# A CHARACTERIZATION OF SPACE-FILLING CURVES

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**Abstract**

A famous theorem discovered in 1936 by H.Steinhaus on a sufficient condition for obtaining the coordinate functions of a curve filling the unit square is revised in the present paper. Here we point out that the converse of the above theorem fails in the Lebesgue curve. A characterization of the space-filling curves by means of a filling condition is proposed. A constructive characterization of this filling condition, in terms of the Borel measures, is also settled.

KEYWORDS.
Borel measures, Jordan content, Space-filling curves, Stochastic independence.

## 1 INTRODUCTION and NOTATION.

In 1890, G.Peano [9] demonstrated that the interval $I = [0, 1]$ could be mapped surjectively and continuously onto the square $Q = [0, 1]^2$. Immediately, furthers examples of such curves by D.Hilbert (1891) [3] , E.H.Moore (1900) [7], H.Lebesgue (1904) [5, 6] and others followed. In spite of each curve was greatly superior in simplicity and ingenuity to the previous, a method for generating them remained unsettled. H.Steinhaus in 1936 [11] solved the problem by means of a surpresively result : " if two continuous non-constant functions on $I$ are stochastically independent with respect to Lebesgue measure, then they are the coordinate functions of a space-filling curve " ( see [10] , *pp*.2 and the original paper of Steinhaus [11]).

The attainment of space-filling curves by means of stochastically independent functions, begun by Steinhaus, was soon forgotten and , apparently, Garsia [1] and others ( see [4] ) arrived to the same conclusions about forty years later. Following the way of the stochastic independence ( in brief, *s.i.* ) , it is neccesary to remark the work of Holbrook in [4].Nevertheless, as we shall prove below, the *s.i.* is a too much strong condition for giving a characterization theorem on space-filling curves, which is exactly the objective of our paper. For this reason we introduce here (Definition 1) a *filling condition* ( in brief *f.c.* ) , which will be appropriate to characterize the space-filling curves.

The *f.c.* is a concept ,implicitly handled in [8] ,that was given to characterize a class of curves that contains to the family of the space-filling , namely the $\alpha - dense\ curves$ in parallelepipeds $H$ of $R^n$.These curves have the property of densifying $H$ ,i.e. for any point of $H$ there is a point of the curve at distance less than or equal that $\alpha \geq 0$.

To avoid that the *f.c.* to be considered as a trivial characterization of space-filling curves, a characterization theorem on the filling condition , in terms of Borel measures, will be also settled..Moreover, this result will point the way to the construction of the coordinate function of a space-filling curve.

In order to facilitate the reading of the text , recall some definitions, contained in [10], concerning to the concepts of space-filling curves, stochastically independent functions and others.

$\Gamma$ will denote the *Cantor set* , $J_n$ the n-dimensional *Jordan content* of a Jordan measurable subset of $R^n$ and $\Lambda_n$ the n-dimensional *Lebesgue measure* of a Lebesgue measurable subset of $R^n$.

A continuous function $f : I \to R^n$ with $n \geq 2$ , is called a $space - filling$ *curve* if $J_n(f(I)) > 0$.

Let $\varphi_1,...,\varphi_n : I \to R$ be measurable functions. Then, $\varphi_1,...,\varphi_n$ are called *stochastically independent* with respect to the Lebesgue measure ( in brief r.L.m.) if, for any measurable sets $A_1, ..., A_n$ of $R$ ,

$$\Lambda_1 \left[\varphi_1^{-1}(A_1) \cap ... \cap \varphi_n^{-1}(A_n)\right] = \Lambda_1 \left[\varphi_1^{-1}(A_1)\right] \times ... \times \Lambda_1 \left[\varphi_n^{-1}(A_n)\right] .$$

A surjective function $f : I \to Q$ is said to be $measure - preserving$ if, for any measurable set $A$ of $Q$,

$$\Lambda_1(f^{-1}(A)) = \Lambda_2(A).$$

## 2 The quasi-stochastic independence as a …lling condition.

The purpose of this section is to prove that the stochastic independence is sufficient but it is not a necessary condition to define space-filling curves. A characterization of these curves will be given by means of the following simple concept .

