{\delta}{2(\|\bar{x}\|+\mu+2)}\right\}$. If $Q_1 \subset Q + \varepsilon B_n$, then for each $t_1 \in Q_1$, there exists $t \in Q$ such that $d(t_1, t) = \|t_1 - t\| < \varepsilon$. Thus

$$\begin{aligned}
(t_1 - s)' \bar{x} &= (t - s)' \bar{x} + (t_1 - t)' \bar{x} \\
&\leq \frac{\|t\|^2 - \|s\|^2}{2} - \delta + \varepsilon \|\bar{x}\| \\
&\leq \frac{\|t_1\|^2 + 2\varepsilon \|t_1\| + \varepsilon^2 - \|s\|^2}{2} - \delta + \varepsilon \|\bar{x}\| \\
&\leq \frac{\|t_1\|^2 - \|s\|^2}{2} - \delta + \varepsilon \|\bar{x}\| + \frac{2\varepsilon(\mu+1) + \varepsilon^2}{2} \\
&\leq \frac{\|t_1\|^2 - \|s\|^2}{2} - \delta + \varepsilon(\|\bar{x}\| + \mu + 2) \\
&\leq \frac{\|t_1\|^2 - \|s\|^2}{2} - \frac{\delta}{2}.
\end{aligned}$$

Hence $\bar{x}$ is a strong Slater point for (Q_1, s). $\square$

4 Perturbations of s

Let us consider the simple case in which only s can be perturbed. Then, $\mathcal{V} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the mapping

$$\begin{aligned}
\mathcal{V}(s_1) &= \left\{x \in \mathbb{R}^n : (t - s_1)' x \leq \tfrac{\|t\|^2 - \|s_1\|^2}{2},\ t \in (T \setminus \{s\}) \cup \{s_1\}\right\} \\
&= \left\{x \in \mathbb{R}^n : (t - s_1)' x \leq \tfrac{\|t\|^2 - \|s_1\|^2}{2},\ t \in T \setminus \{s\}\right\}
\end{aligned}$$

for any $s_1 \in \mathbb{R}^n$.

Theorem 4.1 *The following statements are true for any $s \in T$:*
(i) $\mathcal{V}$ is closed at s.
(ii) $\mathcal{V}$ is lsc at s if and only if $V_T(s) = \{s\}$ or s is an isolated point of T.
(iii) $\mathcal{V}$ is usc at s if $V_T(s)$ is bounded.

Proof. First, we express $\mathcal{V}$ as the composition of a continuous ordinary mapping and a well-known set-valued mapping. To this aim, consider the set of functions

$$\Theta = \left\{(a,b) : a : T \backslash \{s\} \to \mathbb{R}^n, b : T \backslash \{s\} \to \mathbb{R}\right\},$$

only depending on T and n, where $(a,b) \in \Theta$ can be identified with the system $\{a_t' x \leq b_t, t \in T \backslash \{s\}\}$. Following Greenberg and Pierskalla (1975), if (c,d) is the resulting system of perturbing (a,b), the size of this perturbation is measured by means of the *uniform pseudometric*, i.e.,

$$d\left((c,d),(a,b)\right) := \sup_{t \in T \backslash \{s\}} \left\|(c_t,d_t) - (a_t,b_t)\right\|_\infty.$$

Let $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ be the corresponding feasible set mapping, i.e.,

$$\mathcal{F}(a,b) = \{x \in \mathbb{R}^n : a_t' x \leq b_t, t \in T \backslash \{s\}\}.$$

It is known that $\mathcal{F}$ is closed everywhere, and that it is usc at (a,b) whenever $\mathcal{F}(a,b)$ is bounded (see, e.g., Chapter 6 in Goberna and López 1998).
Let $\varphi : \mathbb{R}^n \to \Theta$ be such that $\varphi(u)(t) = \left(t-u, \frac{\|t\|^2 - \|u\|^2}{2}\right)$ for all $u \in \mathbb{R}^n$, $t \in T \backslash \{s\}$. Obviously, $\mathcal{V} = \mathcal{F} \circ \varphi$.

