ON HOW TO BEST INTRODUCE THE CONCEPT OF DIFFERENTIAL IN PHYSICS

R. López-Gay, J. Martínez-Torregrosa, *Dpto. de Didáctica General y Didácticas Específicas, Alacant, Spain*
A., **Gras-Martí,** *Dpto. de Física Aplicada, Universitat d'Alacant, Alacant, Spain*
G. Torregrosa, *Dpto. de Análisis Matemático y Didáctica de las Matemáticas, Alacant, Spain*

1. Introduction

The almost unanimous opinion of 103 high school Physics teachers, and the results of the analysis of 38 Physics textbooks, indicate clearly that Differential Calculus is necessary to teach Physics in the last years of high school, and it becomes indispensable at university level (Lopez-Gay et al., 2001). This necessity is obvious since Differential Calculus allows to study physical situations with a higher degree of complexity than the ones dealt with in elementary courses.
In agreement with this importance, it should be expected that the use with comprehension of Differential Calculus would be a concern of Physics teachers and a result of the learning process; and that it should be perceived as an indispensable tool to progress in the comprehension of Physics. In spite of this, our teaching experience has showed us that the reality of the customary use of Differential Calculus in Physics lectures is very different: textbooks and lecturers use it in an operational and mechanical way, the students don't understand what is being done and why is it done when it is used even in simple situations, both teachers and students have low expectations of it being used with comprehension and autonomy, its use is perceived as an obstacle and a source of reject against Physics. This impression is confirmed by our analysis of the usual teaching practice.
A clear uneasiness among Physics teachers exists, for they feel obligated to use Differential Calculus in their lectures, but are aware that their students don't understand the reason and the meaning of what is being done. Even worse, there is a *diffuse uneasiness* as a result of their incapability to identify clearly the cause of this situation and to find an adequate way to overcome it.
In order to confront this diffuse uneasiness, we have carried out a historical and epistemological study (Martínez-Torregrosa et al., 2001) looking for answers to the following questions: in what kind of situations is Differential Calculus used?, which is the global strategy of Calculus to deal with those situations?, which is the meaning of the different concepts used?, which is the relationship among them? These questions have remained concealed in the customary education practice under a stream of rules and algorithms, which shows a clear disdain towards the more understandable and conceptual aspects (Artigue and Viennot, 1987; Ferrini-Mundy and Gaudard, 1982, NCTM, 2000).
Taking into account that differential expressions constitute the starting point in the reasoning and *mathematisation* of the physical situations, we have centred our study in the concept of differential, although without losing sight of the global strategy of Calculus. The conclusions we have obtained have enabled us to understand better the present deficiencies and to introduce proposals to overcome them.

2. Towards a better understanding of the differential in physics

The concept of differential has not always answered to the same definition, nor has played the same part in the whole of Calculus. In its origin, in the 17th century, the differential was invented to undertake the step from approximation to exactness. For this, the differential of a quantity (dy) was considered as an *approximation* of the increment of that same quantity (Δy), but an increment so small that they tended to be the same ($dy = \Delta y$). This way one could ignore the error made when the (very small) increments were substituted by differentials, or when the terms containing differentials were neglected (which L'Hôpital resumed in the equation: $y + dy = y$).
An analysis of 38 Physics textbooks shows that this concept of the differential -and the arguments that accompany it- is the only one that appears in Physics lectures. The same criticisms that were

made more than 300 years ago about the use of infinitely small quantities, can be formulated nowadays almost in the same terms: how small is the differential?, is it a fixed quantity, or a variable quantity?, is it really possible to obtain an exact result by neglecting terms that are not zero?, what criteria should be used to determine the differential expression in each specific case?

In general, in order to write the differential expression, an *approximate* expression of the increment is written, with the certainty that, when that increment is *very* small, it can already be considered an *exact* expression. For instance, it is said that the variation of the intensity of a plane wave when crossing a material medium of very small thickness is: $\mathrm{d}I = -\alpha \cdot I \cdot \mathrm{d}x$. But this is not true, no matter how small dx or dI are. It is true, however, that when dx is very small, that expression is *very* approximately –although it never coincides with- ΔI, but the same happens to other expressions (for example: $\mathrm{d}I = -\alpha \cdot I^2 \cdot \mathrm{d}x$, $\mathrm{d}I = -\alpha \cdot \mathrm{d}x/I$, $\mathrm{d}I = -\alpha \cdot I \cdot \mathrm{d}x^2$...). If d$I$ never gets to coincide with ΔI, an error will always exist when substituting one for the other. How then get to the exact result? Furthermore, what criteria do we use to select which starting expression is adequate?