**Definition 1** *(Filling condition) We shall say that n measurable functions* $\varphi_1,...,\varphi_n$ : $I \to R$ *are quasi-stochastically independent (in brief q.s.i.) with respect to the Lebesgue measure, if for any open sets* $A_1, ..., A_n$ *of* $R$ *the condition*

$\Lambda_1 \left[\varphi_1^{-1}(A_1)\right] \times ... \times \Lambda_1 \left[\varphi_n^{-1}(A_n)\right] > 0$

*implies*

$\Lambda_1 \left[\varphi_1^{-1}(A_1) \cap ... \cap \varphi_n^{-1}(A_n)\right] > 0.$

Our next result is immediate.

**Proposition 2** *Let* $\varphi_1,...,\varphi_n : I \to R$ *be nonconstant continuous functions such that the curve* $f = (\varphi_1,...,\varphi_n) : I \to R^n$ *fills the parallelepiped* $\sqcap_{i=1}^n \varphi_i(I)$. *Thus* $\varphi_1,...,\varphi_n$ *are quasi-stochastically independent (r.L.m).*

As a generalization of the classical result of Steinhaus ( see [10, p.109] or [4, Prop.1] ) we expose the following theorem.

**Theorem 3** *Let* $\varphi_1,...,\varphi_n : I \to R$ *be quasi-stochastically independent functions ( r.L.m.). Assume also that they are continuous but not constant. Thus the curve defined by* $f = (\varphi_1,...,\varphi_n) : I \to R^n$ *fills the parallelepiped* $\sqcap_{i=1}^n \varphi_i(I)$.

**Proof.** Let $x = (x_i)_{i=1}^n$ be a point of $\sqcap_{i=1}^n \varphi_i(I)$, so there exist $(t_i)_{i=1}^n$ in $I$ such that $x_i = \varphi_i(t_i)$ for each $i = 1, 2, ..., n$ .Given $\varepsilon > 0$, consider the open sets

$$A_i = \left(x_i - \frac{\varepsilon}{\sqrt{n}}, x_i + \frac{\varepsilon}{\sqrt{n}}\right) \text{ for } i = 1, 2, ..., n \, . \qquad (1)$$

By continuinity, $\varphi_i^{-1}(A_i)$ is open in $I$ and contains $t_i$, therefore the condition $\Lambda_1 \left[\varphi_1^{-1}(A_1)\right] \times ... \times \Lambda_1 \left[\varphi_n^{-1}(A_n)\right] > 0$ holds.

Since $\varphi_1,...,\varphi_n$ are q-s.i. ,one has $\Lambda_1 \left[\varphi_1^{-1}(A_1) \cap ... \cap \varphi_n^{-1}(A_n)\right] > 0.$

Thus there exists $t \in I$ such that $\varphi_i(t) \in A_i$ .From (1)

$$|\varphi_i(t) - x_i| < \frac{\varepsilon}{\sqrt{n}} \text{ for } i = 1, 2, ..., n,$$

so the euclidean norm $\|f(t) - x\| < \varepsilon$.This proves that $f(I)$ is dense in $\prod_{i=1}^n \varphi_i(I)$, but $f(I)$ is a compact set, therefore $f(I) = \prod_{i=1}^n \varphi_i(I)$ and the result follows. ■

To expose our next results,we need recall some elementary properties on the Cantor set and the Lebesgue curve.

The binary representation of a number $a \in I$ will be denoted by $0_2, a_1a_2a_3...$ where $a_i \in \{0, 1\}$ .Analogously in the ternary basis, $a \in I$ is written as $0_3, a_1a_2a_3...$ with $a_i \in \{0, 1, 2\}$ .

The Cantor set,or the set of the excluded middle thirds ,can be represented by all numbers of $[0, 1]$ such that ,in the ternary basis, can be written only using the digits 0 and 2, i.e.

$$\Gamma = \{0_3, (2t_1)(2t_2)(2t_3)... \; : t_j = 0 \text{ or } 1\} \, .$$

A continuous mapping $f$ can be defined from $\Gamma$ onto the unit square $Q$ by means of

$$f(0_3,(2t_1)(2t_2)(2t_3)...\ ) = (0_2, t_1t_3t_5...\ ; 0_2, t_2t_4t_6...).$$

H. Lebesgue extended this mapping continuously into $I$ by linear interpolation, obtaining a continuous function $f_l$ defined on the complement $\Gamma^c$ as

$$f_l(t) = \frac{1}{b_n - a_n}\left[(b_n - t).f(a_n) + (t - a_n).f(b_n)\right]$$

, $(a_n, b_n)$ being the interval that is removed in the construction of $\Gamma$ at the $n$th step and $a_n \le t \le b_n$ .