(i) Since φ is continuous at $s \in \mathbb{R}^n$ and $\mathcal{F}$ is closed at $\varphi(s) \in \Theta$, $\mathcal{V} = \mathcal{F} \circ \varphi$ is closed at s.
(ii) If $V_T(s) = \{s\}$, then $\mathcal{V}$ is trivially lsc at s. Alternatively, if s is an isolated point of T, $d(s, T \backslash \{s\}) > 0$. By Proposition 3.1, with $Q = T \backslash \{s\}$, there exists $\overline{x} \in \mathbb{R}^n$ and $\delta > 0$ such that $(t-s)' \overline{x} \leq \frac{\|t\|^2 - \|s\|^2}{2} - \delta$ for all $t \in T \backslash \{s\}$. Then $\mathcal{F}$ is lsc at $\varphi(s)$ by Theorem 6.1 in Goberna and López (1998) and $\mathcal{V} = \mathcal{F} \circ \varphi$ turns out to be lsc at s. For the converse, assume that $\mathcal{V}$ is lsc at s and $V_T(s) \neq \{s\}$. Take any $\overline{x} \in V_T(s)$, $\overline{x} \neq s$, and choose some $u \in \mathbb{R}^n$ with $\|u\| = 1$ and $u'(s - \overline{x}) > 0$. Let $\varepsilon > 0$ be such that $u'(s - x) > 0$ if $\|x - \overline{x}\| < \varepsilon$. Put $W = \overline{x} + \varepsilon B_n$, then $W \cap \mathcal{V}(s) \neq \emptyset$ and there exists some $\rho > 0$ such that $\|s_1 - s\| < 2\rho$ implies that $W \cap \mathcal{V}(s_1) \neq \emptyset$. Consider $s_1 = s + \rho u$ and take some $x_1 \in W \cap \mathcal{V}(s_1)$. Then, for any $t \in T$, $t \neq s$, we have

$$
\begin{aligned}
(t-s)' x_1 &= (t-s_1)' x_1 + \rho u' x_1 \\
&\leq \tfrac{\|t\|^2-\|s_1\|^2}{2} + \rho u' x_1 \\
&= \tfrac{\|t\|^2-\|s\|^2}{2} - \frac{2\rho u' s + \rho^2}{2} + \rho u' x_1 \\
&\leq \tfrac{\|t\|^2-\|s\|^2}{2} - \rho u' (s - x_1),
\end{aligned}
$$

where $\rho u' (s - x_1) > 0$. Hence s is an isolated point of T again by Proposition 3.1.
(iii) Since φ is continuous at s and $\mathcal{F}$ is usc at $\varphi(s)$ (because we are assuming that $\mathcal{F}(\varphi(s)) = V_T(s)$ is bounded), $\mathcal{V} = \mathcal{F} \circ \varphi$ is usc at s. $\square$

It is worth observing that, from Lemma 2.1, the boundedness of $V_T(s)$ entails the boundedness of the images of $\mathcal{V}$ in some neighborhood of s (recall that $V_T(s)$ is bounded if and only if $s \in \operatorname{int} \operatorname{conv} T$), whereas statement (iii) shows that $\mathcal{V}$ is actually locally bounded at s.

Observe also that statements (i) and (iii) are still valid for the restriction of $\mathcal{V}$ to T (in this type of perturbations, s_1 is required to be in T), but the "only if" part in (ii) fails, as the next example shows.

Example 4.2 *Consider $T = \left\{t \in \mathbb{R}^2 : \|t\| = 1\right\}$ and the site $s = (\cos\alpha, \sin\alpha) \in T$, $\alpha \in \mathbb{R}$. Then, $\mathcal{V}(s) = \mathbb{R}_+ s$ is not a singleton and s is an accumulation point in T. On the other hand, if W is an open ball such that $\mathcal{V}(s) \cap W \neq \emptyset$, there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $\alpha_1 < \alpha < \alpha_2$ and $\mathbb{R}_+ (\cos\alpha_i, \sin\alpha_i) \cap W \neq \emptyset$, $i = 1, 2$. Then*

$$
\mathcal{V}((\cos\beta, \sin\beta)) \cap W = \mathbb{R}_+ (\cos\beta, \sin\beta) \cap W \neq \emptyset
$$

whenever $\alpha_1 < \beta < \alpha_2$, so that $\mathcal{V}(s_1) \cap W \neq \emptyset$ for $s_1 \in T$ close enough to s.

