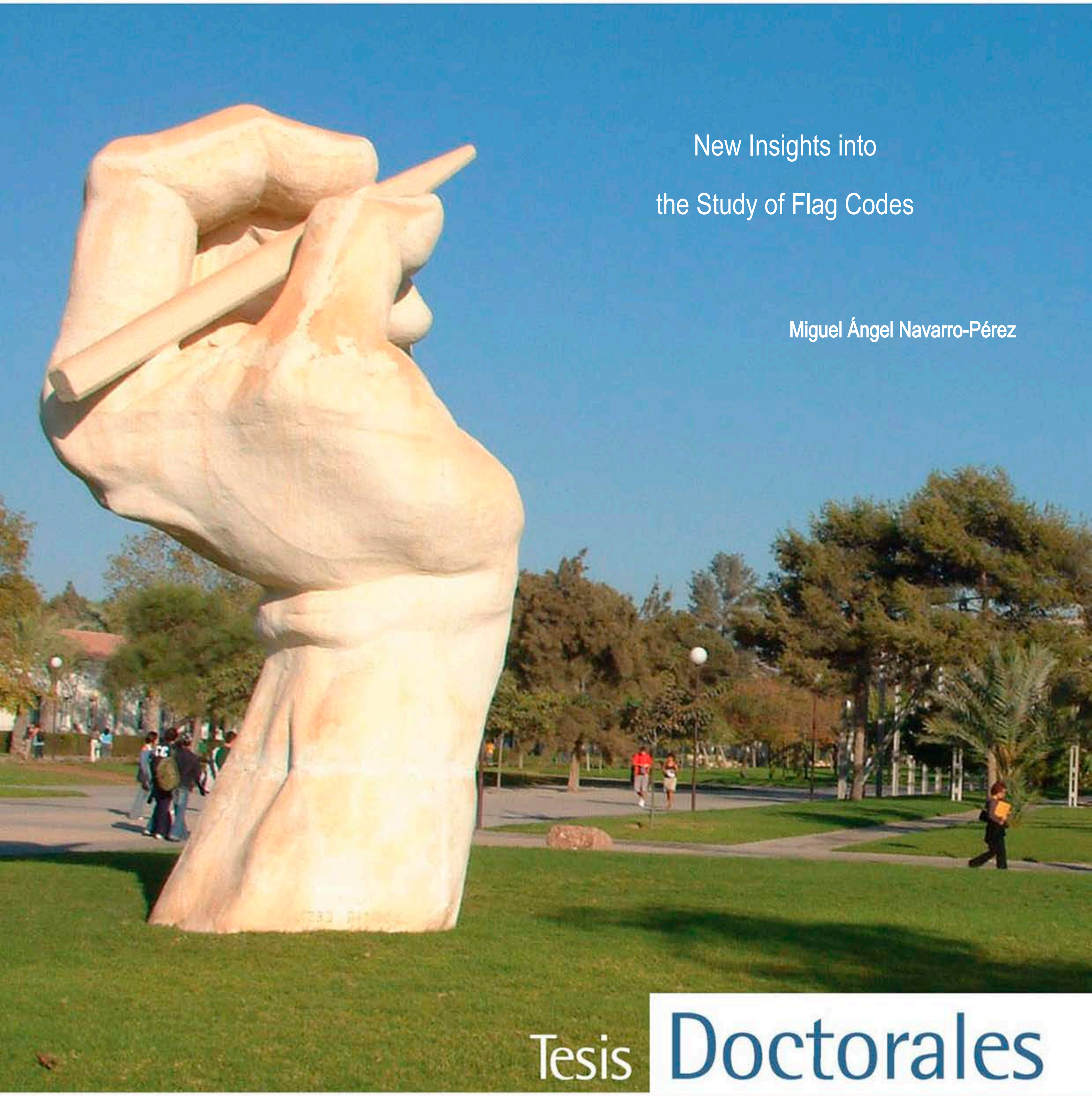

New Insights into the Study of Flag Codes

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Siempre he intentado ser un buen ejemplo para ti...

... y tú siempre me superas.

Esta tesis es para ti.

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Introducción

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El uso de *códigos flag* en el ámbito de la codificación de red fue recientemente propuesto por Liebhold, Nebe and VAzquez-Castro como una generalización de los códigos de dimensión constante. El primer trabajo en esta línea de investigación es [54] y, desde entonces, varios autores han contribuido con los recientes trabajos [2, 3, 4, 5, 6, 7, 8, 9, 23, 47, 52, 53, 59].