Then ,the *Lebesgue curve* (also *Lebesgue function*), denoted by $L$, is defined by

$$\begin{aligned} L(t) &= f(t) \text{ if } t \in \Gamma \\ L(t) &= f_l(t) \text{ if } t \in \Gamma^c. \end{aligned}$$

$L$ is a continuous and surjective function onto the square $Q$, so a space-filling curve ( is also differentiable almost everywhere ; for details see [10], theorem 5.4.2, pp.78 ).

The two following simple propositions shows just how fails the converse of the Steinhaus theorem in the Lebesgue Curve.

**Proposition 4** *The Lebesgue curve is not a measure-preserving function.*

**Proof.** Consider the measurable set $A = \left[0, \frac{1}{2}\right) \times \left[0, \frac{1}{2}\right)$, then we claim that $L^{-1}(A) \subset \left[0, \frac{1}{9}\right)$.

Indeed, let $(x, y)$ be an element belonging to $A$, then in the binary basis

$$x = 0_2, 0r_2r_3... \qquad ; \ y = 0_2, 0s_2s_3... \ ,$$

where $r_i, s_i \in \{0, 1\}$ for any $i \ge 2$ and with some $r_i, s_i = 0$ ( for instance, observe that if all $r_i = 1$ , then $x = \dfrac{1}{2}$ ).

By denoting $t = L^{-1}(x, y)$, we have two cases.

Case 1 : $t \in \Gamma$ , then $t = 0_3, 00(2r_2)(2s_2)...$ and consequently $t \in \left[0, \dfrac{1}{9}\right)$ .

Case 2 : $t \in \Gamma^c$,then we have again that $t \in \left[0, \dfrac{1}{9}\right)$. Indeed, suppose that there is a value $t \in \Gamma^c$ with $t > \dfrac{1}{9}$ and $L(t) \in A$. Let $(a_n, b_n)$ be the interval that has been removed in the construction of $\Gamma$. Thus $a_n < t < b_n$ and noticing that $\dfrac{1}{9} \in \Gamma$ , it follows that $a_n \ge \dfrac{1}{9}$.

On the other hand, as $L(\dfrac{1}{3}\ ) = (\dfrac{1}{2}, 1)$ and $L(\dfrac{2}{3}\ ) = (\frac{1}{2}, 0)$ one has that $t \notin (\dfrac{1}{3}, \dfrac{2}{3})$. Furthermore, since

$L(\frac{1}{9}\ ) = (\frac{1}{2},\frac{1}{2})$ , $L(\frac{2}{9}\ ) = (0,\frac{1}{2})$ , $L(\frac{7}{9}\ ) = (1,\frac{1}{2})$ and $L(\frac{8}{9}) = (\frac{1}{2},\frac{1}{2})$ ,we conclude that $t \notin (\frac{1}{9},\frac{2}{9}) \cup (\frac{7}{9},$ 8 hence

$$\varphi(a) = 0_2, 0r_3r_5... \ \leq \ 0_2, 011... = \frac{1}{2}.$$

Let $a_{\mathsf{n}}$, $b_{\mathsf{n}} \in \left[0, \frac{1}{3}\right) \cap \Gamma$ be the end-points of the interval $(a_{\mathsf{n}}, b_{\mathsf{n}})$ removed in the construction of $\Gamma$, then

$$b_{\mathsf{n}} = 0_3, 0(2s_2)(2s_3)....$$

has the following easy property :

"there exists $k \geq 2$ such that $s_{\mathsf{k}} = 1$ and $s_{\mathsf{i}} = 0$ for all $i > k$ " .As consequence

$$\varphi(b_{\mathsf{n}}) < 0_2, 011... = \frac{1}{2}.$$

For each $t \in \left[0, \frac{1}{3}\right) \setminus \Gamma$ denote by $\lambda$ the number $\dfrac{t - a_{\mathsf{n}}}{b_{\mathsf{n}} - a_{\mathsf{n}}}$, then

$$\varphi(t) = (1 - \lambda)\varphi(a_{\mathsf{n}}) + \lambda\varphi(b_{\mathsf{n}}) \quad \text{ with } \ 0 < \lambda < 1.$$

Since $\varphi(a_{\mathsf{n}}) \leq \frac{1}{2}$ and $\varphi(b_{\mathsf{n}}) < \frac{1}{2}$, we deduce that $\varphi(t) < \frac{1}{2}$ and so $\left[0, \frac{1}{3}\right) \setminus \Gamma \subset \varphi^{-1}(A_1)$ is proved.