In usual teaching these questions are not dealt with. The wrong belief is implied that, whatever the form of the approximate increment, as it is so small, the error will always be practically zero and will not affect the result. This is wrong: there is only one correct differential expression for each situation[1]. Therefore, the criticisms made to the use of the differential as an infinitely small quantity do not only affect formal or philosophical questions, but have practical consequences.

Why are not wrong results obtained more frequently then? On the contrary, the use of Differential Calculus is considered almost as a guarantee that the result is more correct than when it is not used? It is known that by recalling past examples one selects the correct results, and the absurd results that have historically been obtained are soon forgotten (Orton, 1983; Schneider, 1991). In teaching, the starting differential expression is accepted as a dogma, without arguments nor analysis that justify that particular choice and not a lot of other possible ones. This gives rise to the feeling that the simple idea of differential as a very small increment, where everything is allowed, is enough to sustain the whole of Calculus in the physical applications.

The weight of these criticisms and the adverse results lead to the creators of Calculus to doubt of the identification between the differential and an infinitely small quantity [Kline, 1980, p. 480 and 511]. Leibniz left the differential without a rigorous meaning but, convinced of its utility, did not give up the use of the differential: "the differentials may be used as an instrument, in the same way that the algebraists use the imaginary roots with great benefit" [Kline, 1980, p. 509]. We have confirmed this same ambiguity when we study in depth the significance that teachers and students assign to the differential expressions they constantly use in the Physics lectures.

The following 150 years showed the enormous power of Calculus, but no substantial progress was made in understanding correctly what was being done: one worked "in an almost blind way, often guided by a naïve intuition that what they did had to be valid" (Eves, 1981). Early in the 19th century Cauchy laid out the foundations of Calculus; based on a precise definition of the concept of limit, he transformed the concepts of derivative and integral into the pillars of Calculus, leaving the concept of differential relegated to a marginal role, useful to abbreviate certain formal expansions. Even the definition of differential ($\mathrm{d}y = y' \cdot \mathrm{d}x$) depended on the previous definition of derivative (y'). This work, nevertheless, barely affected the use of Calculus *in physical applications,* where the differential expressions continued to play an essential role, as starting points to tackle a great number of problems and situations; not in vain, the Mathematics of the 19th century marks the break up between Physics and Mathematics, considered by some as a *divorce* (Gonzalez-Urbaneja, 1991), and by others as a *decolonization* (Aghadiuno, 1992). It is not strange, then, that the differential has kept in Physics the original meaning of Leibniz, without overcoming its inconsistence, in spite of the precise definition of the concept of limit and of an infinitesimal quantity provided by Cauchy[2].

---

[1] For example, approximating the portion of a sphere by the corresponding cylinder, the volume of the sphere is correctly obtained, but not the area of its surface [Artigue and Viennot, 1987].

[2] Cauchy defines an infinitely small quantity as "a variable whose numerical value decreases indefinitely so that it converges towards the limit zero" [Cauchy, 1998, pp. 26-27]. It is obvious that the increment of any continuous

3. A definition which improves its use with comprehension in physics

At the beginning of the 20th century, in the context of Functional Analysis, the French mathematician Fréchet formulated a new definition of differential, which "resembles the old definition ... and offers all its advantages, but escaping all the objections of rigour that very justly had been made to it", according to a textual quote of the author [Artigue, 1989, p. 34]. This definition is independent of the derivative, it is centred in the original idea of an *approximation*, and provides a precise meaning to any differential expression; for this reason, we have used it –adapted to functions of one variable- to *rebuild* the use of Differential Calculus in a wide variety of physical situations, giving a precise answer to all the enigmas related with its comprehension (López-Gay et al., 2001).

In short, dy (the differential with respect to x) is an estimate of the Δy produced by an Δx; such an estimate is linear with respect to Δx. Among all the possible linear estimates, the differential is the one whose gradient (dy/dx) coincides with the derivative (y'); this guarantees the possibility of obtaining the exact result by integration, that is, that the limit of the total error accumulated when differentials are added is zero. This conception allows us to overcome the deficiencies related with the original concept:

- It clearly shows that it is necessary to resort to the differential in non-linear situations and, for this reason, the differential will never coincide with the increment (it can be bigger, or smaller), no matter how small it may be.
- The differential is a new function of two variables: x, dx. Its value may be big or small, depending on the value assigned to the variables x, dx. What is infinitely small, to a first order, is the difference between differential and increment: $\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$
- The differential expressions acquire a precise meaning. For example, the expression: dp = F·dt is used when the force is variable in the interval of time Δt o dt, and is an estimate of how much would the linear momentum change if the force remained constant during Δt.
- Sometimes, it is easy to determine the differential expression: when the uniform behaviour is known beforehand (for example, when the force is constant we know that: $\Delta p = F\cdot\Delta t$, therefore the corresponding differential expression when it is not constant will be: dp = F·dt). But, in most physical problems, that behaviour is not known beforehand, and the starting differential expression must be advanced as a hypothesis, based on the physical analysis of the situation, and with the only formal requirement of it being linear with respect to the change of variable. The confirmation of that hypothesis can only be done through the result to which it leads.