5 Global perturbations

We suppose in this section that P is a given nonempty compact set which does not contain the fixed site s, and require the maintaining of this property through perturbations. The parameter space Ω is now the family of nonempty compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff distance d_H. The *Hausdorff distance* between two compact sets $P_1, P_2 \subset \mathbb{R}^n$ is

$$d_H(P_1, P_2) = \inf\{\eta \in \mathbb{R}_+ : P_1 \subset P_2 + \eta B_n \text{ and } P_2 \subset P_1 + \eta B_n\},$$

and the set-valued mapping to be analyzed is $\mathcal{V} : \Omega \rightrightarrows \mathbb{R}^n$ such that

$$\mathcal{V}(P_1) = \left\{x \in X : (t-s)' x \leq \frac{\|t\|^2 - \|s\|^2}{2},\ t \in P_1\right\},$$

for $P_1 \in \Omega$, where

$$X := X_P^s = \left\{x \in \mathbb{R}^n : (t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2},\ t \in T \backslash P\right\} \qquad (5.1)$$

will remain fixed along this section. Thus, $\mathcal{V}(P) = V_T(s)$.

Theorem 5.1 *The following statements are true for any $P \subset T$, $P \in \Omega$:*
(i) $\mathcal{V}$ is closed at P.
(ii) $\mathcal{V}$ is lsc at P.
(iii) $\mathcal{V}$ is usc at P if $V_T(s)$ is bounded.

Proof. (i) Let $\{P_k\} \subset \Omega, \{x_k\} \subset \mathbb{R}^n$ be sequences such that $x_k \in \mathcal{V}(P_k)$ for all k, $P_k \to P$, and $x_k \to \bar{x}$, for some $\bar{x} \in \mathbb{R}^n$. Obviously, $\bar{x} \in X$ because X is closed and $\{x_k\} \subset X$. For each positive integer j choose $k_j, k_j > k_{j-1}$, such that $d_H(P_k, P) < 1/j$, for all $k \geq k_j$. Now, fix $t \in P$ and take $t_{k_j} \in P_{k_j}$ with $d\left(t_{k_j}, t\right) < 1/j$. Then $\left(t_{k_j} - s\right)' x_{k_j} \leq \frac{\|t_{kj}\|^2 - \|s\|^2}{2}$ for all j, which implies that $(t-s)' \bar{x} \leq \frac{\|t\|^2 - \|s\|^2}{2}$. So $\bar{x} \in \mathcal{V}(P)$.
(ii) Since we are assuming that $s \notin P = \operatorname{cl} P$, by Proposition 3.1, we can take a strong Slater point for (P, s), $\bar{x} \in X$ with

associated scalar δ. Let W be any open subset in $\mathbb{R}^n$ with $W \cap \mathcal{V}(P) \neq \emptyset$, and take some $y \in W \cap \mathcal{V}(P)$. Consider any $0 < \lambda < 1$ small enough so that $w = (1-\lambda)\, y + \lambda \bar{x} \in W$. Then $w \in X$ and

$$(t-s)' w = (1-\lambda)(t-s)' y + \lambda (t-s)' \bar{x} \leq \frac{\|t\|^2 - \|s\|^2}{2} - \lambda\delta$$

