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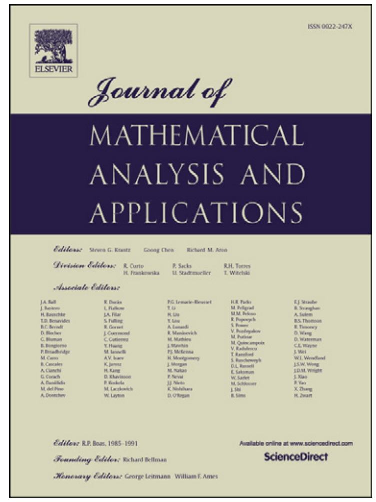

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LIPSCHITZ COMPACT OPERATORS

A. JIMÉNEZ-VARGAS, J.M. SEPULCRE, AND MOISÉS VILLEGAS-VALLECILLOS

Abstract. We introduce the notion of Lipschitz compact (weakly compact, finite-rank, approximable) operators from a pointed metric space X into a Banach space E. We prove that every strongly Lipschitz p-nuclear operator is Lipschitz compact and every strongly Lipschitz p-integral operator is Lipschitz weakly compact. A theory of Lipschitz compact (weakly compact, finite-rank) operators which closely parallels the theory for linear operators is developed. In terms of the Lipschitz transpose map of a Lipschitz operator, we state Lipschitz versions of Schauder type theorems on the (weak) compactness of the adjoint of a (weakly) compact linear operator.

Introduction

Let X be a pointed metric space with a base point denoted by 0 and let E be a Banach space over the field of real or complex numbers $\mathbb{K}$. In the case that X is a normed space, the base point of X will be the origin. The Lipschitz space $\mathrm{Lip}_0(X,E)$ is the Banach space of all Lipschitz mappings f from X to E that vanish at 0, under the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup\left\{\frac{\|f(x)-f(y)\|}{d(x,y)} : x,y \in X,\ x \neq y\right\}.$$

The elements of $\mathrm{Lip}_0(X,E)$ are referred to as Lipschitz operators and the space $\mathrm{Lip}_0(X,\mathbb{K})$, denoted by $X^{\#}$, is called the Lipschitz dual of X. A Lipschitz mapping $f\colon X \to E$ which satisfies the local flatness condition:

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{\|f(x)-f(y)\|}{d(x,y)} = 0,$$

is called a little Lipschitz function, and the little Lipschitz space $\mathrm{lip}_0(X,E)$ is the closed subspace of $\mathrm{Lip}_0(X,E)$ formed by all little Lipschitz functions. In the case $E=\mathbb{K}$, it is usual to write $\mathrm{lip}_0(X)$. For a complete study on these spaces, we suggest the Weaver's book [17].

Recently, Lipschitz versions of different types of bounded linear operators have been investigated by various authors. Farmer and Johnson [7] introduced the notion of Lipschitz p-summing operators and the notion of Lipschitz p-integral operators between metric spaces and proved a nonlinear version of the Pietsch factorization theorem. The Farmer–Johnson factorization theorem was used by Chen and Zheng in [4] to give a nonlinear version of Maurey's extrapolation theorem and deduce a nonlinear form of the Grothendieck's theorem. Moreover, Chen and Zheng [5] introduced and studied strongly Lipschitz p-nuclear operators and Lipschitz p-nuclear operators. Chávez-Domínguez introduced and investigated Lipschitz (p,r,s)-summing operators and Lipschitz (q,p)-mixing operators in [2] and [3], respectively.

In this paper, we introduce natural notions of Lipschitz compact operators, Lipschitz weakly compact operators, Lipschitz finite-rank operators and Lipschitz approximable operators. The concept of a free Banach space $F(X)$ over a pointed metric space X such that every Lipschitz operator $f\in \mathrm{Lip}_0(X,E)$ has an extension to a bounded linear operator $T_f \in \mathcal{L}(F(X),E)$ was introduced by Pestov [16] (see also

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[1]). This procedure provides the linearization of Lipschitz operators and so we can apply the methods of Banach space theory.

The plan of the paper is as follows. Section 1 contains generalities on free Banach spaces that will be needed later. The principal tool is Theorem 1.2, a result proved independently by Pestov [16], Weaver [17] and Kalton [13]. Another essential tool is obtained in Lemma 1.1 where the preduality problem of $X^{\#}$ is analyzed by applying the Dixmier–Ng theorem, and this approach permits to describe the closed unit ball of the Lipschitz-free space of X as the closed convex balanced hull in $(X^{\#})^*$ of the called Lipschitz evaluation functionals.

Section 2 focuses on the notions of Lipschitz compact (weakly compact, finite-rank, approximable) operators from X to E and the extension of the theory of compact (weakly compact, finite-rank bounded) linear operators to the setting of Lipschitz operators. We prove that every strongly Lipschitz p-nuclear operator from X to E is a Lipschitz compact operator and every strongly Lipschitz p-integral operator from X to E is a Lipschitz weakly compact operator. Chen and Zheng [5] introduced a class of Lipschitz operators called strongly Lipschitz p-integral operators which differ from the strongly Lipschitz p-integral operators discussed here (see Definition 2.4 and Remark 2.1). We also address slightly the problem as to when $X^{\#}$ has the approximation property.

In Section 3, and in terms of the Lipschitz transpose map of a Lipschitz operator, we formulate a Lipschitz version of the (Gantmacher's) Schauder's theorem on the (weak) compactness of the adjoint of a (weakly) compact linear operator.

Notation. Given Banach spaces E and F, we denote by $\mathcal{L}((E,\mathcal{T}_E);(F,\mathcal{T}_F))$ the space of all continuous linear operators from $(E,\mathcal{T}_E)$ to $(F,\mathcal{T}_F)$, where $\mathcal{T}_E$ and $\mathcal{T}_F$ are topologies on E and F, respectively. We will not write $\mathcal{T}_E$ whenever it is the norm topology of E. Hence $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E into F with the canonical norm of operators. As is customary, E^* stands for $\mathcal{L}(E,\mathbb{K})$, S_E for the unit sphere of E, B_E for the closed unit ball of E and J_E for the canonical isometric embedding from E into E^{**}. As usual, w^*, w and bw^* denote the weak* topology, the weak topology and the bounded weak* topology, respectively. Finally, $\mathcal{K}(E,F)$, $\mathcal{W}(E,F)$ and $\mathcal{F}(E,F)$ represent the spaces of compact linear operators, weakly compact linear operators and finite-rank bounded linear operators from E into F, respectively.

