Generic primal-dual solvability in continuous linear semi-infinite programming

M.A. GOBERNA †* , M.I. TODOROV** ‡

† Department of Statistics and Operations Research, University of Alicante,
03071 Alicante, Spain

‡ Operations Research Center, Miguel Hernández University of Elche,
03202 Elche (Alicante), Spain

In this paper we consider the space of all the linear semi-infinite programming (LSIP) problems with a given infinite compact Hausdorff index set, a given number of variables, and continuous coefficients, endowed with the topology of the uniform convergence. These problems are classified as inconsistent, solvable with bounded optimal set, bounded (i.e., finite valued) but either unsolvable or having an unbounded optimal set, and unbounded (i.e., with infinite optimal value), giving rise to the so-called refined primal partition of the space of problems. The mentioned LSIP problems can be also classified with a similar criterion applied to the corresponding Haar's dual problems, which provides the refined dual partition of the space of problems. We characterize the interior of the elements of the refined primal and dual partitions as well as the interior of the intersections of the elements of both partitions (the so-called refined primal-dual partition). These characterizations allow to prove that most (primal or dual) bounded problems have simultaneously primal and dual non-empty bounded optimal set. Consequently, most bounded continuous LSIP problems are primal and dual solvable.

1   Introduction

Throughout this paper we consider given an infinite compact Hausdorff topological space T and a natural number $n \geq 2$. We associate with each triple $\pi = (a, b, c) \in \Pi := \mathcal{C}(T)^n \times \mathcal{C}(T) \times \mathbb{R}^n$ a (continuous) linear semi-infinite programming (LSIP) problem

$$
\begin{aligned}
P : \operatorname{Min} \; & c'x \\
\text{s.t.} \; & a_t'x \geq b_t, \;\; t \in T,
\end{aligned}
$$

and its corresponding Haar's dual problem

$$
\begin{aligned}
D : \operatorname{Max} \; & \sum_{t \in T} \lambda_t b_t \\
\text{s.t.} \; & \sum_{t \varepsilon T} \lambda_t a_t = c, \\
& \lambda_t \geq 0, \; t \in T,
\end{aligned}
$$

where $\lambda : T \mapsto \mathbb{R}$ satisfies that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices. The decision space of D is denoted by $\mathbb{R}^{(T)}$ (the linear space of generalized finite sequences) and its positive cone by $\mathbb{R}_+^{(T)}$. The feasible (optimal) sets of D and P are denoted by Λ and F (Λ^* and F^*, respectively)

Among the well-known applications of continuous LSIP let us mention that problems like P arise in functional approximation, separation, pollution control, finance, Bayesian statistics and the design of telecommunications networks, whereas problems like D have been used in robust Bayesian analysis and optimization under uncertainty. These applications are described in [8, Chapters 1-2], [7], and the references therein. Recall that, in contrast with ordinary linear programming, in continuous LSIP the bounded (primal or dual) problems are not necessarily solvable (i.e., there exist finite-valued problems with no optimal solution).

We consider $\mathbb{R}^{(T)}$ equipped with one of the norms l_∞ or l_1. The space of parameters Π can be interpreted as the set of triples (or dual pairs of problems) obtained by means of arbitrary perturbations performed on a fixed triple $\pi = (a, b, c)$, provided such perturbations preserve the structure of π, i.e., the number of variables, the index set and the continuity of the coefficient functions. We say that a certain desirable property *holds generically* in a certain set $\Delta \subset \Pi$ when this property holds for all the elements of an open dense subset of Δ (for the topology of the uniform convergence on Π defined in Section 2). Typical desirable properties of convex optimization problems are uniqueness of the optimal solution (strong uniqueness in the case of linear objective func-

tion) or at least non-emptyness and boundedness of the optimal set. This last property plays a crucial role in convergence analysis (see, e.g., [1, Chapter 3]).

We denote by Π_C^P, Π_{IC}^P, Π_B^P, and Π_{UB}^P (Π_C^D, Π_{IC}^D, Π_B^D, and Π_{UB}^D) the classes of parameters providing primal (dual) consistent, inconsistent, bounded (i.e., with finite optimal value), and unbounded problems, respectively. We are concerned with those parameters which are stable for the corresponding property in the sense that sufficiently small perturbations of the parameter preserve its membership. The interior of Π_C^P, Π_{IC}^P, Π_C^D, and Π_{IC}^D where characterized in [9] and [10], whereas [3], [12] and [13] have characterized the interior of the elements of the *primal partition* (of Π), $\left\{\Pi_{IC}^P, \Pi_B^P, \Pi_{UB}^P\right\}$, the *dual partition*, $\left\{\Pi_{IC}^D, \Pi_B^D, \Pi_{UB}^D\right\}$, and the *primal-dual partition*, which is formed by the non-empty pairwise intersections of the primal and the dual partitions, identifying those elements of the mentioned partitions which have non-empty interior and showing that each non-empty interior is a dense subsets in the corresponding class. Concerning the primal-dual partition, let us observe that its elements are formed by those parameters sharing the same *duality state* (see [2], [17], [21] and [18] for duality states in mathematical programming, the last one dealing with LSIP and semi-definite programming).