On the other hand, if $a \in \Gamma$ with $a \geq \frac{1}{3}$, one has that either $a = 0_3, 022..$ .( for $a = \frac{1}{3}$) or $a = 0_3, 2(2r_2)(2r_3)...$ ( for $a > \frac{1}{3}$). Therefore we get $\varphi(a) \geq \frac{1}{2}$.

Assume $t \in \left(\frac{1}{3}, 1\right] \setminus \Gamma$ , then there exist $a_{\mathsf{n}}$, $b_{\mathsf{n}} \in \Gamma$ with $\frac{1}{3} \leq a_{\mathsf{n}} < t < b_{\mathsf{n}}$ and so

$$\varphi(t) = (1 - \lambda)\varphi(a_{\mathsf{n}}) + \lambda\varphi(b_{\mathsf{n}}) \geq \frac{1}{2}.$$

Therefore $\varphi^{-1}(A_1) \subset \left[0, \frac{1}{3}\right)$ and it proves i). Finally, as conclusion, since $\Lambda_1(\Gamma) = 0$, it follows that

$$\Lambda_1(\varphi^{-1}(A_1)) = \frac{1}{3}. \tag{3}$$

For proving ii), first observe that if $a \in \left(\left[0, \frac{1}{9}\right) \cup \left[\frac{2}{3}, \frac{7}{9}\right)\right) \cap \Gamma$

thus

$$\psi(a) \leq \frac{1}{2}. \tag{4}$$

Indeed, a number $a \in \left[0, \frac{1}{9}\right)$ is expressed as

$$a = 0_3, 00(2r_3)(2r_4)...,$$

with some $r_i = 0$ for $i \geq 3$. Therefore $\psi(a) = 0_2, 0r_4r_6... \leq 0_2, 011... = \frac{1}{2}$.
Analogously, if $a \in \left[\frac{2}{3}, \frac{7}{9}\right)$ then $a = 0_3, 20(2r_3)(2r_4)...$ with some $r_i = 0$ for $i \geq 3$ and so $\psi(a) = 0_2, 0r_4r_6... \leq 0_2, 011... = \frac{1}{2}$.

On the other hand, if $a \in \left(\left[\frac{1}{9}, \frac{2}{3}\right) \bigcup \left[\frac{7}{9}, 1\right)\right) \cap \Gamma$

thus

$$\psi(a) \geq \frac{1}{2}. \tag{5}$$

Indeed, whether $a \in \left[\frac{1}{9}, \frac{2}{3}\right)$ its expression is given by

$a = 0_3, 02(2r_3)(2r_4)...$ and then $\psi(a) = 0_2, 1r_4... \geq 0_2, 1 = \frac{1}{2}$.

If $a \in \left[\frac{7}{9}, 1\right)$, thus $a = 0_3, 2(2r_2)(2r_3)(2r_4)...$ .Now ,noticing that $\frac{7}{9} = 0_3, 2022...$ we have that, either $r_2 = 1$ ( $a > \frac{7}{9}$ ) or $r_2 = 0$ with $r_i = 1$ for all $i \geq 3$ ( $a = \frac{7}{9}$ ) . Therefore, in both cases

$$\psi(a) = 0_2, r_2r_4... \geq 0_2, 1 = \frac{1}{2}.$$

Whether $[a_{\mathsf{n}}, b_{\mathsf{n}}] = \left[\frac{1}{3}, \frac{2}{3}\right]$, since $\frac{1}{2} < t < \frac{2}{3}$, one has

$$\begin{aligned}\psi(t) &= \\ (1-\lambda)\psi(a_{\mathsf{n}}) + \lambda\psi(b_{\mathsf{n}}) &= (1-\lambda)\psi(\frac{1}{3}) + \lambda\psi(\frac{2}{3}) = 1-\lambda = \\ 1 - \frac{t-a_{\mathsf{n}}}{b_{\mathsf{n}}-a_{\mathsf{n}}} &= \frac{\frac{2}{3}-t}{\frac{2}{3}-\frac{1}{3}} < \frac{1}{2}\end{aligned}$$

and this shows the first part of ii).