1. Free Banach spaces

This section contains some functional analytic results on free Banach spaces.

The space $X^{\#}$ is a dual Banach space, that is, it is isometrically isomorphic to the dual of some Banach space. The earliest reference to a predual of $X^{\#}$ is the Arens–Eells space Æ(X) defined as the completion of the vector space of molecules with respect to a natural norm. This space was called and denoted so by Weaver in [17], but it was introduced by Arens and Eells in [1]. Without any reference to molecules, Johnson [11] proved that the closed linear subspace of $(X^{\#})^*$ spanned by the evaluation functionals $\delta_x\colon X^{\#}\to\mathbb{K}$, given by

$$\delta_x(f)=f(x)\qquad \left(f\in X^{\#}\right)$$

with $x\in X$, is a predual of $X^{\#}$. The terminology Lipschitz-free Banach space of X and the notation $\mathcal{F}(X)$ for this predual of $X^{\#}$ are due to Godefroy and Kalton [8].

By the Ng–Dixmier theorem [15], $X^{\#}$ is a dual space since $B_{X^{\#}}$ is a compact subset of $X^{\#}$ for the topology of pointwise convergence τ_p by the Ascoli theorem. The content of the next lemma is surely known but we include it here for future references. It is showed that the predual of $X^{\#}$ provided by the Ng–Dixmier theorem coincides with the Lipschitz-free Banach space of X. This justifies the previous use in the statement of the lemma of the symbol $\mathcal{F}(X)$ for denoting the Ng–Dixmier's predual of $X^{\#}$. Furthermore, by applying the bipolar theorem, we give a precise description of $B_{\mathcal{F}(X)}$ by means of the

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Lipschitz evaluation functionals

$$\delta_{(x,y)} = \frac{\delta_x - \delta_y}{d(x,y)}$$

defined on $X^{\#}$, where (x, y) runs through $\widetilde{X} = \{(x, y) \in X^2 \colon x \neq y\}$.

Before going to this, it is worth noting that if E is a Banach space, the polar set of a subset $M \subset E$ is

$$M^\circ = \{e^* \in E^* \colon |e^*(e)| \leq 1, \ \forall e \in M\},$$

and the prepolar set of a subset $N \subset E^*$ is

$$N_\circ = \{e \in E \colon |e^*(e)| \leq 1, \ \forall e^* \in N\}.$$

The bipolar of M is the set $(M^\circ)_\circ$. We will denote by $\mathrm{lin}(M)$, $\overline{\mathrm{lin}}(M)$ and $\overline{\mathrm{aco}}(M)$ the linear hull, the closed linear hull and the closed, convex, balanced hull of M in E, respectively.

Lemma 1.1. *Let X be a pointed metric space.*

(i) *The space $\mathcal{F}(X)$ of all linear functionals γ on $X^{\#}$ such that γ is τ_p-continuous on $B_{X^{\#}}$ is a Banach space (in fact, a closed subspace of $(X^{\#})^*$).*

(ii) *The evaluation map $Q_X \colon X^{\#} \to \mathcal{F}(X)^*$ defined by*

$$Q_X(f)(\gamma) = \gamma(f) \qquad \big(f \in X^{\#}, \ \gamma \in \mathcal{F}(X)\big)$$

is an isometric isomorphism.

(iii) *The closed unit ball of $\mathcal{F}(X)$ is the closed, convex, balanced hull of the set $\left\{\delta_{(x,y)} \colon (x, y) \in \widetilde{X}\right\}$ in $(X^{\#})^*$.*

(iv) *The space $\mathcal{F}(X)$ is the closed linear hull of the set $\{\delta_x \colon x \in X\}$ in $(X^{\#})^*$.*

Proof. (i) If $\gamma \in \mathcal{F}(X)$, then $\gamma(B_{X^{\#}})$ is the continuous image of the τ_p-compact set $B_{X^{\#}}$, so is compact and hence bounded. Therefore γ is continuous on $X^{\#}$ and so $\gamma \in (X^{\#})^*$. This proves that $\mathcal{F}(X) \subset (X^{\#})^*$.

We next prove that $\mathcal{F}(X)$ is a closed subspace of $(X^{\#})^*$. For it, let $\{\gamma_n\}$ be a sequence in $\mathcal{F}(X)$ converging in $(X^{\#})^*$ to $\gamma \in (X^{\#})^*$ and we must show that $\gamma \in \mathcal{F}(X)$. Let $f_0 \in B_{X^{\#}}$ and $\varepsilon > 0$ be given. There exists $m \in \mathbb{N}$ such that $\|\gamma_m - \gamma\| < \varepsilon/3$. Since $\gamma_m \in \mathcal{F}(X)$, there is a τ_p-neighborhood G of f_0 such that if $f \in G \cap B_{X^{\#}}$, then $|\gamma_m(f) - \gamma_m(f_0)| < \varepsilon/3$. For any $f \in G \cap B_{X^{\#}}$, we have

$$|\gamma(f) - \gamma(f_0)| \leq |\gamma(f) - \gamma_m(f)| + |\gamma_m(f) - \gamma_m(f_0)| + |\gamma_m(f_0) - \gamma(f_0)| < \varepsilon,$$

and this proves that γ is continuous at f_0 with respect to the relative τ_p-topology on $B_{X^{\#}}$.