In this paper we extend the previous analysis to the *refined partitions* which result of classifying any bounded problem as having a non-empty bounded optimal set or not satisfying this desirable property (in which case the bounded problem can be either unsolvable or solvable with an unbounded optimal set). This criterion splits Π_B^P into the sets Π_S^P and Π_N^P. Similarly, from the dual perspective, we decompose Π_B^D into the sets of parameters providing dual problems with bounded optimal sets, Π_S^D, and its complement relative to Π_B^D, denoted by and Π_N^D. Thus, we obtain the *refined primal partition*, $\left\{\Pi_{IC}^P, \Pi_S^P, \Pi_N^P, \Pi_{UB}^P\right\}$, the *refined dual partition*, $\left\{\Pi_{IC}^D, \Pi_S^D, \Pi_N^D, \Pi_{UB}^D\right\}$, and the *refined primal-dual partition*, which is formed by the non-empty pairwise intersections of the last two partitions.

In [22] was shown that Π_B^P contains a G_δ (intersection of a countable family of open sets) dense subset of parameters with strongly unique primal optimal solution. In [6] the G_δ-set was replaced by an open set, i.e., it was shown that the strong uniqueness holds generically in Π_B^P. The generic property analyzed in this paper is the primal-dual solvability (i.e., of both associated problems), with bounded optimal sets. This property is weaker than the previous one in the sense that it does not imply the primal strongly uniqueness of the approximating problems, but it is also stronger in the sense that the approximating problems and their dual problems are simultaneously solvable with bounded optimal sets.

The new results appear in Sections 3 and 4. In Section 3 we identify the elements of the refined primal-dual partition, characterizing their respective interiors whereas in Section 4 we provide similar results for the elements of the

refined primal and dual partitions. From the characterizations of the interiors of these sets we prove, in both sections, that primal-dual solvability with bounded optimal set is a generic property in $\Pi_B^P \cup \Pi_B^D$ (the class of parameters for which at least one of the two associated problems is bounded). A suitable counterexample shows that the last statement is not true in general LSIP.

2 Preliminaries

The null-vector in $\mathbb{R}^p$ is denoted by 0_p and the j th element of the canonical basis by e_j. Given a non-empty set $X \subset \mathbb{R}^p$, conv X and cone X denote the convex hull of X and the conical convex hull of $X \cup \{0_p\}$, respectively (so that cone $\emptyset = \{0_p\}$). If X is convex, dim X denotes its dimension. From the topological side, if X is a subset of any topological space, int X, cl X, bd X, and acc X represent the interior, the closure, the boundary, and the set of ω-accumulation points of X, respectively. Finally, $\lim_r x_r = x$ (in short $x_r \to x$) must be interpreted as $\lim_{r\to\infty} x_r = x$.

The next lemma guarantees the existence of continuous real functions on T satisfying certain conditions. Such elements of $\mathcal{C}(T)$ will allow us to construct ad hoc elements of Π.

LEMMA 2.1 *Let T be an infinite compact Hausdorff space and $m \in \mathbb{N}$. Then there exists a point $\bar{t} \in \mathrm{acc}\, T$, m subsets of T, $S_1, ..., S_m$, and m functions in $\mathcal{C}(T)$ and values in $[0,1]$, $\varphi_1, ..., \varphi_m$, such that $\varphi_i(S_i) \subset \mathbb{R}_{++}$ and $\varphi_i(\bar{t}) = 0 \in \mathrm{acc}\, \varphi_i(S_i)$, for all $i = 1, ..., m$, and $\varphi_i(S_j) = \{0\}$ for all $i, j = 1, ..., m$ such that $i \neq j$.*

Proof We use the following consequence of Urisohn's Lemma (already used in the proof of [10, Theorem 6.4(iii)]): there exists a point $\bar{t} \in \mathrm{acc}\, T$ and a sequence of non-repeated indices $\{t_r\}_{r=1}^\infty \subset T$, with $t_r \to \bar{t}$, and a sequence $\{f_r\} \subset \mathcal{C}(T)$, with $f_r : T \to [0,1]$, such that $f_r(t_r) = 1$ and $f_r(t_k) = 0$ for all $k \neq r$.