Finally, if $\psi(x) < \frac{1}{2}$ and $x = a \in \Gamma$, from (6) we deduce that

$$x \in \left[0, \frac{1}{9}\right) \bigcup \left[\frac{2}{3}, \frac{7}{9}\right) \subset \left[0, \frac{1}{9}\right) \bigcup \left(\frac{1}{2}, \frac{7}{9}\right).$$

On the other hand, if $\psi(x) < \frac{1}{2}$ and $x = t \notin \Gamma$, we obtain the same conclusion. Indeed, by supposing that $x \notin \left[0, \frac{1}{9}\right) \cup \left[\frac{2}{3}, \frac{7}{9}\right)$ and by applying again (5) to the end-points $a_{\mathsf{n}}$ and $b_{\mathsf{n}}$ of the removed interval $(a_{\mathsf{n}}, b_{\mathsf{n}})$, with $a_{\mathsf{n}} < t < b_{\mathsf{n}}$, we are led to $\psi(t) \geq \frac{1}{2}$ ,which is a contradiction.

The above involves that for any $x$ with $\psi(x) < \frac{1}{2}$, one follows that $x \in \left[0, \frac{1}{9}\right) \cup \left[\frac{2}{3}, \frac{7}{9}\right) \subset \left[0, \frac{1}{9}\right) \cup \left(\frac{1}{2}, \frac{7}{9}\right)$and therefore ii) is proved.

Now, i) and ii) imply that :

$$\Lambda_1(\psi^{-1}(A_2)) = \frac{7}{18}$$

and from (3) one has

$$\Lambda_1(\varphi^{-1}(A_1)).\Lambda_1(\psi^{-1}(A_2)) = \frac{7}{54}.$$

On the other hand, since $\varphi^{-1}(A_1) \cap \psi^{-1}(A_2) \subset \left[0, \frac{1}{9}\right)$, one deduces

$$\Lambda_1\left(\varphi^{-1}(A_1) \cap \psi^{-1}(A_2)\right) \leq \frac{1}{9} < \frac{7}{54} = \Lambda_1(\varphi^{-1}(A_1)).\Lambda_1(\psi^{-1}(A_2)),$$

concluding that $\varphi$ and $\psi$ are not stochastically independent. ■

From this the following is clear.

**Corollary 6** *There are space-filling curves whose coordinate functions are not stochastically independent.*

**Corollary 7** *There are Q.S.I. functions that are not stochastically independent.*

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# 3 CHARACTERIZATION OF THE Q.S.I. CONDITION.

Though it is obvious that the Steinhaus theorem ( Theorem 1 ) is not a trivial result, the characterization of space-filling curves by means of the Q.S.I. condition could seem it..Therefore, in this section we are going to prove that the Borel measures characterize the Q.S.I condition in such a way that the coordinate functions of a space-filling curve can be easily determined.

From the countably additivity of the Lebesgue measure and Theorem 3, the easy technical lemma follows immediately.

**Lemma 8** *Let $\varphi_1, \varphi_2, ..., \varphi_n : I \rightarrow R$ be nonconstant continuous functions .Suppose also they are Q.S.I. Then the set function $\mu$ defined by*

$$\mu(\prod_{i=1}^{n} B_i) = \Lambda_1 \left[\cap_{i=1}^{n} \varphi_i^{-1}(B_i)\right],$$

*is countably additive on the class $C_H$ of all cubes $C = \prod_{i=1}^{n} B_i$ contained in the parallelepiped $H = \prod_{i=1}^{n} \varphi_i(I)$ . Furthermore ,$\mu(H) = 1$.*

With the help of this lemma we finally have what we wanted all along, the connection between the curves filling a parallelepiped and the Borel measures defined on it.