This clarification not only leads to a better understanding of the nature and the significance of the differential expressions, but also of the general strategy of Calculus to tackle a physical problem, the reason for the steps that are taken when applying that strategy, and the meaning of other basic concepts (derivative and integral) and the relationships among them. With this clarification we have identified some indicators of what is an adequate comprehension of the differential in Physics:

1. **To know when and why its use becomes necessary**, that is, to know which is the problem that makes ordinary calculus insufficient (i.e., it is necessary to resort to the differential when we need the Δy produced in an Δx, and the relationship between both is not linear).
2. **To understand the strategy that Calculus offers to solve that problem** and comprehend the reason of the different steps that are taken, i.e.:
   a. To be able to explain with accuracy and physical sense the meaning of the expressions.
   b. To know and justify the relationship that exists between the differential and the derivative $y' = dy/dx$, and accept without ambiguity the reasoning in which this relationship is used.
   c. To know the meaning of the integral and to know how to justify the so called Fundamental Theorem, i.e., why the definite integral requires the calculation of *antiderivatives* or primitive functions.

---

function obeys this definition, and therefore it is not useful to characterize the differential expression. Nor can one say that it is the limit of the increment when it tends to zero as, if it is continuous, the differential would always be zero.

d. To use that strategy with full knowledge of the physical content in situations and problems.

3. **To be aware of the hypothetical and exploratory nature**, in almost all physical situations, of the starting differential expression, and to know that the validity of that hypothesis cannot be checked directly but through the result to which it leads.
4. **To value positively** the role of the differential in learning Physics. This axiological component should be a natural consequence when the crucial role that the differential plays in the treatment of physical situations of interest is understood.

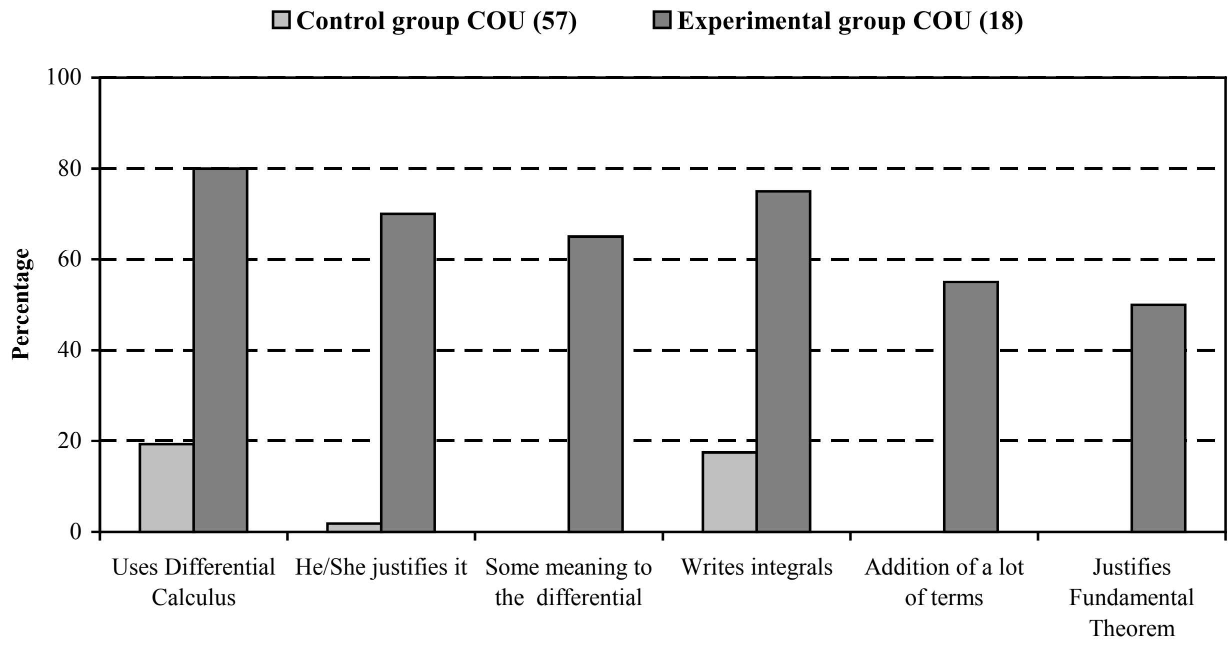
These indicators have been of use to us as a guide to analyse the use of Calculus in the common teaching practices, and to elaborate an alternative proposal.