Esta tesis, dedicada al estudio de diferentes aspectos de la teoría de códigos *flag*, se presenta como el compendio de los siguientes trabajos, que el lector puede encontrar, en este orden, en los Capítulos 1-9.

- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes from planar spreads in Network Coding*, Finite Fields and their Applications, Vol. 68 (2020), 101745.
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum distance flag codes from spreads via perfect matchings in graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An orbital construction of optimum distance flag codes*, Finite Fields and their Applications, Vol. 73 (2021), 101861.
- C. Alonso-González and M. A. Navarro-Pérez, *Cyclic orbit flag codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2331-2356.
- C. Alonso-González and M. A. Navarro-Pérez, *Consistent flag codes*, Mathematics, Vol. 8(12) (2020), 2234.
- M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes of maximum distance and constructions using Singer groups*, https://arxiv.org/abs/2109.00270 (preprint)
- C. Alonso-González and M. A. Navarro-Pérez, *On generalized Galois cyclic orbit flag codes*, https://arxiv.org/abs/2111.09615 (preprint).
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes: distance vectors and cardinality bounds*, https://arxiv.org/abs/2111.00910 (preprint).
- C. Alonso-González and M. A. Navarro-Pérez, *A combinatorial approach to flag codes*, https://arxiv.org/abs/2111.15388 (preprint).

En esta primera parte, detallamos los resultados más importantes de dichos trabajos, haciendo especial énfasis en las conexiones que pueden establecerse entre ellos. Para ello, recordaremos algunas definiciones y resultados conocidos de la teoría de códigos de dimensión constante, así como el estado del arte de la teoría de códigos *flag*, en el que este tesis se enmarca.

Códigos de dimensión constante

El término *codificación de red*, introducido en [1], hace referencia al método de envío de información a través de redes, modeladas como multigrafos dirigidos acíclicos con, al menos, un emisor y un receptor, en las que los nodos intermedios, en lugar de simplemente reenviar los paquetes de información (vectores) que reciben, tienen la capacidad de realizar combinaciones lineales de dichos paquetes. En el mismo trabajo se prueba que este nuevo comportamiento de los nodos intermedios permite mejorar la velocidad de la comunicación, como podemos observar en el siguiente ejemplo, en el que consideramos el envío de mensajes a través de la conocida *red de mariposa*.

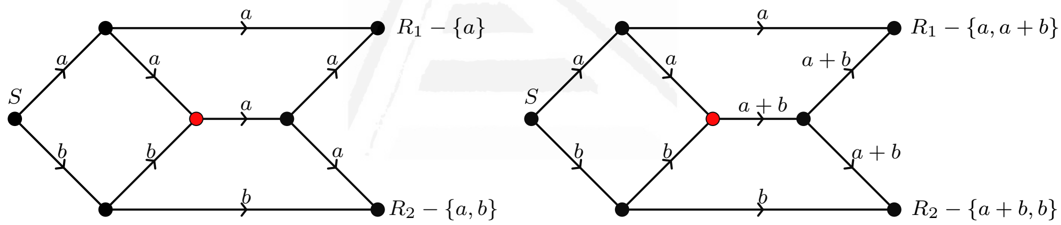

Figure 1: Red de mariposa

Notemos que en el primer caso, en el que los nodos intermedios simplemente reenvían la información recibida, un solo uso del canal es insuficiente si queremos que los dos receptores, R_1 y R_2, reciban ambos mensajes a y b. Esto se debe a que el nodo señalado en rojo actúa como “cuello de botella” y, aunque reciba los dos mensajes a y b, solo puede reenviar uno de ellos en cada uso del canal. Sin embargo, la segunda situación, en la que permitimos que este nodo calcule y envíe combinaciones lineales de a y b, los dos receptores pueden recuperar a y b en un solo uso del canal.