**Theorem 9** *Assume $\varphi_1, \varphi_2, ..., \varphi_n : I \rightarrow R$ are continuous nonconstant functions verifying the Q.S.I.condition. Then the set function $\mu(\prod_{i=1}^{n} B_i) = \Lambda_1 \left[\cap_{i=1}^{n} \varphi_i^{-1}(B_i)\right]$, on the class $C_H$ of all cubes $\prod_{i=1}^{n} B_i$ contained in $H = \prod_{i=1}^{n} \varphi_i(I)$, defines a Borel measure on $H$ such that*

*$\mu(H) = 1$ and $\mu(C) > 0$ for any cube $C$ of $C_H$ with $int(C) \neq \emptyset$ . (1) Reciprocally, any Borel measure $\mu$ on a parallelepiped $H = \prod_{i=1}^{n} [a_i; b_i]$ $(a_i < b_i, i = 1, 2, ..., n)$ satisfying (1) defines $n$ continuous nonconstant functions that are Q.S.I.*

**Proof.** Suposse that the functions $\varphi_1, \varphi_2, ..., \varphi_n$ are Q.S.I., then ,by Theorem 3, the curve $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ fills $H = \prod_{i=1}^{n} \varphi_i(I)$. On the other hand, by the previous Lemma, $\mu$ is countably additive on the class $C_H$ and satisfies $\mu(H) = 1$. Clearly, then, $\mu$ defines on the ring $\Re(K)$, of all finite disjoint unions of sets of $C_H$, a unique finite measure which is extended to finite meas0previo3thaatio

This allows us to define on $I$ the $n$ functions

$$\begin{aligned} h_1^{(M)}(t) &= x_{(p)1}^{(M)} \text{ if } t \in I_p^{(M)}, \\ h_2^{(M)}(t) &= x_{(p)2}^{(M)} \text{ if } t \in I_p^{(M)}, ..., \\ h_n^{(M)}(t) &= x_{(p)n}^{(M)} \text{ if } t \in I_p^{(M)}, \\ 1 &\leq p \leq 2^{Mn}. \end{aligned}$$

(2)

Now, we are going to prove that the limits

$\lim_{M\to\infty} h_1^{(M)}$, $\lim_{M\to\infty} h_2^{(M)}$, ..., $\lim_{M\to\infty} h_n^{(M)}$

there exist, define functions that are continuous ( observe that $h_1^{(M)}, h_2^{(M)}, ..., h_n^{(M)}$ are not ) and satisfy the Q.S.I. condition.

Indeed,let us take an index $i$ with $1 \leq i \leq n$ and denote by $L_i$ the length of the interval $[a_i, b_i]$ .Directly from the definition of $h_i^{(M)}$ ,

$$\left| h_i^{(M)}(t) - h_i^{(M+1)}(t) \right| = \frac{1}{4} L_i 2^{-M} \text{ for any } t \in I. \tag{3}$$

For $N > M$

$$\begin{aligned} \left| h_i^{(N)}(t) - h_i^{(M)}(t) \right| \leq & \left| h_i^{(N)}(t) - h_i^{(N-1)}(t) \right| + \left| h_i^{(N-1)}(t) - h_i^{(N-2)}(t) \right| + ... \\ & + \left| h_i^{(M+1)}(t) - h_i^{(M)}(t) \right|. \end{aligned}$$

Hence, given $\varepsilon > 0$, there exists a large enough $M_0$ such that for $M \geq M_0$,

$$\left| h_i^{(N)}(t) - h_i^{(M)}(t) \right| \leq \frac{1}{4} L_i \sum_{j=M}^{N} 2^{-j} < \varepsilon \ . \tag{4}$$

This proves that $\left\{ h_i^{(N)}(t) \right\}_{N=1;2;:::}$ is a Cauchy sequence for any $t \in I$ .Hence, there exists the pointwise limit

$$h_i(t) = \lim_{N\to\infty} h_i^{(N)}(t). \tag{5}$$

Taking limits in inequality (4) when $N \to \infty$, one has

$$\left| h_i(t) - h_i^{(M)}(t) \right| \leq \frac{1}{4} L_i \sum_{j=M}^{\infty} 2^{-j} < \varepsilon \text{ for all } M \geq M_0 \tag{6}$$

and , certainly, then, the limit (5) is also uniform.