4. Improving the use comprehension: first results

Although we are aware that it is arguable which is the adequate school year to introduce the use of Calculus, we are convinced that, once that decision is made, it is necessary to do it right, without expecting the comprehension to be acquired in the future, out of the blue. In our case, we begin the use of Calculus in the 3rd year of BUP (16 years old) and complet it in the year of COU (17 years old). We have designed programmes of activities for those two years based on the idea of the differential as a linear estimation of the increment, paying attention to the indicators of an adequate comprehension as listed above. Only in the 2nd year the concept of integral is introduced to all the students, therefore we are not taking into account the indicators 2c and 2d in the 1st year. We have carried out a detailed experimental design to confirm that the incorporation, from the beginning, of the *new* proposal on the differential, improves the teaching and learning. The following diagram shows the results of the analysis of a problem solved by students of COU, in which the use of Differential Calculus was necessary. The students of the experimental group have had one of us (RLG) as their teacher during two years.

We may mention also some results in relation with the attitude that students adopt. 63% of the students of control groups (COU) *do not pay attention when Differential Calculus is used in Physics for they know they are not going to understand it and only pay attention to the final formula.* This clearly reflects the preponderance of the mechanicism in standard teaching practice. On the other hand, 62% of the students of experimental groups (3rd of BUP) taught by teachers trained by us, and 85.4% of the students of one us (RLG) (3rd of BUP), plainly reject this idea, which we interpret as an indication of the use with comprehension which is promoted in their Physics lectures.

In general, the comparative analysis of the results obtained by experimental and control groups show clearly that significant differences exist among them in all the indicators of an adequate comprehension, always in favour of the experimental groups, whether of one of us or of *teachers trained by us.* The differences become bigger with students who have used the new teaching materials for two years. We find ourselves, then, in an adequate direction to improve the use with comprehension of Differential Calculus in the teaching of Physics.



References

Aghadiuno M.C.K., Mathematics: history, philosophy and applications to science, *International Journal for Mathematical Education in Science and Technology*, 23, (5), (1992), 683-690.

Artigue M., and Viennot L., Some aspects of students' conceptions and difficulties about differentials, *Misconceptions and Edu. Strategies in Sci. & Math.* Cornell, Ithaca, USA, (1987).

Artigue M., Le passage de la différentielle totale à la notion d'application linéaire tangente, en: *Procedures différentielles ... (Annexe I)*. Université Paris 7: IREM et LDPES, (1989).

Cauchy A.L., *Cours d'Analyse de L'École Royale Polytechnique*, Facsímil de la 1ª ed., Sevilla: SAEM Thales, (1998), 1821

Eves H., *Great moments in Mathematics (After 1690),* Washington, D.C.: The Mathematical Association of America, Dolciani Mathematical Expositions, 7, (1981).

Ferrini-Mundy J., and Gaudard M., Secondary school calculus: preparation or pitfall in the study of college calculs?, *Journal for Research in Mathematics Education*, 23 (1), (1992), 56-71.

Gonzalez-Urbaneja P.M.,. Historia de la matemática: génesis de los conceptos y orientación de su enseñanza, *Enseñanza de las Ciencias*, 9 (3), (1991), 281-289.

Kline M., *Math. Thought from Ancient to Modern Times*, New York,Oxford, Univ. Press, (1980).

López-Gay R., Martínez-Torregrosa J., and Gras-Martí A, What is the meaning and use of this expresion: $\mathrm{d}N=-\alpha\cdot N\cdot t^2\cdot \mathrm{d}t$?, *GIREP 2000.* (2001) Selected Contributions. R. Pintó & S. Surinach (eds.). Paris: Elsevier editions; and *Enseñanza de las Ciencias*, (to be published)

NCTM, (2000) (National Council of Teachers of Mathematics). *Principle and Standards for School Mathematics*, http://standards.nctm.org/document.

Orton A., Students' understanding of differentiation, *Educ. Stud. in Math.*, 14, (1983), 235-250.

Schneider M., Un obstacle épistémologique soulevé par des "découpages infinis" des surfaces et des solides, *Recherches en Didactique des Mathématiques,* (1991).