It is easy to see that the sets $S_i := \{t_{mr+i}, r = 1, 2, ...\}$ and the functions $\varphi_i := \sum_{r=1}^\infty 2^{-(mr+i)} f_{mr+i}$, $i = 1, ..., m$, satisfy all the requirements. $\square$

We denote by $v^P(\pi)$ $(v^D(\pi))$ the optimal value of P (D), defining as usual $v^P(\pi) = +\infty$ $(v^D(\pi) = -\infty$, respectively) when the corresponding problem is inconsistent. Since P and D can be either inconsistent (IC) or bounded (B) or unbounded (UB), we get at most nine possible duality states, which are reduced to six by the weak duality theorem: $v^D(\pi) \leq v^P(\pi)$. The first row in Table 1 contains the elements of the primal partition, the first column are the elements of the dual partition, and the sets in the remaining cells, $\Pi_1, ..., \Pi_6$, are

the non-empty intersections of the corresponding entries (i.e., $\Pi_1 = \Pi_B^P \cap \Pi_B^D$, etc.), so that they are the elements of the primal-dual partition.

	Π_{IC}^P	Π_B^P	Π_{UB}^P
Π_{IC}^D	Π_4	Π_5	Π_2
Π_B^D	Π_6	Π_1	
Π_{UB}^D	Π_3		

Table 1

As most of the works on perturbation theory in continuous LSIP (e.g., the classical paper [20]), we consider Π equipped with the metric of the uniform convergence, i.e., given $\pi^i = \left(a^i, b^i, c^i\right) \in \Pi$, $i = 1, 2$, the distance between π^1 and π^2 is

$$d(\pi^1, \pi^2) = \max \left\{ \left\| c^1 - c^2 \right\|_\infty, \max_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\|_\infty \right\}. \tag{1}$$

From now on, the same subscripts or superscripts that distinguish the elements of Π will distinguish also their corresponding objects: $\pi^r = (a^r, b^r, c^r)$, D_r, P_r, Λ_r, F_r, and so on. Thus, according to Farkas lemma, if $\pi^1, \pi^2 \in \Pi_C^P$, then $F_1 = F_2$ if and only if $\operatorname{cl} K_1 = \operatorname{cl} K_2$.

We will show that some results in this paper are not valid for general LSIP, where T is an arbitrary infinite set (not even a topological space) and the functions in the triple $\pi = (a, b, c)$ are also arbitrary. To do that we need some additional notation. In this general setting the space of parameters is $\Theta := (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ equipped with the pseudometric which generates in Θ the topology of the uniform convergence (replacing "max" with "sup" in (1)) introduced in [14]. We denote the relevant subsets of Θ with the same subscripts and upperscripts as for the corresponding subsets of Π (i.e., Θ_C^P stands for the parameters with consistent primal problem, etc.).

Next we recall some concepts and basic results we will use on general LSIP (all the proofs and references can be found in [8]). We associate with $\pi = (a, b, c)$ its *first* and *second moment cones* of π, $M := \operatorname{cone}\{a_t, t \in T\}$ and $N := \operatorname{cone}\{(a_t, b_t), t \in T\}$, and its *characteristic cone*, $K := N + \mathbb{R}_+ \{(0_n, -1)\}$. Moreover, if P is consistent and N is closed, then K is closed too. If D is consistent and K is closed, then $\sup\{\alpha \in \mathbb{R} \mid (c, \alpha) \in K\}$ is attained and, so, D is solvable. The existence theorem establishes that P is consistent if and only if $(0_n, 1) \notin \operatorname{cl} K$. In such a case, the non-homogeneous Farkas lemma establishes that the inequality $c'x \geq d$ holds for all $x \in F$ if and only if $(c, d) \in \operatorname{cl} K$.

The above results are also valid in continuous LSIP, where the Slater con-

straint qualification plays a crucial role. Recall that $\pi = (a, b, c)$ satisfies the *Slater condition* if there exists $\overline{x} \in \mathbb{R}^n$ such that $a_t' \overline{x} > b_t$ for all $t \in T$. The Slater condition holds if and only if $0_{n+1} \notin \operatorname{conv} \{(a_t, b_t), t \in T\}$. If π satisfies the *Slater condition*, then N is closed.

We use the following characterizations of Π_S^P and Π_S^D (see [8, Corollary 9.3.1 and Theorem 9.8]).

LEMMA 2.2 *(i)* $\pi \in \Pi_S^P$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$ *and* $c \in \operatorname{int} M$.
(ii) $\pi \in \Pi_S^D$ *if and only if* $c \in M$ *and Slater condition holds.*

The next example shows that Lemma 2.2 is not true in general LSIP

Example 2.3 Let $T = \mathbb{N}^2$, $n = 2$ and $\pi = (a, b, c) \in \Theta$ be such that a is the identity map on T, $b = -1$ and $c = (1, 1)$. It is easy to see that $K_1 = \left(\mathbb{R}_{++}^2 \times \mathbb{R}_-\right) \cup \{0_2\}$, $c^1 \in \operatorname{int} M_1 = \mathbb{R}_{++}^2$, the Slater condition holds (take $\overline{x} = 0_2$), $F_1^* = \{0_2\}$, and $\sup \left\{\alpha \in \mathbb{R} \mid \left(c^1, \alpha\right) \in K_1\right\} = 0$ is not attained for all $\pi^1 = \left(a^1, b^1, c^1\right) \in \Theta$ such that $d\left(\pi^1, \pi\right) < \frac{1}{2}$. Thus we have an open subset of Π_S^P where statement (ii) in Lemma 2.2 fails.