(ii) It is easy to check that $Q_X \colon X^{\#} \to \mathcal{F}(X)^*$ is linear, injective and continuous (in fact, $\|Q_X(f)\| \leq \mathrm{Lip}(f)$ for all $f \in X^{\#}$). Since each $\gamma \in \mathcal{F}(X)$ is τ_p-continuous on $B_{X^{\#}}$, the restriction $Q_X|_{B_{X^{\#}}}$ is continuous with respect to the relative τ_p-topology and the w^*-topology $\sigma(\mathcal{F}(X)^*, \mathcal{F}(X))$. Since $B_{X^{\#}}$ is τ_p-compact, it follows that $Q_X(B_{X^{\#}})$ is $\sigma(\mathcal{F}(X)^*, \mathcal{F}(X))$-compact. Also, $Q_X(B_{X^{\#}})$ is convex and balanced. By the bipolar theorem, $Q_X(B_{X^{\#}}) = (Q_X(B_{X^{\#}})_\circ)^\circ$ with respect to the duality $(\mathcal{F}(X)^*, \mathcal{F}(X))$. Note that

$$\begin{aligned} Q_X(B_{X^{\#}})_\circ &= \{\gamma \in \mathcal{F}(X) \colon |Q_X(f)(\gamma)| \leq 1, \ \forall f \in B_{X^{\#}}\} \\ &= \{\gamma \in \mathcal{F}(X) \colon |\gamma(f)| \leq 1, \ \forall f \in B_{X^{\#}}\} \end{aligned}$$

which is the closed unit ball of $\mathcal{F}(X)$, and hence $(Q_X(B_{X^{\#}})_\circ)^\circ$ is the closed unit ball of $\mathcal{F}(X)^*$. Hence $Q_X(B_{X^{\#}}) = B_{\mathcal{F}(X)^*}$. It follows that $Q_X \colon X^{\#} \to \mathcal{F}(X)^*$ is a surjective isometry.

(iii) It is an elementary check that each evaluation functional δ_x with $x \in X$ defined on $X^{\#}$ belongs to $\mathcal{F}(X)$, and hence so is every Lipschitz evaluation functional $\delta_{(x,y)}$ with $(x, y) \in \widetilde{X}$. Since Q_X maps

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$X^\#$ onto $\mathcal{F}(X)^*$, we have

$$\begin{aligned} Q_X\left(B_{X^\#}\right) &= \left\{Q_X(f)\colon f\in X^\#,\ \left|\delta_{(x,y)}(f)\right|\le 1,\ \forall (x,y)\in\widetilde{X}\right\} \\ &= \left\{Q_X(f)\colon f\in X^\#,\ \left|Q_X(f)\left(\delta_{(x,y)}\right)\right|\le 1,\ \forall (x,y)\in\widetilde{X}\right\} \\ &= \left\{F\in\mathcal{F}(X)^*\colon\ \left|F\left(\delta_{(x,y)}\right)\right|\le 1,\ \forall (x,y)\in\widetilde{X}\right\} \\ &= \left\{\delta_{(x,y)}\colon (x,y)\in\widetilde{X}\right\}^\circ \end{aligned}$$

and therefore

$$Q_X(B_{X^\#})_\circ = \left(\left\{\delta_{(x,y)}\colon (x,y)\in\widetilde{X}\right\}^\circ\right)_\circ .$$

Moreover, $Q_X(B_{X^\#})_\circ = B_{\mathcal{F}(X)}$, as noted in (ii). Hence $B_{\mathcal{F}(X)}$ is equal to $\overline{\mathrm{aco}}\left(\left\{\delta_{(x,y)}\colon (x,y)\in\widetilde{X}\right\}\right)$ by the bipolar theorem, as desired.

(iv) From (iii) we infer that $\mathcal{F}(X)$ is the closed linear hull in $(X^\#)^*$ of the set $\left\{\delta_{(x,y)}\colon (x,y)\in\widetilde{X}\right\}$. Then (iv) follows since the linear hulls of this set and the set $\{\delta_x\colon x\in X\}$ coincide. Notice that

$$\delta_x = \delta_x - \delta_0 = d(x,0)\delta_{(x,0)} \qquad (x\in X,\ x\neq 0).$$

□

A different approach to the preduality problem of $X^\#$ was taken with the next known result.

Theorem 1.2. [16, 17, 13] *Let X be a pointed metric space. Then there exists a Banach space $F(X)$ and an isometric embedding $e\colon X\to F(X)$ satisfying the following universal property: For each Banach space E and each map $f\in\mathrm{Lip}_0(X,E)$, there is a unique operator $T_f\in\mathcal{L}(F(X),E)$ such that $T_f\circ e = f$ that is, the diagram*

$$\begin{tikzcd} X \arrow[d, "e"'] \arrow[dr, "f"] & \\ F(X) \arrow[r, dashed, "T_f"'] & E \end{tikzcd}$$

commutes, and $\|T_f\|\le \mathrm{Lip}(f)$. This property characterizes the pair $(F(X),e)$ uniquely up to an isometric isomorphism. The mapping $f\mapsto T_f$ is an isometric isomorphism from $\mathrm{Lip}_0(X,E)$ onto $\mathcal{L}(F(X),E)$.

This theorem was independently proved by Pestov in [16, Theorem 1]; Weaver in [17, Theorem 2.2.4] with $(F(X),e) = (\text{Æ}(X),\iota_X)$ where $\text{Æ}(X)$ is the Arens–Eells space of X and ι_X is the isometric embedding from X into $\text{Æ}(X)$ that maps each point x to the atom m_{x0}; and Kalton in [13, Lemma 3.2] with $(F(X),e) = (\mathcal{F}(X),\delta_X)$ where $\mathcal{F}(X)$ is the Lipschitz-free Banach space of X and δ_X is the map $x\mapsto\delta_x$ from X into $\mathcal{F}(X)$.

2. Compactness for Lipschitz operators

If X is a metric space and E is a Banach space, by the Lipschitz image of a mapping $f\colon X\to E$ we mean the set $\{(f(x)-f(y))/d(x,y)\colon x,y\in X,\ x\neq y\}$. It is immediate that $f\colon X\to E$ is a Lipschitz mapping if its Lipschitz image is a bounded subset of E, which motivates the following definition.

Definition 2.1. *Let X be a pointed metric space and E a Banach space. We say that a base-point preserving map $f\colon X\to E$ is Lipschitz compact (Lipschitz weakly compact) if its Lipschitz image is relatively compact (respectively, relatively weakly compact) in E.*

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We denote by $\mathrm{Lip}_{0K}(X, E)$ and $\mathrm{Lip}_{0W}(X, E)$ the sets of Lipschitz compact operators and Lipschitz weakly compact operators from X to E, respectively. Plainly,

$$\mathrm{Lip}_{0K}(X, E) \subset \mathrm{Lip}_{0W}(X, E) \subset \mathrm{Lip}_0(X, E).$$

Note that $\mathrm{Lip}_{0K}(X, E)$ and $\mathrm{Lip}_{0W}(X, E)$ are linear subspaces of $\mathrm{Lip}_0(X, E)$.