Now, it remains to show that the $h_i(t)$ are continuous. Indeed, given $\varepsilon > 0$, let $M > 1$ be such that $L_i 2^{-M+1} < \varepsilon$..If $t_0$ is a fixed point of $I$, there exists some $p$ for which $t_0 \in I_p^{(M)}$.Choose a number $\delta$ so that $0 < \delta < Min \left\{ \mu(C_p^{(M)}) : 1 \leq p \leq 2^{Mn} \right\}$.

Clearly, then ,for $t$ such that $|t - t_0| < \delta$ one has that either

$t \in I_{p}^{(M)}$ or $t \in I_{p-1}^{(M)}$ or $t \in I_{p+1}^{(M)}$.

Anyway, from (6) and (2) we have

$$\begin{aligned} |h_i(t) - h_i(t_0)| &\le \\ \left|h_i(t) - h_i^{(M)}(t)\right| + \left|h_i^{(M)}(t) - h_i^{(M)}(t_0)\right| + \left|h_i^{(M)}(t_0) - h_i(t_0)\right| &\le \\ L_i 2^{-M-1} + L_i 2^{-M} + L_i 2^{-M-1} &= L_i 2^{-M+1} < \varepsilon. \end{aligned}$$

This shows the continuity of $h_i$ for all $i = 1, 2, ..., n$.

Finally, let $\{A_i : i = 1, ..., n\}$ be an arbitrary open sets of $R$ such that the condition

$\Lambda_1 \left[h_1^{-1}(A_1)\right] \times ... \times \Lambda_1 \left[h_n^{-1}(A_n)\right] > 0$

holds.Evidently, then, there exists a closed cube $C$ in $H$ with $int(C) \neq \emptyset$, such that $C \subset A = \sqcap_{i=1}^{n} A_i$. Given $C$, determine a cube $C_p^{(M)}$ of a certain partition $P_M$ so that $C_p^{(M)} \subset C$. Denoting by $h$ the function defined by $(h_1, h_2, ..., h_n)$, we are going to prove that $I_p^{(M)}$ ( the corresponding interval to the cube $C_p^{(M)}$ ) verifies

$$I_p^{(M)} \subset h^{-1}(C). \tag{6}$$

Indeed, let $t$ be a point of $I_p^{(M)}$.From (1),the function $h^{(M)}$ ,defined as $\left(h_1^{(M)}, h_2^{(M)}, ..., h_n^{(M)}\right)$, satisfies

$$h^{(M)}(t) = P_p^{(M)}. \tag{7}$$

In view of the given partitions, there exists a cube $C_{p_1}^{(M+1)} \subset C_p^{(M)}$ such that $t \in I_{p_1}^{(M+1)}$ and then,

$h^{(M+1)}(t) = \left(h_1^{(M+1)}(t), h_2^{(M+1)}(t), ..., h_n^{(M+1)}(t)\right) = P_{p_1}^{(M+1)}$.

In this way, we can inductively determine a sequence of cubes

$$... \subset C_{p_N}^{(M+N)} \subset ... \subset C_{p_1}^{(M+1)} \subset C_p^{(M)} \subset C$$

and a sequence $\left\{h^{(M+N)}(t) : N = 1, 2, ..\right\}$ of points of $R^n$. Now, taking the limit, we have

$$\lim_{N\to\infty} h^{(M+N)}(t) = h(t) = \lim_{N\to\infty} P_{p_N}^{(M+N)} = P \in C.$$

Therefore $t \in h^{-1}(C)$ and so (7) is showed.Consequently, we have

$\Lambda_1 \left[\cap_{i=1}^{n} h_i^{-1}(A_i)\right] = \Lambda_1 \left[h^{-1}(A)\right] \ge \Lambda_1(I_p^{(M)})$. (8)

By using (1) ,$\Lambda_1(I_p^{(M)}) = \mu(C_p^{(M)})$. Because of the assumption on the measure $\mu$,

$\mu(C_p^{(M)}) > 0$ . Hence ,from (8), $\Lambda_1 \left[\cap_{i=1}^{n} h_i^{-1}(A_i)\right] > 0$ and the theorem is demonstrated.

**Corollary 10** *Under the same conditions as that of Theorem 2, a Borel measure on a parallelepiped $H$ defines a curve that fills it.*

■

**Proof.** . ■

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