Let us interpret the topological interior of some elements of the partitions considered in this paper. A LSIP problem is called *ill-posed in the feasibility sense* if arbitrarily small perturbations provide both consistent and inconsistent problems (the distance to ill-posedness in Θ has been characterized in [3]). Consequently, the sets $\operatorname{int} \Pi_C^P$ and $\operatorname{int} \Pi_{IC}^P$ ($\operatorname{int} \Pi_C^D$ and $\operatorname{int} \Pi_{IC}^D$) can be seen as the set of primal (dual) stable problems (these interiors have been characterized in [9], [8] and [10]). On the other hand, [19] defines a conic programming problem to be *ill-posed (in primal-dual feasibility sense)* when it lays on the boundary of the set of parameters providing consistent primal and dual problems. This class of primal-dual ill-posed parameters is, in our setting, $\operatorname{bd} \Pi_1$. The following lemma summarizes results on the primal-dual partition which appeared in [22] (where $\operatorname{int} \Pi_1$ was characterized), [3], [12, Section 4] (taking into account that N can be replaced with K in all the characterizations) and [13].

LEMMA 2.4 *The elements of the primal-dual partition are neither open nor closed and satisfy the following statements:*
(i) $\pi \in \Pi_1$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$ *and* $c \in M$. *In particular,* $\pi \in \operatorname{int} \Pi_1$ *if and only if Slater condition holds and* $c \in \operatorname{int} M$. *Moreover,* $\operatorname{int} \Pi_1$ *is dense in* Π_1.
(ii) $\pi \in \Pi_2$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$, $c \notin M$ *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} K = \emptyset$. *In particular,* $\pi \in \operatorname{int} \Pi_2$ *if and only if there exists* $y \in \mathbb{R}^n$ *such that* $c'y < 0$ *and* $a_t'y > 0$ *for all* $t \in T$. *Moreover,* $\operatorname{int} \Pi_2$ *is dense in* Π_2.
(iii) $\pi \in \Pi_3$ *if and only if* $(0_n, 1) \in \operatorname{cl} K$, $c \in M$ *and* $\{c\} \times \mathbb{R} \subset K$. *In particular,* $\pi \in \operatorname{int} \Pi_3$ *if and only if* $(0_n, 1) \in \operatorname{int} K$. *Moreover,* $\operatorname{int} \Pi_3$ *is dense in* Π_3.

(iv) $\pi \in \Pi_4$ *if and only if* $(0_n, 1) \in \operatorname{cl} K$ *and* $c \notin M$. *Moreover,* $\operatorname{int} \Pi_4 = \emptyset$.
(v) $\pi \in \Pi_5$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$, $c \notin M$ *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} K \neq \emptyset$.
Moreover, $\operatorname{int} \Pi_5 = \emptyset$.
(vi) $\pi \in \Pi_6$ *if and only if* $(0_n, 1) \in \operatorname{cl} K$, $c \in M$ *and* $\{c\} \times \mathbb{R} \nsubseteq K$. *Moreover,*
$\operatorname{int} \Pi_6 = \emptyset$.
(vii) $\Pi_i \subset \operatorname{cl} \operatorname{int} \Pi_j$ *for all* $(i,j) \neq (4,1)$, $i = 4,5,6$, $j = 1,2,3$.

Let us observe that the recent paper [5] provides characterizations of the interior, the boundary and the exterior of the sets Θ_C^D and Θ_1 in general LSIP. Obviously, these characterizations become sufficient conditions in the context of continuous LSIP. In general LSIP, $\operatorname{int} \Theta_B^P$ coincides with the interior of the class of those $\pi \in \Theta$ such that the corresponding primal problem is solvable ([4, Theorem 1]).

3 The refined primal-dual partition

Table 2 is the counterpart of Table 1 for the refined partitions, although some of the crossed intersections could be empty:

	Π_{IC}^P	Π_S^P	Π_N^P	Π_{UB}^P
Π_{IC}^D	Π_4		Π_5	Π_2
Π_S^D		Π_1^1	Π_1^3	
Π_N^D	Π_6	Π_1^2	Π_1^4	
Π_{UB}^D	Π_3			

Table 2

We must justify the empty cells in Table 2, i.e., that $\Pi_S^P \cap \Pi_{IC}^D = \Pi_{IC}^P \cap \Pi_S^D = \emptyset$.

First, if $\pi \in \Pi_S^P \cap \Pi_{IC}^D$, then $c \in \operatorname{int} M$ and $c \notin M$ (contradiction). Thus $\Pi_S^P \cap \Pi_{IC}^D = \emptyset$ and, so, $\Pi_N^P \cap \Pi_{IC}^D = \Pi_5$.