Observe that if X and E are Banach spaces and $f : X \to E$ is a linear (weakly) compact operator, then f is a Lipschitz (weakly) compact operator since the Lipschitz image of f is justly $f(S_X)$. So the notion of Lipschitz (weakly) compact operators is really a generalization of (weakly) compact operators in this context.

We next study the relation between the compactness of a Lipschitz operator $f \in \mathrm{Lip}_0(X, E)$ and the compactness of its linearization $T_f \in \mathcal{L}(\mathcal{F}(X), E)$.

Proposition 2.1. *Let X be a pointed metric space, E a Banach space and $f \in \mathrm{Lip}_0(X, E)$. If T_f is the operator in $\mathcal{L}(\mathcal{F}(X), E)$ corresponding to f under the identification in Theorem 1.2, then f is Lipschitz compact if and only if T_f is compact.*

Proof. Consider the map $\delta_{\widetilde{X}} : (x, y) \mapsto \delta_{(x,y)}$ from $\widetilde{X}$ to $(X^\#)^*$, take its image $\delta_{\widetilde{X}}(\widetilde{X})$ and observe that

$$T_f(\delta_{\widetilde{X}}(\widetilde{X})) = \left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X,\ x \neq y \right\}.$$

Since $B_{\mathcal{F}(X)} = \overline{\mathrm{aco}}(\delta_{\widetilde{X}}(\widetilde{X}))$ by Lemma 1.1, the proposition follows from the inclusions

$$T_f(\delta_{\widetilde{X}}(\widetilde{X})) \subset T_f(\overline{\mathrm{aco}}(\delta_{\widetilde{X}}(\widetilde{X})) \subset \overline{\mathrm{aco}}(T_f(\delta_{\widetilde{X}}(\widetilde{X})).$$

□

A remarkable factorization theorem due to Davis, Figiel, Johnson and Pełczynski [6] asserts that any weakly compact linear operator factors through a reflexive Banach space. We next show that Lipschitz weakly compact operators also factor through reflexive spaces.

Proposition 2.2. *Let X be a pointed metric space, E a Banach space and $f \in \mathrm{Lip}_0(X, E)$. The following are equivalent:*

- (i) *The Lipschitz operator f is Lipschitz weakly compact.*
- (ii) *The corresponding operator T_f in $\mathcal{L}(\mathcal{F}(X), E)$ is weakly compact.*
- (iii) *There exist a reflexive Banach space F, a bounded linear operator $T \in \mathcal{L}(F, E)$ and a Lipschitz operator $g \in \mathrm{Lip}_0(X, F)$ such that $f = T \circ g$.*

Proof. The proof of Proposition 2.1 is valid to show the equivalence between (i) and (ii). If (ii) holds, applying the Davis-Figiel-Johnson-Pełczynski theorem, there exists a reflexive Banach space F and operators $T \in \mathcal{L}(F, E)$ and $S \in \mathcal{L}(\mathcal{F}(X), F)$ such that $T_f = T \circ S$. Let $g = S \circ \delta_X$. Clearly, $g \in \mathrm{Lip}_0(X, F)$ and $f = T_f \circ \delta_X = T \circ S \circ \delta_X = T \circ g$, and this proves (iii). Finally, (iii) implies (i) is trivial. □

We now show the ideal property for these new classes of Lipschitz operators.

Proposition 2.3. *Let Y and X be pointed metric spaces and let E and F be Banach spaces. Let $h \in \mathrm{Lip}_0(Y, X)$ and $S \in \mathcal{L}(E, F)$. If $f \in \mathrm{Lip}_{0K}(X, E)$ ($\mathrm{Lip}_{0W}(X, E)$), then $Sfh \in \mathrm{Lip}_{0K}(Y, F)$ (respectively, $\mathrm{Lip}_{0W}(Y, F)$).*

Proof. By [13, Lemma 3.1], there exists a unique operator $\widehat{h} \in \mathcal{L}(\mathcal{F}(Y), \mathcal{F}(X))$ such that $\widehat{h}\delta_Y = \delta_X h$. Clearly, $Sfh \in \mathrm{Lip}_0(Y, F)$. We have the equality $Sfh = ST_f\delta_X h = ST_f\widehat{h}\delta_Y$. Since $ST_f\widehat{h} \in \mathcal{L}(\mathcal{F}(Y), F)$ and, by Theorem 1.2, T_{Sfh} is the unique operator in $\mathcal{L}(\mathcal{F}(Y), F)$ satisfying that equality, it follows that $T_{Sfh} = ST_f\widehat{h}$.

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Assume now that $f \in \mathrm{Lip}_{0K}(X,E)$. Then $T_f \in \mathcal{K}(\mathcal{F}(X),E)$ by Proposition 2.1. Since $\mathcal{K}(\mathcal{F}(X),E)$ is a Banach operator ideal, then $T_{Sfh} \in \mathcal{K}(\mathcal{F}(Y),F)$ which implies that $Sfh \in \mathrm{Lip}_{0K}(Y,F)$ again by Proposition 2.1.

The another case can be proved similarly, but we prefer a new approach. If $f \in \mathrm{Lip}_{0W}(X,E)$, then f factors as Tg through a reflexive Banach space G with $g \in \mathrm{Lip}_0(X,G)$ and $T \in \mathcal{L}(G,E)$ by Proposition 2.2, and so $Sfh = STgh$ which implies that $Sfh \in \mathrm{Lip}_{0W}(Y,F)$ by the same proposition. □

By analogy with the preceding notions, we introduce the following.