for any $t \in P$. Hence w is a strong Slater point for (P, s). From Proposition 3.2 we get the existence of $\varepsilon > 0$ such that w is also a strong Slater point for (P_1, s) whenever $P_1 \subset P + \varepsilon B_n$, which entails $w \in W \cap \mathcal{V}(P_1)$. Therefore $\mathcal{V}$ is lsc at P.
(iii) Assume that $V_T(s)$ is bounded. Let $\mu > 0$ be such that $\mathcal{V}(P) \subset \mu B_n$. If $\mathcal{V}$ is not locally bounded at P, then there are sequences $\{P_k\}$ and $\{y_k\}$ such that $P_k \to P$, $y_k \in \mathcal{V}(P_k) \backslash \mu B_n$ for all k, and $\|y_k\| \to \infty$. Since the whole segment $[s, y_k]$ is contained in $\mathcal{V}(P_k)$, we can take $z_k \in [s, y_k]$ with $\|z_k\| = \mu$. We may assume w.l.o.g. (by taking a subsequence if necessary) that $z_k \to z$. The closedness of $\mathcal{V}$ gives $z \in \mathcal{V}(P)$. But then $\|z\| = \mu$ contradicts the assumption $\mathcal{V}(P) \subset \mu B_n$. Therefore, $\mathcal{V}$ is locally bounded at P, since it is also closed by (i), it follows that $\mathcal{V}$ is usc at P. $\square$

Concerning (iii), the converse statement does not hold when $V_T(s)$ is unbounded and $V_{T\backslash P}(s) = V_T(s)$ because, in such a case, $\mathcal{V}(P_1) \subset \mathcal{V}(P)$ for any $P_1 \in \Omega$, i.e., $\mathcal{V}$ is usc at P. The next examples show situations guaranteeing that $V_{T\backslash P}(s) = V_T(s)$.

Example 5.2 *Let T, $s \in T$, and $P \subset T$ be such that $T\backslash P$ is dense in T. Then $T \subset \operatorname{cl}(T\backslash P)$, $K_T(s) \subset \operatorname{cl} K_{T\backslash P}(s)$, and so $\operatorname{cl} K_{T\backslash P}(s) = \operatorname{cl} K_T(s)$, i.e., $V_{T\backslash P}(s) = V_T(s)$.*

Example 5.3 *Let T, $s \in T$, and $P \subset T$ be such that $]s, t[\cap (T\backslash P) \neq \emptyset$ for all $t \in P$. To see that $V_{T\backslash P}(s) = V_T(s)$ it is enough to show that, given $t_1 \in P$ there exists $t_2 \in T\backslash P$ such that $d(x, s) \leq d(x, t_2)$ entails $d(x, s) \leq d(x, t_1)$. By the assumption, there exist $t_2 \in T\backslash P$ and $0 < \lambda < 1$ such that $t_2 = \lambda s + (1-\lambda)\, t_1$. By using the convexity of the $\|\cdot\|^2$ function, and writing $x - t_2 = \lambda(x-s) + (1-\lambda)(x-t_1)$, one gets that $\|x - t_2\|^2 \leq$*

$\lambda \left\| x-s \right\|^2 + (1-\lambda) \left\| x-t_1 \right\|^2$, *which gives that* $d(x,s) \leq d(x,t_2)$ *implies* $d(x,s) \leq d(x,t_1)$. *Therefore* $V_{T\setminus P}(s) = V_T(s)$. *For instance, if* $T = \{(0,0),(1,0),(2,0)\}$, $P = \{(2,0)\}$, *and* $s = (0,0)$, $V_T(s)$ *is unbounded but* $(1,0) = \frac{1}{2}(s+(2,0))$, *so that* $\mathcal{V}$ *is usc at* P.

In this section we have considered a compact set P whose perturbations provide compact sets too. Replacing compactness by just boundedness, d_H becomes a pseudometric on the enlarged parameters set, whose topology is no longer Hausdorff as far as $d_H(A,B) = 0$ if and only if $\operatorname{cl} A = \operatorname{cl} B$. Moreover, the extended Voronoi cell mapping fails to be lsc everywhere because the possible location of s on the boundary of P is an important source of instability, as the next example shows. Observe that the condition $s \notin P$ for the elements of Ω precludes $s \in \operatorname{bd} P$ because any $P \in \Omega$ is closed.