Second, if $\pi \in \Pi_{IC}^P \cap \Pi_S^D$, then π is primal inconsistent and satisfies Slater condition (contradiction). Thus $\Pi_{IC}^P \cap \Pi_S^D = \emptyset$ and $\Pi_{IC}^P \cap \Pi_N^D = \Pi_6$.

The next result shows that all the intersections in Table 2 are non-empty ($\Pi_i \neq \emptyset$, $i = 2, ...6$, was shown in [12]).

THEOREM 3.1 $\Pi_1^j \neq \emptyset$, $j = 1, .., 4$.

Proof The basic tool of the proof is Lemma 2.1.

$\underline{\Pi_1^1 \neq \emptyset}$: Let $\bar{t} \in T$, $S_i^k \subset T$ and $\varphi_i^k \in \mathcal{C}(T)$, $i = 1, ..., n$, $k = 1, 2$, such that

$$\varphi_i^k(\bar{t}) = 0 \in \mathrm{acc}\, \varphi_i^k(S_i^k) \subset \mathbb{R}_{++},$$

for all i and k, and $\varphi_i^k(S_j^l) = \{0\}$ if $(i, k) \neq (j, l)$. Consider $\pi = (a, b, c)$ such that $a := (\varphi_1^1 - \varphi_1^2, ..., \varphi_n^1 - \varphi_n^2)$, $b := -1$ and $c \in \mathbb{R}^n$ arbitrary. Given $i \in \{1, ..., n\}$, since $a(S_i^1) = \{0_{i-1}\} \times \varphi_i^1(S_i^1) \times \{0_{n-i}\}$, we have $\mathrm{cone}\,\{a_t, t \in S_i^1\} = \mathbb{R}_+ e_i$; similarly, $\mathrm{cone}\,\{a_t, t \in S_i^2\} = -\mathbb{R}_+ e_i$. Then

$$\sum_{i=1}^n \mathrm{cone}\,\{a_t, t \in S_i^1\} + \sum_{i=1}^n \mathrm{cone}\,\{a_t, t \in S_i^2\} = \sum_{i=1}^n \mathbb{R}_+ e_i - \sum_{i=1}^n \mathbb{R}_+ e_i = \mathbb{R}^n,$$

and so $M = \mathbb{R}^n$ and $c \in \mathrm{int}\, M = \mathbb{R}^n$. Moreover, $\{(a_t, b_t), t \in T\} \subset \mathbb{R}^n \times \{-1\}$, so that $0_{n+1} \notin \mathrm{conv}\,\{(a_t, b_t), t \in T\}$. Thus $\pi \in \Pi_1^1$.

$\underline{\Pi_1^2 \neq \emptyset}$: Let $\bar{t} \in T$, $S_i^k \subset T$, $\varphi_i^k \in \mathcal{C}(T)$, $i = 1, ..., n$, $k = 1, 2$, and a be as in the previous proof. Let $b := -\sum_{i=1}^n (\varphi_i^1 + \varphi_i^2)^2$, and $\pi := (a, b, c)$, where $c \in \mathbb{R}^n$ is taken arbitrarily. We have again $c \in \mathrm{int}\, M = \mathbb{R}^n$. On the other hand, $K \subset \mathbb{R}^n \times \mathbb{R}_-$ (because $b \leq 0$) and $(a_{\bar{t}}, b_{\bar{t}}) = 0_{n+1}$, so that Slater condition fails. Thus $\pi \in \Pi_1^2$.

$\underline{\Pi_1^3 \neq \emptyset}$: Let $\bar{t} \in T$, $S \subset T$ and $\varphi \in \mathcal{C}(T)$ be such that $\varphi(\bar{t}) = 0 \in \mathrm{acc}\, \varphi(S) \subset \mathbb{R}_{++}$. Let $\pi = (a, b, c)$, where $a := (\varphi, 0_{n-1})$, $b := -1$ and $c := e_1$. Then we have $c \in M = \mathbb{R}_+ e_1$, $\mathrm{int}\, M = \emptyset$ and $\{(a_t, b_t), t \in T\} \subset \mathbb{R}^n \times \{-1\}$, so that Slater condition holds. Thus $\pi \in \Pi_1^3$.

$\underline{\Pi_1^4 \neq \emptyset}$: Let $\bar{t}$, S, φ and a be as in the last proof. Let $\pi = (a, b, c)$, where $b := -\varphi^2$ and $c := e_1$. We have again $c \in M$ and $\mathrm{int}\, M = \emptyset$. Moreover, $(a_{\bar{t}}, b_{\bar{t}}) = 0_{n+1}$ and $K \subset \mathbb{R}^n \times \mathbb{R}_-$ (because $b \leq 0$), so that Slater condition fails and $(0_n, 1) \notin \mathrm{cl}\, K$. Hence $\pi \in \Pi_1^4$. □

Concerning the parameter $\pi \in \Pi_1^1$ constructed in the first part of the proof of Theorem 3.1, we cannot assert the uniqueness of the primal and dual optimal solutions. For particular compact Hausdorff spaces T, it is possible to give examples where this double uniqueness holds.