Definition 2.2. *Let X be a pointed metric space and E a Banach space. A Lipschitz operator $f \in \mathrm{Lip}_0(X,E)$ has Lipschitz finite dimensional rank if the linear hull of its Lipschitz image is a finite dimensional subspace of E. In that case we define the Lipschitz rank* $\mathrm{Lrank}(f)$ *of f to be the dimension of this subspace.*

This concept is closely related to the following. Let us recall that if X is a set and E is a linear space, then a map $f: X \to E$ is said to have finite dimensional rank if the linear hull of its image is a finite dimensional subspace of E in whose case the rank of f, denoted by $\mathrm{rank}(f)$, is defined as the dimension of $\mathrm{lin}(f(X))$.

Proposition 2.4. *Let X be a pointed metric space, E a Banach space and $f \in \mathrm{Lip}_0(X,E)$. The following are equivalent:*

(i) *The map f has finite dimensional Lipschitz rank.*
(ii) *The map f has finite dimensional rank.*
(iii) *The linearization $T_f \in \mathcal{L}(\mathcal{F}(X),E)$ has finite rank.*

In that case, $\mathrm{lin}(f(X)) = T_f(\mathcal{F}(X))$ *and* $\mathrm{Lrank}(f) = \mathrm{rank}(f) = \mathrm{rank}(T_f)$.

Proof. In Lemma 1.1 (iv), we have proved that $\mathrm{lin}\left\{\delta_{(x,y)} : (x,y) \in \widetilde{X}\right\} = \mathrm{lin}\left\{\delta_x : x \in X\right\}$ and therefore

$$\mathrm{lin}\left\{\frac{f(x)-f(y)}{d(x,y)} : (x,y) \in \widetilde{X}\right\} = \mathrm{lin}\left\{f(x) : x \in X\right\}$$

for any function $f \in \mathrm{Lip}_0(X,E)$. The equivalence between (i) and (ii) and that $\mathrm{Lrank}(f) = \mathrm{rank}(f)$ follow from this observation. We now prove that (ii) is equivalent to (iii). If f has finite dimensional rank, then $\mathrm{lin}(f(X))$ is finite dimensional and therefore closed in E. Invoking Lemma 1.1 and Theorem 1.2, we have

$$\begin{aligned} T_f(\mathcal{F}(X)) &= T_f(\overline{\mathrm{lin}}(\delta_X(X))) \\ &\subset \overline{T_f(\mathrm{lin}(\delta_X(X)))} \\ &= \overline{\mathrm{lin}}(T_f(\delta_X(X))) \\ &= \overline{\mathrm{lin}}(f(X)) \\ &= \mathrm{lin}(f(X)) \end{aligned}$$

and hence T_f has finite rank. Conversely, if T_f has finite rank, then f has finite dimensional rank since

$$\begin{aligned} \mathrm{lin}(f(X)) &= \mathrm{lin}(T_f(\delta_X(X))) \\ &= T_f(\mathrm{lin}(\delta_X(X))) \\ &\subset T_f(\overline{\mathrm{lin}}(\delta_X(X))) \\ &= T_f(\mathcal{F}(X)). \end{aligned}$$

□

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We denote by $\mathrm{Lip}_{0F}(X,E)$ the set of all Lipschitz finite-rank operators from X to E. Note that $\mathrm{Lip}_{0F}(X,E)$ is a linear subspace of $\mathrm{Lip}_{0K}(X,E)$. It seems natural to introduce the following class of Lipschitz operators.

Definition 2.3. *Let X be a pointed metric space and let E be a Banach space. A Lipschitz operator $f \in \mathrm{Lip}_0(X,E)$ is said to be approximable if it is the limit in the Lipschitz norm* Lip *of a sequence of Lipschitz finite-rank operators from X to E.*

It is clear that every Lipschitz approximable operator from X to E is Lipschitz compact by applying Theorem 1.2 and Propositions 2.1 and 2.4.

We recall that a Banach space E is said to have the approximation property if given a compact set $K \subset E$ and $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(E,E)$ such that $\|Tx - x\| < \varepsilon$ for every $x \in K$. The approximation property was introduced by Grothendieck [10], who proved that a dual Banach space E^* has the approximation property if and only if given a Banach space F, an operator $S \in \mathcal{K}(E,F)$ and $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(E,F)$ such that $\|T - S\| < \varepsilon$. Concerning the approximation property in Lipschitz function spaces, we can cite the papers by Johnson [12] and Godefroy and Ozawa [9].

Combining the aforementioned result of Grothendieck with Theorem 1.2 and Propositions 2.1 and 2.4, we next deduce that a necessary and sufficient condition for $X^{\#}$ to have the approximation property is that, for each Banach space E, every Lipschitz compact operator from X to E is Lipschitz approximable.

Corollary 2.5. *Let X be a pointed metric space. Then $X^{\#}$ has the approximation property if and only if for each Banach space E, $\varepsilon > 0$ and $f \in \mathrm{Lip}_{0K}(X,E)$, there exists $g \in \mathrm{Lip}_{0F}(X,E)$ such that* $\mathrm{Lip}(f-g) < \varepsilon$.

From Theorem 1.2 and Propositions 2.1, 2.2 and 2.4 we infer the following identifications.

Corollary 2.6. *Let X be a pointed metric space and E a Banach space. Then the map $f \mapsto T_f$ is an isometric isomorphism between the following spaces:*

(i) *From $\mathrm{Lip}_{0K}(X,E)$ onto $\mathcal{K}(\mathcal{F}(X),E)$.*
(ii) *From $\mathrm{Lip}_{0W}(X,E)$ onto $\mathcal{W}(\mathcal{F}(X),E)$.*
(iii) *From $\mathrm{Lip}_{0F}(X,E)$ onto $\mathcal{F}(\mathcal{F}(X),E)$.*

There is a plentiful supply of Lipschitz compact operators and Lipschitz weakly compact operators as we see next.

Let us recall (see [5, Theorem 2.2]) that given a pointed metric space X and a Banach space E, a Lipschitz operator $f \in \mathrm{Lip}_0(X,E)$ is said to be strongly Lipschitz p-nuclear $(1 \le p < \infty)$ if there exist operators $A \in \mathcal{L}(\ell_p, E)$ and $b \in \mathrm{Lip}_0(X, \ell_\infty)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ induced by a sequence $\lambda \in \ell_p$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "b"'] & E \\ \ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[u, "A"'] \end{tikzcd}$$

The triple (A, b, λ) is called a strongly Lipschitz p-nuclear factorization of f.