Example 5.4 *Take* $T = \left\{ t \in \mathbb{R}^2 : \|t - e_1\| = 1 \right\}$, $s = 0_2$, *and* $P = T \setminus \{0_2\}$ *(i.e., all sites except* 0_2 *are perturbable). For any* ε, $0 < \varepsilon < 1$, *let* $P_\varepsilon = P - \varepsilon(1,0)$. *Obviously,* $d_H(P_\varepsilon, P) = \varepsilon$. *Moreover, from Example 1.1,*

$$\mathcal{V}(P_\varepsilon) = \left\{ x \in \mathbb{R}^2 : \|x\| + \|x - (1-\varepsilon) e_1\| \leq 1 \right\},$$

with $\mathcal{V}(P_\varepsilon) \cap [(-2,0) + B_2] = \emptyset$, *whereas* $\mathcal{V}(P) \cap [(-2,0) + B_2] \neq \emptyset$, *so that* $\mathcal{V}$ *is not lsc at* P, *which is not closed.*

Concerning the boundedness required to the elements of the parameter space, it is a technical assumption which allows to measure the size of the perturbations in a natural way by means of d_H. Nevertheless, it is possible to replace Ω with a hyperspace of closed sets in $\mathbb{R}^n$ equipped with a suitable metric (as the Attouch-Wets one).

6 Pointwise perturbations

In this section we assume that the nominal set of perturbable sites P is bounded, and allow for pointwise perturbations on its elements, while $s \notin P$ remains fixed.

Consider the Banach space $l_{\infty}^{n}(P)$ of all bounded functions from $P \subset \mathbb{R}^n$ to $\mathbb{R}^n$ equipped with the norm

$$\|f\|_{\infty} := \sup_{t \in P} \|f(t)\|_{\infty}, \text{ for all } f \in l_{\infty}^{n}(P).$$

The set P is then represented by the identity mapping on it, say i_P, and the result of perturbing P is represented by the (bounded) range, $f(P)$, of certain $f \in l_{\infty}^{n}(P)$. In this way the parametric space is $l_{\infty}^{n}(P)$ and the set-valued mapping to be analyzed is $\mathcal{V} : l_{\infty}^{n}(P) \rightrightarrows \mathbb{R}^n$ defined by

$$\mathcal{V}(f) = \left\{ x \in X : (f(t) - s)' x \leq \frac{\|f(t)\|^2 - \|s\|^2}{2}, \ t \in P \right\},$$

for all $f \in l_{\infty}^{n}(P)$, where $X := X_P^s$ (as in (5.1)). Observe that the set X is again a fixed closed convex set such that $s \in X$. Obviously, $\mathcal{V}(i_P) = V_T(s)$. Unlikely the global perturbations, in this framework we have:

(a) If T is finite, then $\mathcal{V}(f) = V_{(T \backslash P) \cup f(P)}(s)$ is always a polyhedral convex set because $(T \backslash P) \cup f(P)$ is finite; and

(b) If T is discrete, $(T \backslash P) \cup f(P)$ is discrete too because it is the union of discrete sets, so that $\mathcal{V}(f)$ is quasipolyhedral for all $f \in l_{\infty}^{n}(P)$.

On the other hand it is also true in this framework that the boundedness of $V_T(s)$ is maintained in some neighborhood of i_P, which follows readily by putting $c_t = f(t)$, if $t \in P$, and $c_t = t$, if $t \in T \backslash P$, for any $f \in l_{\infty}^{n}(P)$ such that $\|f - i_P\|_{\infty}$ is small enough and applying Lemma 2.1.

In Example 5.3, $P=\{(2,0)\}$ and $s=0_2$, so that $l_\infty^n(P)=\mathbb{R}^2$ and $\mathcal{V}(i_P)=\left\{x\in\mathbb{R}^2 : x_1\leq\frac{1}{2}\right\}$ has a unique facet and no extreme point. If $f(2,0)=(a,b)\in\mathbb{R}^2$, we have

$$\mathcal{V}(f)=\left\{x\in\mathbb{R}^2 : x_1\leq\frac{1}{2}, ax_1+bx_2\leq\frac{a^2+b^2}{2}\right\},$$

and this polyhedral convex set has two facets and one extreme point whenever $b\neq 0$. Thus, the number of extreme points (or facets) of a Voronoi cell can change even for arbitrarily small pointwise perturbations.