Example 3.2 Let $T = \{t \in \mathbb{R}^n \mid \|t\| = 1\}$ and $\pi = (a, b, c)$ such that a is the identity mapping on T, $b := -1$ and $c := e_1$. Then $F = T$, $F^* = \{-e_1\}$, $v^P(\pi) = v^D(\pi) = -1$ and $\Lambda^* = \{\lambda^*\}$, where $\lambda_t^* = 1$ if $t = e_1$ and $\lambda_t^* = 0$ otherwise.

Lemma 2.4 characterizes $\Pi_1, ..., \Pi_6$ and their corresponding interiors, showing that $\operatorname{int} \Pi_i \neq \emptyset$ is dense in Π_i, $i = 1, 2, 3$, whereas $\operatorname{int} \Pi_i = \emptyset$, $i = 4, 5, 6$. Now we consider the partition of Π_1, i.e., the sets Π_1^j, $j = 1, ..., 4$.

THEOREM 3.3 *(i) $\pi \in \Pi_1^1$ if and only if $c \in \operatorname{int} M$ and the Slater condition holds. Moreover, Π_1^1 is an open dense subset of Π_1.*
(ii) $\pi \in \Pi_1^2$ if and only if $(0_n, 1) \notin \operatorname{cl} K$, $c \in \operatorname{int} M$, and the Slater condition fails. Moreover, Π_1^2 is neither closed nor open and $\operatorname{int} \Pi_1^2 = \emptyset$.
(iii) $\pi \in \Pi_1^3$ if and only if Slater condition holds and $c \in M \backslash (\operatorname{int} M)$. Moreover, Π_1^3 is neither closed nor open and $\operatorname{int} \Pi_1^3 = \emptyset$.
(iv) $\pi \in \Pi_1^4$ if and only if $(0_n, 1) \notin \operatorname{cl} K$, $c \in M \backslash (\operatorname{int} M)$, and the Slater condition fails. Moreover, Π_1^4 is neither closed nor open and $\operatorname{int} \Pi_1^4 = \emptyset$.

Proof The characterizations of the four sets Π_1^j, $j = 1, ..., 4$, are consequence of Lemmas 2.2 and 2.4. Since these sets are cones, with the null triple belonging to Π_1^4, only Π_1^4 could be closed. Once we prove that Π_1^1 is an open dense subset of Π_1, we must have $\operatorname{int} \Pi_1^j = \emptyset$, $j = 2, 3, 4$, and, by Theorem 3.1, none of the sets Π_1^j, $j = 2, 3, 4$, can be open. Thus we just have to study Π_1^1 and Π_1^4.

Analyzing Π_1^1: It is open because, by Lemma 2.4, $\Pi_1^1 = \operatorname{int} \Pi_1$, with this set being a dense subset of Π_1. Thus, Π_1^1 is a dense subset of Π_1.

Analyzing Π_1^4: We have just to prove that Π_1^4 is non-closed. By Lemma 2.1, there exists $S \subset T$ and $\varphi \in \mathcal{C}(T)$ be such that $0 \in \operatorname{acc} \varphi(S) \subset \mathbb{R}_{++}$. Let $\{\mu_r\} \subset \mathbb{R}_{++}$ be an increasing sequence such that $\mu_r \to \frac{\pi}{2}$ and consider the sequence $\{\pi^r\} \subset \Pi$ such that $\pi^r := (a^r, b^r, c^r)$, where $a^r = \left(\varphi \cos \mu_r + \varphi^2 \sin \mu_r, 0_{n-1} \right)$, $b^r = \varphi \sin \mu_r - \varphi^2 \cos \mu_r$ and $c^r = e_1$, $r = 1, 2, ...$. We have $c \in M_r$ and $\operatorname{int} M_r = \emptyset$. Moreover, $\left(a_{\bar{t}}^r, b_{\bar{t}}^r \right) = 0_{n+1}$, so that Slater condition fails, and $(0_n, 1) \notin \operatorname{cl} K_r$. Hence $\{\pi^r\} \subset \Pi_1^4$ and $\pi^r \to \pi = (a, b, c)$, with $a = \left(\varphi^2, 0_{n-1} \right)$, $b = \varphi$ and $c = e_1$. Since $(0_n, 1) \in \operatorname{cl} K$, $c \in M$, and $\{c\} \times \mathbb{R} \subset K$, we get $\pi \in \Pi_3$. Hence Π_1^4 is non-closed. $\square$

In Example 2.3, the given neighborhood of π is contained in Θ_1^2 although the Slater condition holds for all its elements. Thus Theorem 3.3(ii) fails in general LSIP.