Proposition 2.7. *Let X be a pointed metric space, E a Banach space and $1 \le p < \infty$. Every strongly Lipschitz p-nuclear operator from X to E is Lipschitz compact.*

Proof. Let $f\colon X \to E$ be a strongly Lipschitz p-nuclear operator and take a strongly Lipschitz p-nuclear factorization

$$f = AM_\lambda b\colon X \xrightarrow{b} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{A} E.$$

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We can find sequences $\alpha \in c_0$ and $\tau \in \ell_p$ such that $\lambda_n = \alpha_n \tau_n$ for every $n \in N$. Consider the diagonal operators $M_\alpha \colon \ell_\infty \to c_0$ and $M_\tau \colon c_0 \to \ell_p$. Then we have the factorization

$$f = AM_\tau M_\alpha b \colon X \xrightarrow{b} \ell_\infty \xrightarrow{M_\alpha} c_0 \xrightarrow{M_\tau} \ell_p \xrightarrow{A} E.$$

Note that $M_\alpha \in \mathcal{K}(\ell_\infty, c_0)$ and therefore $M_\alpha \in \mathrm{Lip}_{0K}(\ell_\infty, c_0)$ by a remark following Definition 2.1. Then $f \in \mathrm{Lip}_{0K}(X, E)$ by Proposition 2.3. $\square$

In analogy with the definition of strongly Lipschitz p-nuclear operator, we introduce the following.

Definition 2.4. *Let X be a pointed metric space, E a Banach space and $1 \leq p < \infty$. A Lipschitz operator $f \in \mathrm{Lip}_0(X, E)$ is called a strongly Lipschitz p-integral operator if there exists a finite measure space (Ω, Σ, μ), a bounded linear operator $A \in \mathcal{L}(L_p(\mu), E^{**})$ and a Lipschitz operator $b \in \mathrm{Lip}_0(X, L_\infty(\mu))$ giving rise to the commutative diagram:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "b"'] & E \arrow[r, "J_E"] & E^{**} \\
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[u, "A"']
\end{tikzcd}$$

where $I_{\infty,p} \colon L_\infty(\mu) \to L_p(\mu)$ is the formal inclusion operator. The triple (A, b, μ) is called a strongly Lipschitz p-integral factorization of f.

Remark 2.1. *Chen and Zheng [5] introduced a class of Lipschitz operators called strongly Lipschitz p-integral operators which differ from the strongly Lipschitz p-integral operators discussed here. A Lipschitz mapping between Banach spaces $f \colon X \to E$ is strongly Lipschitz p-integral in the terminology of Chen and Zheng if f has a factorization $AI_{\infty,p}b$ but requiring, justly backward as in Definition 2.4, that $A \colon L_p(\mu) \to E^{**}$ is a Lipschitz mapping and $b \colon X \to L_\infty(\mu)$ is a bounded linear operator.*

Proposition 2.8. *Let X be a pointed metric space, E a Banach space and $1 \leq p < \infty$. Every strongly Lipschitz p-integral operator from X to E is Lipschitz weakly compact.*

Proof. Let $f \colon X \to E$ be a Lipschitz p-integral operator and select a strongly Lipschitz p-integral factorization

$$J_E f = AI_{\infty,p}b \colon X \xrightarrow{b} L_\infty(\mu) \xrightarrow{I_{\infty,p}} L_p(\mu) \xrightarrow{A} E^{**}.$$

If $p > 1$, then $L_p(\mu)$ is reflexive, hence $J_E f$ is Lipschitz weakly compact by Proposition 2.2, and so is also f by Proposition 2.3 (or Proposition 2.2). For the case $p = 1$, take any $q > 1$ and factor the operator $I_{\infty,1} \colon L_\infty(\mu) \to L_1(\mu)$ through the space $L_q(\mu)$ in the form

$$I_{\infty,1} = I_{q,1} I_{\infty,q} \colon L_\infty(\mu) \xrightarrow{I_{\infty,q}} L_q(\mu) \xrightarrow{I_{q,1}} L_1(\mu),$$

where $I_{q,1}$ and $I_{\infty,q}$ are the canonical injections. Then we arrive at the same conclusion. $\square$

3. Schauder type theorems for Lipschitz operators

For each $f \in \mathrm{Lip}_0(X, E)$, the Lipschitz adjoint map $f^\# \colon E^\# \to X^\#$, given by $f^\#(g) = g \circ f$ for all $g \in E^\#$, is a continuous linear operator and $\left\| f^\# \right\| = \mathrm{Lip}(f)$. The restriction of $f^\#$ to E^* defines a continuous linear operator into $X^\#$ called the Lipschitz transpose map of f and denoted here by f^t. By means of this map, we may identify the space $\mathrm{Lip}_0(X, E)$ with the closed subspace of $\mathcal{L}(E^*, X^\#)$ formed

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by all continuous linear operators from (E^*, w^*) to $(X^\#, w^*)$. Recall that we can consider the weak* topology on $X^\#$, that is, the topology

$$\left\{Q_X^{-1}(U) \colon U \text{ is open in } (\mathcal{F}(X)^*, w^*)\right\},$$

the isometric isomorphism $Q_X \colon X^\# \to \mathcal{F}(X)^*$ being as in Lemma 1.1.