Since we only admit pointwise perturbations on P, we can obtain stability results concerning the closedness and the lower and upper semicontinuity of $\mathcal{V}$ from the general known theory applied to some feasible set mapping by expressing $\mathcal{V}$ as the composition of a continuous ordinary mapping and a well-known set-valued mapping, as in the proof of Theorem 4.1.

Theorem 6.1 *Given $s\in T\backslash P$, the following statements are true:*
(i) $\mathcal{V}$ is closed at i_P.
(ii) $\mathcal{V}$ is lsc at i_P if and only if $V_T(s)=\{s\}$ or P satisfies SSC for s.
(iii) $\mathcal{V}$ is usc at i_P if $V_T(s)$ is bounded.

Proof. Consider the set of functions

$$\Theta=\{(a,b) : a : P\to\mathbb{R}^n, b : P\to\mathbb{R}\},$$

only depending on P and n, where $(a,b)\in\Theta$ can be identified with the system $\{a_t'x\leq b_t, t\in P\}$. We measure again the size of the perturbations in Θ by means of the uniform pseudometric. Let $\mathcal{F}:\Theta\rightrightarrows\mathbb{R}^n$ be the corresponding feasible set mapping, i.e.,

$$\mathcal{F}(a,b)=\{x\in\mathbb{R}^n : a_t'x\leq b_t, t\in P\},$$

and let $\mathcal{F}^X:\Theta\rightrightarrows\mathbb{R}^n$ be the set-valued mapping $\mathcal{F}^X(a,b)=\mathcal{F}(a,b)\cap X$. With this notation, $\mathcal{V}=\mathcal{F}^X\circ\varphi$, where $\varphi: l_\infty^n(P)\to\Theta$ is the ordinary mapping $\varphi(f)(t)=\left(f(t)-s, \frac{\|f(t)\|^2-\|s\|^2}{2}\right)$, which is continuous at i_P because P is bounded. Indeed, if $P \subset \mu B_n$ for some $\mu > 0$, taking any $\varepsilon > 0$ and $f \in l_\infty^n(P)$ such that $\|f - i_P\|_\infty < \varepsilon$, and given $t \in P$, we have $|f_i(t) - t_i| < \varepsilon$, $i = 1, ..., n$, so that

$$\frac{\|f(t)\|^2 - \|t\|^2}{2} < n\varepsilon(\mu + \varepsilon).$$

Hence $\|f - i_P\|_\infty < \varepsilon$ implies $d(\varphi(f), \varphi(i_P)) < \max\{\varepsilon, n\varepsilon(\mu + \varepsilon)\}$. Thus φ is continuous at i_P.

(i) $\mathcal{F}^X$ is closed at $\varphi(f)$ because X is closed (by Corollary 12 in Amaya et al 2008), so that $\mathcal{V} = \mathcal{F}^X \circ \varphi$ is closed at i_P.
(ii) Since $V_T(s) = s + V_{T-s}(0_n)$, we can assume w.l.o.g. that $s = 0_n$.
If $\mathcal{V}(i_P) = \{0_n\}$, $\mathcal{V}$ is trivially lsc at i_P. Alternatively, if $\bar{x}$ is a strong Slater point for (P, s), then $\mathcal{F}^X$ is lsc at $\varphi(i_P)$ (according to Corollary 18 in Amaya et al 2008) because X is a closed convex set), so that $\mathcal{V} = \mathcal{F}^X \circ \varphi$ is lsc at i_P.
For the converse, suppose that $\mathcal{V}$ is lsc at i_P and P does not satisfy SSC for 0_n. Then there is a sequence $\{t_k\}$ in P converging to 0_n and with decreasing norms, $\|t_1\| > \|t_2\| > ... > 0$. For each positive integer k we consider the function $f_k : P \to \mathbb{R}^n$ that sends the $t_j's$ with $j > k$ into the canonical vectors e_i and their opposite vectors $-e_i$, multiplied by convenient scalars, and leaving the remaining $t's$ unchanged:

$$\begin{aligned}
&f_k(t) = t, && \text{if } t \notin \{t_{k+1}, t_{k+2}, \ldots\}, \\
&f_k(t_{j+i}) = \|t_{j+i}\| e_i, && \text{if } i = 1, \ldots, n; \quad j = k, k+2n, k+4n, \ldots, \\
&f_k(t_{j+n+i}) = -\|t_{j+n+i}\| e_i, && \text{if } i = 1, \ldots, n; \quad j = k, k+2n, k+4n, \ldots.
\end{aligned}$$

