On the construction of difference schemes for numerical solutions of generalized diffusion equations with delay

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ABSTRACT

This work deals with the construction of difference schemes for numerical solutions of initial-boundary-value problems of the type

$$(1)\ u_t(t,x)=a^2u_{xx}(t,x)+b^2u_{xx}(t-\tau,x),\ t>\tau,\ 0\le x\le l$$

$$(2)\ u(t,0)=0, u(t,l)=0, t\ge 0$$

$$(3)\ u(t,x)=\phi(t,x), 0\le x\le l,\ 0\le t\le \tau$$

where a and b are real constants.

We obtain highly efficient numerical schemes for partial differential equations with delay, analyzing their convergence properties, and showing numerical examples.

DIFFERENCE SCHEMES

We have used the following schemes, where

$$\alpha=a^2\frac{k}{h^2},\ c=\frac{b}{a},\ hP=l,\ kN=\tau,$$

$$\Delta_t u_{n,j}=u_{n+1,j}-u_{n,j},\ \delta_x^2 u_{i,p}=u_{i,p-1}-2u_{i,p}+u_{i,p+1}$$

In Figure 1 we show the diagrams of these schemes.

Explicit

$$(4)\ \Delta_t u_{n,p}=\alpha\delta_x^2 u_{n,p}+\alpha c^2\delta_x^2 u_{n-N,p}$$

Implicit Crank-Nicolson

$$(5)\ \Delta_t u_{n,p}=\frac{\alpha}{2}\left(\delta_x^2 u_{n,p}+\delta_x^2 u_{n+1,p}\right)+\frac{\alpha c^2}{2}\left(\delta_x^2 u_{n-N,p}+\delta_x^2 u_{n+1-N,p}\right)$$

Implicit Richtmyer 12

$$(6)\ \frac{1}{12}\Delta_t u_{n,p-1}+\frac{5}{6}\Delta_t u_{n,p}+\frac{1}{12}\Delta_t u_{n,p+1}=\frac{\alpha}{2}\left(\delta_x^2 u_{n,p}+\delta_x^2 u_{n+1,p}\right)+$$
$$+\frac{\alpha c^2}{2}\left(\delta_x^2 u_{n-N,p}+\delta_x^2 u_{n+1-N,p}\right)$$

$$p=1,2,\ldots,P-1,\ n\ge 0$$

Initial-boundary conditions

The initial-boundary conditions for the three schemes above are

$$(7)\ \begin{array}{ll} u_{n+1,0}=0,\ u_{n+1,P}=0, & n\ge 0 \\ u_{0,p}=\phi(kn,ph), & n=0,1,\ldots,N,\ p=0,1,\ldots,P \end{array}$$

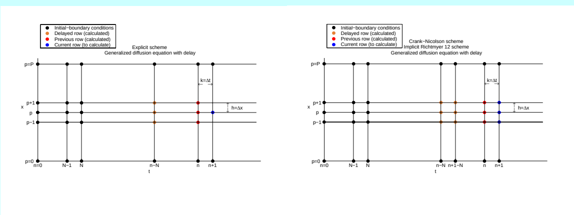

Figure 1: Explicit scheme diagram (left). Implicit Crank-Nicolson and Richtmyer 12 schemes diagram (right).

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RESULTS

Stability

It can be proved that, when $c^2<1$, if $\alpha<\alpha_1$ then the problem defined by (4) and (7) is stable, however if $\alpha>\alpha_2$ then it is unstable, where $\alpha_1=0.5\left(1+c^2\right)^{-1}$ and $\alpha_2=\left(4c^2\sin\left(0.5\pi(P-1)/P\right)\right)^{-1}$

In Figures 2 and 3 we show a stable case and an unstable one. The initial function in these examples is

$$\phi(t,x)=\begin{cases} e^{-t}x & 0\le x<0.5 \\ e^{-t}(1-x) & 0.5\le x<1 \end{cases},\quad 0\le t\le 1$$

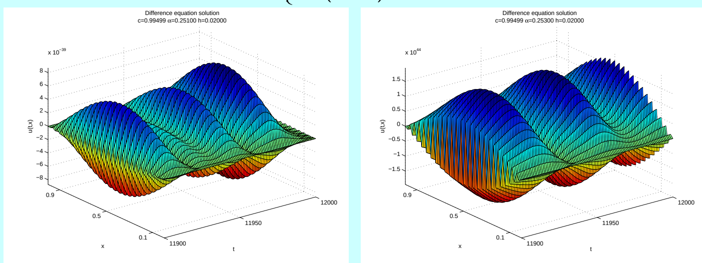

Figure 2: Approximate solution to problem (1)-(3) with *t* in [1190,1200] obtained by using the difference scheme defined by (4) and (7): (left) stable and (right) unstable cases.

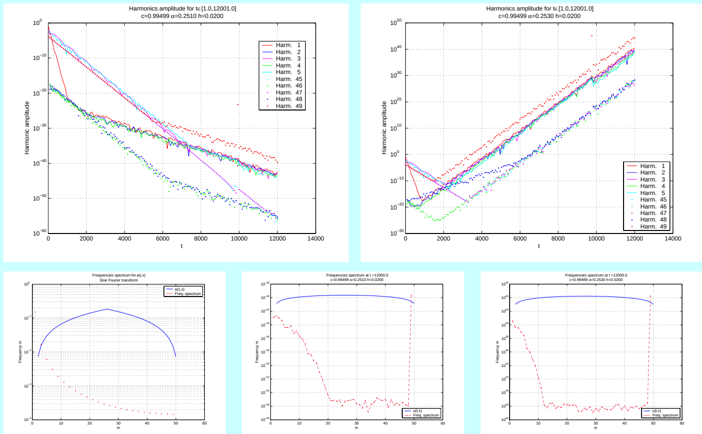

Figure 3: Harmonic analysis. Whole interval (upper left and right). Frequency spectrum: initial function (bottom left) and end of interval (bottom centre and right).

Order of accuracy

The following figure show that the implicit Richtmyer 12 scheme has a higher order of accuracy than the (also implicit) Crank-Nicolson one, while the explicit one has the lowest order of accuracy. The initial function is given by $\phi(t,x)=te^{-t}\sin(\pi x),\ 0\le t\le 1,\ 0\le x\le 1.$

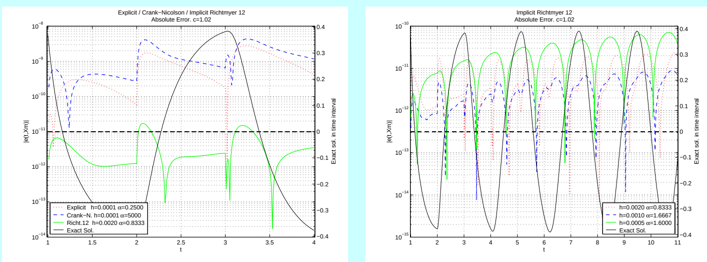

Figure 4: Difference between the solution of problem (1)-(3) and the solution of difference problems (4), (5) and (6) with initial-boundary conditions (7) (left). Influence on the error of the value of *h* (right).

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