Theorem 3.1. *Let X be a pointed metric space and E a Banach space. Then the map $f \mapsto f^t$ is an isometric isomorphism from $\mathrm{Lip}_0(X, E)$ onto $\mathcal{L}((E^*, w^*); (X^\#, w^*))$.*

Proof. Let $f \in \mathrm{Lip}_0(X, E)$. We have

$$\begin{aligned} Q_X(f^t(e^*))(\delta_x) &= f^t(e^*)(x) \\ &= e^*(f(x)) \\ &= e^*(T_f(\delta_x)) \\ &= (T_f)^*(e^*)(\delta_x) \end{aligned}$$

for any $e^* \in E^*$ and $x \in X$. Since $\mathcal{F}(X) = \overline{\mathrm{lin}}(\delta_X(X))$, we infer that $Q_X f^t = (T_f)^*$ and, consequently, $f^t = Q_X^{-1}(T_f)^*$. Let us write

$$\begin{array}{ccccccc} \mathrm{Lip}_0(X,E) & \to & \mathcal{L}(\mathcal{F}(X),E) & \to & \mathcal{L}((E^*,w^*);(\mathcal{F}(X)^*,w^*)) & \to & \mathcal{L}((E^*,w^*);(X^\#,w^*)) \\ f & \mapsto & T_f & \mapsto & (T_f)^* & \mapsto & Q_X^{-1}(T_f)^* \end{array}$$

where each mapping is an isometric isomorphism. This proves the theorem. □

Our next aim is to get a Lipschitz version of the Gantmacher's theorem on the weak compactness of the adjoint of a weakly compact linear operator. First, we state a general result for Banach spaces. One may refer to Megginson's book [14] for definitions and properties of the weak* topology, the weak topology and the bounded weak* topology.

Lemma 3.2. *Let E, F be Banach spaces. Then:*

(i) $\mathcal{W}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*)) = \mathcal{L}((E^*, w^*); (F^*, w))$.
(ii) $\mathcal{K}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*)) = \mathcal{L}((E^*, bw^*); F^*)$.

Proof. (i) Let $T \in \mathcal{L}((E^*, w^*); (F^*, w))$. Obviously, $T \in \mathcal{L}((E^*, w^*); (F^*, w^*))$ and therefore $T = S^*$ for some $S \in \mathcal{L}(F, E)$. Since $S^* \in \mathcal{L}((E^*, w^*); (F^*, w))$, it turns out that $S \in \mathcal{W}(F, E)$ by Gantmacher–Nakamura's theorem [14, 3.5.14]. Then Gantmacher's theorem [14, 3.5.13] says us that $T = S^* \in \mathcal{W}(E^*, F^*)$ and so $T \in \mathcal{W}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*))$.

Conversely, let $T \in \mathcal{W}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*))$. Then $T = S^*$ for some $S \in \mathcal{L}(F, E)$. Invoking again the two aforementioned theorems, we obtain that $S \in \mathcal{W}(F, E)$ and $T = S^* \in \mathcal{L}((E^*, w^*); (F^*, w))$.

(ii) Let $T \in \mathcal{L}((E^*, bw^*); F^*)$. Then $J_F(u) \circ T \in \mathcal{L}((E^*, bw^*); \mathbb{K})$ for all $u \in F$ and, by [14, 2.7.8], $J_F(u) \circ T \in \mathcal{L}((E^*, w^*); \mathbb{K})$ for all $u \in F$, that is, $T \in \mathcal{L}((E^*, w^*); (F, w^*))$. Hence $T = S^*$ for some $S \in \mathcal{L}(F, E)$. Then $S^* \in \mathcal{L}((E^*, bw^*); F^*)$ which implies that $S \in \mathcal{K}(F, E)$ by [14, 3.4.16]. It follows that $T = S^* \in \mathcal{K}(E^*, F^*)$ by Schauder's theorem, and so $T \in \mathcal{K}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*))$. For the reverse inclusion, take $T \in \mathcal{K}(E^*, F^*) \cap \mathcal{L}((E^*, w^*); (F^*, w^*))$. Then there is a $S \in \mathcal{L}(F, E)$ such that $S^* = T$. By Schauder's theorem, $S \in \mathcal{K}(E, F)$, and, by [14, 3.4.16], we conclude that $T = S^* \in \mathcal{L}((E^*, bw^*); F^*)$. □

In particular, Lemma 3.2 yields the following result. Note that the spaces of weakly compact and compact linear operators between Banach spaces are Banach operator ideals and that the isometric isomorphism $Q_X \colon X^\# \to \mathcal{F}(X)^*$ is continuous with respect to the weak* topologies, weak topologies and norm topologies.

Lemma 3.3. *Let X be a pointed metric space and E a Banach space. Then:*

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(i) $\mathcal{W}(E^*, X^\#) \cap \mathcal{L}((E^*, w^*); (X^\#, w^*)) = \mathcal{L}((E^*, w^*); (X^\#, w))$.
(ii) $\mathcal{K}(E^*, X^\#) \cap \mathcal{L}((E^*, w^*); (X^\#, w^*)) = \mathcal{L}((E^*, bw^*); X^\#)$.

We now are ready to state the announced result.

Proposition 3.4. *Let X be a pointed metric space, E a Banach space and $f \in \mathrm{Lip}_0(X, E)$. The following are equivalent:*

(i) *f is weakly compact Lipschitz.*
(ii) *f^t is weakly compact from E^* to $X^\#$.*
(iii) *f^t is continuous from (E^*, w^*) to $(X^\#, w)$.*

Proof. $(i) \Leftrightarrow (ii)$ follows from

$$\begin{aligned} f \in \mathrm{Lip}_{0W}(X, E) &\Leftrightarrow T_f \in \mathcal{W}(\mathcal{F}(X), E) \\ &\Leftrightarrow (T_f)^* \in \mathcal{W}(E^*, \mathcal{F}(X)^*) \\ &\Leftrightarrow Q_X^{-1}(T_f)^* \in \mathcal{W}(E^*, X^\#) \\ &\Leftrightarrow f^t \in \mathcal{W}(E^*, X^\#) \end{aligned}$$

by applying Proposition 2.2, the Gantmacher's theorem, the fact that the space of weakly compact linear operators is a Banach operator ideal and the equality $f^t = Q_X^{-1}(T_f)^*$ as noted in the proof of Theorem 3.1.

$(ii) \Leftrightarrow (iii)$ turns out inmediately from the equality (i) in Lemma 3.3. □

We now formulate a Lipschitz version of the Schauder's theorem on the compactness of the adjoint of a compact linear operator.