Then $\{f_k\} \subset l_\infty^n(P)$ and $\|f_k - i_P\|_\infty \leq \|t_k\| + \|t_k\|_\infty \leq (1 + \sqrt{n}) \|t_k\| \to 0$. Now, if $x \in \mathcal{V}(f_k)$, in particular we have for $j \geq k$ and $i = 1, \ldots, n$, that

$$\|t_{j+i}\| x_i = \|t_{j+i}\| e_i' x = f_k(t_{j+i})' x \leq \frac{\|t_{j+i}\|^2}{2},$$
$$-\|t_{j+n+i}\| x_i = -\|t_{j+n+i}\| e_i' x = f_k(t_{j+n+i})' x \leq \frac{\|t_{j+n+i}\|^2}{2},$$

dividing by $\|t_{j+i}\|$ and $\|t_{j+n+i}\|$, respectively, and letting $j \to \infty$ we get that $x = 0_n$. So $\mathcal{V}(f_k) = \{0_n\}$. Finally the lsc of $\mathcal{V}$ at i_P gives that $\mathcal{V}(i_P) = \{0_n\}$.
(iii) Assume that $V_T(s)$ is bounded, i.e., that $\left(\mathcal{F}^X \circ \varphi\right)(i_P)$ is bounded. Then, according to Proposition 20 in Amaya et al 2008, $\mathcal{F}^X$ is usc at $\varphi(f)$ because X is closed and convex. Thus $\mathcal{V} = \mathcal{F}^X \circ \varphi$ is usc and locally bounded at i_P. $\square$

As in Section 5, $\mathcal{V}$ is usc at i_P whenever $V_{T \setminus P}(s) = V_T(s)$, so that the converse statement of (iii) does not hold.

Example 6.2 *Let* $T = \left\{t \in \mathbb{R}^2 : \|t - e_1\| = 1\right\}$, $s = 0_2$ *and* $P = \{(t_1, t_2) \in T : t_2 \notin \mathbb{Q}\}$. *Then* $\mathcal{V}$ *is usc at* i_P *in spite of the unboundedness of* $V_T(s) = \left\{(x_1, 0) \in \mathbb{R}^2 : x_1 \leq 1\right\}$ *because* $T \setminus P = \{(t_1, t_2) \in T : t_2 \in \mathbb{Q}\}$ *is dense in* T *(observe that* P *does not satisfy SSC for* (P, s), *so that* $\mathcal{V}$ *is not lsc at* i_P*).*

The next example shows that the lsc and the usc properties may fail simultaneously in cases where $\operatorname{cl} K_T(s)$ is not a halfspace, P is a non closed set with $s \in \operatorname{bd} P$, and $s \notin \operatorname{int} \operatorname{conv} T$.

Example 6.3 *Take* T *and* s *as in the Example 6.2 above, and* $P = T \setminus \{0_2\}$.
(a) For any ε, $0 < \varepsilon < 1$, *let* $f_\varepsilon : P \to \mathbb{R}^2$ *be such that* $f_\varepsilon(t_1, t_2) = (t_1 - \varepsilon, t_2)$. *Obviously,* $\|f_\varepsilon - i_P\|_\infty = \varepsilon$. *Then,* $\mathcal{V}$ *is not lsc at* i_P *by the same argument of the mentioned Example 6.2.*
(b) For any ε, $0 < \varepsilon < 1$, *let* $f_\varepsilon : P \to \mathbb{R}^2$ *be such that* $f_\varepsilon(t) = (1 - \cos\varepsilon, \sin\varepsilon)$, *if* $t_1 < 0$ *and* $0 < t_2 < \sin\varepsilon$, *and* $f_\varepsilon(t) = t$, *otherwise. Then* $\|f_\varepsilon - i_P\|_\infty < \sqrt{2(1 - \cos\varepsilon)} \to 0$ *as* $\varepsilon \searrow 0$ *whereas*

$$\mathcal{V}(f_\varepsilon) = \left\{x \in \mathbb{R}^2 : x_1 + \left(\cot\frac{\varepsilon}{2}\right) x_2 \leq 1, x_2 \geq 0\right\}$$

is not contained in $\mathcal{V}(i_P) + B_2$ *for any* ε*. Therefore* $\mathcal{V}$ *is not usc at* P*.*