As a consequence of Theorem 3.3, the elements of the refined primal-dual partition are neither open nor closed, with the unique exception of Π_1^1, which is open. The density of Π_1^1 in Π_1 means that primal-dual solvability, with bounded optimal set, is a generic property in $\Pi_1 = \Pi_B^P \cap \Pi_B^D$. The next corollary improves this generic result.

COROLLARY 3.4 *Primal-dual solvability, with bounded optimal sets, is a generic property in the set $\Pi_B^P \cup \Pi_B^D$.*

Proof By Theorem 3.3(i), Π_1^1 is open and dense in Π_1. By Lemma 2.4(vii), $\Pi_5 \cup \Pi_6 \subset \operatorname{cl} \Pi_1 \subset \operatorname{cl} \Pi_1^1$. Therefore $\Pi_B^P \cup \Pi_B^D = \Pi_1 \cup \Pi_5 \cup \Pi_6 \subset \operatorname{cl} \Pi_1^1$. $\qquad\square$

Example 2.3 shows that the continuity assumption is essential for the validity of this generic result. In fact, the open neighborhood of π is contained in Θ_S^P whereas all its elements are contained in $\Theta_B^D \backslash \Theta_S^D$, i.e., int Θ_S^P contains an open set of parameters which are not even dual solvable. Hence the primal-dual solvability is not a generic property in Θ_B^P (or in Θ_B^D) and Corollary 3.4 is not true in general LSIP.

4   The refined primal and dual partitions

COROLLARY 4.1 *The elements of the refined primal partition are neither open nor closed and satisfy the following statements:*
(i) $\pi \in \Pi_{IC}^P$ *if and only if* $(0_n, 1) \in \operatorname{cl} K$. *Moreover,* int $\Pi_{IC}^P = $ int Π_3 *is dense in* Π_{IC}^P.
(ii) $\pi \in \Pi_S^P$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$ *and* $c \in$ int M. *Moreover,* int $\Pi_S^P = \Pi_1^1$ *is dense in* Π_B^P.
(iii) $\pi \in \Pi_N^P$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$ *and* $c \notin$ int M, *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} K \neq \emptyset$ *if* $c \in M$. *Moreover,* int $\Pi_N^P = \emptyset$.
(iv) $\pi \in \Pi_{UB}^P$ *if and only if* $(0_n, 1) \notin \operatorname{cl} K$, $c \notin M$ *and* $(\{c\} \times \mathbb{R}) \cap \operatorname{cl} K = \emptyset$. *Moreover,* int $\Pi_{UB}^P = $ int Π_2 *is dense in* Π_{UB}^P.

Proof (i) It is straightforward consequence of the existence theorem and [13, Theorem 2(i)] (it also follows from Lemma 2.4, parts (iii), (iv), (vi), (vii)).

(ii) The characterization of Π_S^P was established in Lemma 2.2(i). This, together with Table 2 and Lemma 2.4(i) gives $\Pi_1^1 \subset \Pi_S^P \subset \Pi_1$. On the other hand, by Lemma 2.4(i) and Theorem 3.3, int $\Pi_1 = \Pi_1^1$. Hence int $\Pi_S^P = \Pi_1^1$. The density of Π_1^1 in Π_B^P follows from the density of Π_1^1 in Π_1 (again by Lemma 2.4(i)) and the density of Π_1 in $\Pi_1 \cup \Pi_5 = \Pi_B^P$ (by Lemma 2.4(vii)).

(iii) The characterization of Π_N^P is consequence of those corresponding to the remaining elements of the refined primal partition. Concerning int Π_N^P, observe that $\Pi_N^P = \Pi_5 \cup \Pi_1^3 \cup \Pi_1^4$. Since $\Pi_5 \subset \operatorname{cl}$ int Π_1 and $\Pi_1^3 \cup \Pi_1^4 \subset \Pi_1$, we have $\Pi_N^P \subset \operatorname{cl}$ int $\Pi_1 = \operatorname{cl} \Pi_1^1$, with $\Pi_N^P \cap \Pi_1^1 = \emptyset$. Then int $\Pi_N^P = \emptyset$.

(iv) It is Lemma 2.4(ii) because $\Pi_{UB}^P = \Pi_2$.

It remains to prove that the elements of the refined primal partition are neither open nor closed.

The elements of the refined primal partition are cones, and the null parameter belongs to Π_N^P, so that the remaining elements are non-closed. On the other hand, since there exists a sequence $\{\pi^r\} \subset \Pi_1^4 \subset \Pi_N^P$ such that $\pi^r \to \pi \in \Pi_3 \subset \Pi_{IC}^P$ (recall the last part of the proof of Theorem 3.3), Π_N^P is non-closed too.