Proposition 3.5. *Let X be a pointed metric space, E a Banach space and $f \in \mathrm{Lip}_0(X, E)$. The following three statements are equivalent:*

(i) *f is compact Lipschitz.*
(ii) *f^t is compact from E^* to $X^\#$.*
(iii) *f^t is continuous from (E^*, bw^*) to $X^\#$.*

Proof. $(i) \Leftrightarrow (ii)$: From Proposition 2.1, the Schauder's theorem and the fact that the space of compact linear operators between Banach spaces is a Banach operator ideal, we deduce that

$$\begin{aligned} f \in \mathrm{Lip}_{0K}(X, E) &\Leftrightarrow T_f \in \mathcal{K}(\mathcal{F}(X), E) \\ &\Leftrightarrow (T_f)^* \in \mathcal{K}(E^*, \mathcal{F}(X)^*) \\ &\Leftrightarrow f^t = Q_X^{-1}(T_f)^* \in \mathcal{K}(E^*, X^\#) \end{aligned}$$

$(ii) \Leftrightarrow (iii)$ follows clearly from the equality (ii) in Lemma 3.3. □

From the results obtained above we deduce the ensuing identifications.

Corollary 3.6. *Let X be a pointed metric space and E a Banach space. Then the map $f \mapsto f^t$ is an isometric isomorphism between the following spaces:*

(i) *From $\mathrm{Lip}_{0W}(X, E)$ onto $\mathcal{L}((E^*, w^*); (X^\#, w))$.*
(ii) *From $\mathrm{Lip}_{0K}(X, E)$ onto $\mathcal{L}((E^*, bw^*); X^\#)$.*

Proof. In order to prove (i), we only have to check that the map $f \mapsto f^t$ is surjective according to Theorem 3.1 and Proposition 3.4. Let $T \in \mathcal{L}((E^*, w^*); (X^\#, w))$. Then $Q_X T \in \mathcal{L}((E^*, w^*); (\mathcal{F}(X)^*, w))$ and this set is contained in $\mathcal{L}((E^*, w^*); (\mathcal{F}(X)^*, w^*))$. It follows that $Q_X T = S^*$ for some $S \in \mathcal{L}(\mathcal{F}(X), E)$. Hence $S^* \in \mathcal{L}((E^*, w^*); (\mathcal{F}(X)^*, w))$ and, by the Gantmacher–Nakamura's theorem, $S \in \mathcal{W}(\mathcal{F}(X), E)$. Now,

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$S = T_f$ for some $f \in \mathrm{Lip}_0(X, E)$ by Theorem 1.2, and also $f \in \mathrm{Lip}_{0W}(X, E)$ by Proposition 3.4. Finally, $T = Q_X^{-1} S^* = Q_X^{-1}(T_f)^* = f^t$.

(ii) follows analogously from Theorem 3.1, Proposition 3.5 and Theorem 1.2 by taking into account the equality (ii) in Lemma 3.3.

□

In the case that X is compact, the same map identifies $\mathrm{lip}_0(X, E)$ with the space of continuous linear operators from (E^*, bw^*) to $\mathrm{lip}_0(X)$.

Proposition 3.7. *Let X be a pointed compact metric space and E a Banach space. Then the map $f \mapsto f^t$ is an isometric isomorphism from $\mathrm{lip}_0(X, E)$ onto $\mathcal{L}((E^*, bw^*); \mathrm{lip}_0(X))$.*

Proof. Let $f \in \mathrm{Lip}_0(X, E)$. For any $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned} \frac{\|f(x) - f(y)\|}{d(x, y)} &= \sup_{e^* \in B_{E^*}} \frac{|e^*(f(x) - f(y))|}{d(x, y)} \\ &= \sup_{e^* \in B_{E^*}} \frac{|f^t(e^*)(x) - f^t(e^*)(y)|}{d(x, y)}. \end{aligned}$$

We may deduce that if $f \in \mathrm{lip}_0(X, E)$, then for each $e^* \in B_{E^*}$ the function $f^t(e^*)$ is in $\mathrm{lip}_0(X)$ and $\mathrm{Lip}(f^t(e^*)) \leq \mathrm{Lip}(f)$. Hence $f^t(B_{E^*})$ is a bounded subset of $\mathrm{lip}_0(X)$. Moreover,

$$\lim_{d(x,y) \to 0} \sup_{e^* \in B_{E^*}} \frac{|f^t(e^*)(x) - f^t(e^*)(y)|}{d(x, y)} = 0.$$

Then the set $f^t(B_{E^*})$ is relatively compact in $\mathrm{lip}_0(X)$ by [11, Theorem 3.2], that is, $f^t \in \mathcal{K}(E^*, \mathrm{lip}_0(X))$. Consequently, f^t is in $\mathcal{K}(E^*, X^{\#}) \cap \mathcal{L}((E^*, w^*); (X^{\#}, w^*))$ that coincides with $\mathcal{L}((E^*, bw^*); X^{\#})$ by the equality (ii) in Lemma 3.3. Since $f^t(E^*) \subset \mathrm{lip}_0(X)$, then $f^t \in \mathcal{L}((E^*, bw^*); \mathrm{lip}_0(X))$. Hence the map $f \mapsto f^t$ is well defined from $\mathrm{lip}_0(X, E)$ to $\mathcal{L}((E^*, bw^*); \mathrm{lip}_0(X))$. By Theorem 3.1, it is a linear isometry. To check the surjectivity, let $T \in \mathcal{L}((E^*, bw^*); \mathrm{lip}_0(X))$. We can see T as a continuous linear operator from (E^*, bw^*) to $X^{\#}$. By Corollary 3.6, $T = f^t$ for some $f \in \mathrm{Lip}_{0K}(X, E)$. Since $T \in \mathcal{K}(E^*, X^{\#})$ and $T(E^*) \subset \mathrm{lip}_0(X)$, we have $T \in \mathcal{K}(E^*, \mathrm{lip}_0(X))$. Then, by applying again [11, Theorem 3.2] we obtain

$$\begin{aligned} \lim_{d(x,y) \to 0} \frac{\|f(x) - f(y)\|}{d(x, y)} &= \lim_{d(x,y) \to 0} \sup_{e^* \in B_{E^*}} \frac{|f^t(e^*)(x) - f^t(e^*)(y)|}{d(x, y)} \\ &= \lim_{d(x,y) \to 0} \sup_{e^* \in B_{E^*}} \frac{|T(e^*)(x) - T(e^*)(y)|}{d(x, y)} = 0, \end{aligned}$$

and so $f \in \mathrm{lip}_0(X, E)$. This completes the proof. □

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