Corollary 6.4 *If* $\mathcal{V}(i_P) \neq \{s\}$*, then* $s \notin \operatorname{cl} P$ *if and only if for any sequence* $\{f_k\} \subset l_\infty^n(P)$*,*

$$\liminf_{k\to\infty} \mathcal{V}(f_k) = \limsup_{k\to\infty} \mathcal{V}(f_k) = \mathcal{V}(i_P).$$

Proof. The assumption $\mathcal{V}(i_P) \neq \{s\}$ gives that $s \notin \operatorname{cl} P$ is equivalent to $\mathcal{V}$ being lsc at i_P. Since $\mathcal{V}$ is closed, we can apply the Painlevé-Kuratowski characterization (2.1) to get the result. $\square$

Corollary 6.5 *If* $V_T(s) = \mathcal{V}(i_P) \neq \{s\}$*, then* $s \notin \operatorname{cl} P$ *if and only if for any sequence* $\{f_k\} \subset l_\infty^n(P)$*,* $V_T(s)$ *is the set formed by all the possible cluster points of sequences* $\{y_k\}$ *with* $y_k \in \mathcal{V}(f_k)$*, and for any* $x \in V_T(s)$ *there exists a convergent sequence* $\{x_k\}$ *such that* $x_k \in \mathcal{V}(f_k)$ *and* $lim_{k\to\infty} x_k = x$*.*

Proof. It follows immediately from the definitions of the limits $\limsup_{k\to\infty} \mathcal{V}(f_k)$ and $\liminf_{k\to\infty} \mathcal{V}(f_k)$. $\square$

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References

[1] Amaya J, Bosch P, Goberna MA (2008) Stability of the feasible set mapping of linear systems with an exact constraint set. Set-Valued Anal. 16: 621-635

[2] Blanquero R, Carrizosa E (2002) A D.C. biobjective location model. J Global Optim 23: 139-154.

[3] Chawla S, Rajan U, Ravi R, Sinha A (2006) Min–Max payoffs in a two-player location game. Oper Res Letters 34: 499-507

[4] Delaunay B. (1934) Sur la sphère vide: À la mémoire de Georges Voronoi (French). Izv. Akad. Nauk SSSR, Otdelenie Matematicheskih i Estestvennyh Nauk 7: 793-800

[5] Goberna MA, López MA (1998) Linear semi-infinite optimization. J. Wiley, Chichester

[6] Goberna MA, Rodríguez MML, Vera de Serio VN (2010) Voronoi cells of arbitrary sets via linear inequality systems. Preprint, Dep. Stat. Oper. Res., University of Alicante

[7] Greenberg HJ, Pierskalla WP (1975) Stability theorems for infinitely constrained mathematical programs. J Optimization Theory Appl 16: 409-428

[8] Gruber PM (2007) Convex and discrete geometry. Springer-Verlag, Berlin

[9] Jooyandeh M, Mohades A, Mirzakhah M (2009) Uncertain Voronoi diagram. Inform Process Lett. 109: 709–712

[10] Okabe A, Boots B, Sugihara K, Chiu SN (2000) Spatial tessellations: concepts and applications of Voronoi diagrams, 2nd edition. J. Wiley, Chichester

[11] Okabe A, Suzuki A (1997) Locational optimization problems solved through Voronoi diagrams. European J Oper Res 98: 445-456

[12] Rockafellar R, Wets RB (1998) Variational analysis. Springer-Verlag, Berlin

[13] Voigt I (2008) Voronoizellen diskreter Punktmengen (German). Dissertation, TU Dortmund University

[14] Voigt I, Weis S (2010) Polyhedral Voronoi cells, Contr. Beiträge Algebra Geom., to appear.