On the other hand, since the null parameter belongs to $\Pi_N^P \backslash \left(\operatorname{int} \Pi_N^P\right)$, Π_N^P is

non-open. Concerning Π_S^P, (ii) asserts that int $\Pi_S^P = \Pi_1^1 \neq \Pi_S^P$ because $\Pi_1^2 \neq \emptyset$ (Theorem 3.1), so that Π_S^P cannot be open. Finally, $\Pi_{UB}^P = \Pi_2$ is non-open by Lemma 2.4(ii) and the same applies to Π_{IC}^P by [13, Proposition 2]. □

The proof of the next result is similar to the last one and will be omitted.

COROLLARY 4.2 *The elements of the refined dual partition are neither open nor closed and satisfy the following statements:*
(i) $\pi \in \Pi_{IC}^D$ *if and only if* $c \notin M$. *Moreover,* int $\Pi_{IC}^D = $ int Π_2 *is dense in* Π_{IC}^D.
(ii) $\pi \in \Pi_S^D$ *if and only if Slater condition holds and* $c \in M$. *Moreover,* int $\Pi_S^D = \Pi_1^1$ *is dense in* Π_B^D.
(iii) $\pi \in \Pi_N^D$ *if and only if Slater condition fails and* $c \notin M$, *and* $\{c\} \times \mathbb{R} \nsubseteq K$ *if* $(0_n, 1) \in $ cl K. *Moreover,* int $\Pi_N^D = \emptyset$.
(iv) $\pi \in \Pi_{UB}^D$ *if and only if* $(0_n, 1) \in $ cl K, $c \in M$ *and* $\{c\} \times \mathbb{R} \subset K$. *Moreover,* int $\Pi_{UB}^D = $ int Π_3 *is dense in* Π_{UB}^D.

Observe that, from Corollaries 4.1 and 4.2, part (ii), we get that Π_1^1 is dense in $\Pi_B^P \cup \Pi_B^D$. This is an alternative proof of Corollary 3.4.

Corollary 4.2(ii) fails in the context of general LSIP, as Example 2.3 shows. The situation is much better in the open and closed linear subspace of the parameters with bounded data, say Γ, which is formed by those triples $(a, b, c) \in \Theta$ such that the functions a and b are bounded. Obviously, if T is a compact Hausdorff space, then $\Pi \subset \Gamma$. In [11] (using ad hoc tools for Γ introduced in [15]) and in [16] it is proved that the strong uniqueness of the primal problem and the primal-dual solvability are generic properties in Γ_B^P and in int Γ_B^P, respectively. The next example shows that the generic property proved in this paper fails in Γ.

Example 4.3 Let $T = [0, 1]$ and $n = 2$. Let $\pi = (a, b, c)$ be such that $a_t = (t, 1)$ for all $t \in T$,

$$b_t = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } 0 < t < 1, \\ -1 \text{ if } t = 1, \end{cases}$$

and $c = (\frac{1}{3}, 1)$. Since $F = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0, x_2 \geq 1\}$, $x^* = (-1, 1)$ is strongly unique solution of π and $\pi \in \Gamma_S^P$. Even more, since $c \in $ int M, we have $\pi \in $ int Γ_S^P. Nevertheless, since there is no duality gap (because $c \in $ int M) and $(\{c\} \times \mathbb{R}) \cap K$ is not closed, we have $\Lambda^* = \emptyset$ and, so, $\pi \in \Gamma_B^D$ but $\pi \notin \Gamma_1^1$. This means that int $\Gamma_S^P \neq \Gamma_1^1$ (i.e., Theorem 4.2(ii) fails). Consider the sequence $\pi^r = (a^r, b^r, c^r)$, $r = 2, ...,$ such that $a^r = a$, $c^r = c$ and $b_t^r = b_t$ if $t \neq \frac{r-1}{r}$ and $b_t^r = \frac{1}{r}$ if $t = \frac{r-1}{r}$, $r = 1, 2, ...$ Given $r \geq 2$, we have $F_r^* = \{x^*\}$ (because

cl $K_r =$ cl K is equivalent to $F_r = F$) and $\Lambda_r^* = \{\lambda^r\}$, with

$$\lambda_t^r = \begin{cases} \frac{2r-3}{3(r-1)}, & \text{if } t = 0, \\ \frac{r}{3(r-1)}, & \text{if } t = \frac{r-1}{r}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{\pi^r\} \subset \Gamma_1^1$ and $\pi^r \to \pi$, we have $\pi \in \left(\text{cl } \Gamma_1^1\right) \backslash \Gamma_1^1 \subset \text{bd } \Gamma_1^1$ (as expected, taking into account the mentioned generic results).

Concerning the dual problem in continuous LSIP, examples have been given in [23] of compact Hausdorff spaces T such that, in an open set, the dual problems have more than one optimal solutions, so that the generic results on unicity of saddle points fail. This implies that the problem of obtaining generic results for the dual problem is very difficult even in the continuous case. The extension of the generic results in this paper from continuous LSIP to LSIP with bounded data will be the object of further study.

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