Sin embargo, un aspecto negativo de la codificación de red es al alta vulnerabilidad ante la propagación de errores: basta observar que la presencia de errores en un determinado paquete afecta a también a todos aquellos obtenidos como combinaciones lineales que lo involucren. Otra desventaja de este tipo de redes que podemos apreciar en el ejemplo anterior es que los receptores necesitan conocer qué combinaciones lineales se han producido durante la comunicación para poder recuperar a y b a partir de los paquetes recibidos. Como solución a este problema, en [45], Koetter and Kschischang proponen el uso de subespacios vectoriales (en lugar de vectores) como *palabras código* y, puesto que estos objetos son invariantes bajo el uso de combinaciones lineales, ni el emisor ni los receptores necesitan saber cómo funciona la red. En [45], encontramos el primer estudio (desde un punto de vista algebraico) sobre codificación de red a través de redes *no-coherentes*, es decir, aquellas en las que los nodos intermedios realizan combinaciones lineales aleatorias y la topología de la red no es necesariamente conocida por los usuarios. En dicho artículo, los autores presentan la noción de *código de subespacio* y definen una métrica adecuada para el conteo de errores y borrados que puedan producirse durante el proceso de comunicación en este nuevo escenario.

A continuación, recordamos algunos elementos clave en la teoría de códigos de subespacio. Para ello, primero, fijamos la notación que utilizaremos durante el resto de esta memoria:

- Sea q una potencia de primo,
- $\mathbb{F}_q$ denota el *cuerpo finito* con q elementos,
- k y n son enteros positivos satisfaciendo $1 \leqslant k < n$.
- $\mathbb{F}_q^n$ representa el espacio vectorial n-dimensional sobre el cuerpo $\mathbb{F}_q$.
- La *variedad de Grassamann* (o la *Grassmanniana*) $\mathcal{G}_q(k, n)$ es el conjunto de $\mathbb{F}_q$-subespacios vectoriales k-dimensionales de $\mathbb{F}_q^n$.
- La *geometría proyectiva* de $\mathbb{F}_q^n$ es el conjunto $\mathcal{P}_q(n)$ que consta de todos los $\mathbb{F}_q$-subespacios vectoriales de $\mathbb{F}_q^n$.

La geometría proyectiva $\mathcal{P}_q(n)$ es un espacio métrico. En general, dados dos subespacios $\mathcal{U}, \mathcal{V} \in \mathcal{P}_q(n)$, su *distancia de subespacio* es

$$d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}). \tag{1}$$

En caso de que los subespacios $\mathcal{U}$ y $\mathcal{V}$ tengan la misma dimensión, pongamos $\dim(\mathcal{U}) = \dim(\mathcal{V}) = k$, el cálculo de su distancia se reduce a

$$d_S(\mathcal{U}, \mathcal{V}) = 2\big(k - \dim(\mathcal{U} \cap \mathcal{V})\big). \tag{2}$$

Esta métrica sobre $\mathcal{P}_q(n)$ abre la puerta al uso de la siguiente familia de códigos correctores de errores, introducida en [45].

Un *código de subespacio* de longitud n es un subconjunto no vacío $\mathcal{C} \subseteq \mathcal{P}_q(n)$. Si todas las palabras código (subespacios de $\mathbb{F}_q^n$) tienen la misma dimensión $1 \leqslant k < n$, decimos que $\mathcal{C}$ es un *código de dimensión constante* (en $\mathcal{G}_q(k, n)$).

Dado un código de dimensión constante $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$ con $|\mathcal{C}| \geqslant 2$, su *distancia mínima* se define como

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U}, \mathcal{V}) \mid \mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\}.$$

Si $|\mathcal{C}| = 1$, convenimos $d_S(\mathcal{C}) = 0$. En general, y de acuerdo con (2), la distancia $d_S(\mathcal{C})$ es un entero par acotado por

$$0 \leqslant d_S(\mathcal{C}) \leqslant \begin{cases} 2k & \text{si} \quad 2k \leqslant n, \\ 2(n-k) & \text{si} \quad 2k \geqslant n. \end{cases} \tag{3}$$

El cardinal de un código de dimensión constante indica la cantidad de mensajes diferentes que este puede codificar. Por ello, una vez fijados el resto de parámetros, existe un gran interés en obtener códigos de dimensión constante de gran tamaño. En concreto, dados $1 \leqslant k < n$ y un posible valor para la distancia d, escribimos $A_q(n, d, k)$ para denotar el máximo cardinal posible para códigos en $\mathcal{G}_q(k, n)$ con distancia mínima d. Obtener el valor exacto $A_q(n, d, k)$ para cada elección de los parámetros no es, en absoluto, una tarea sencilla. Como consecuencia, el estudio de cotas (tanto inferiores como superiores) para este valor ha dado lugar a multitud de trabajos en los últimos años como, por ejemplo, [22, 35, 41, 43, 48, 66, 67, 70, 71, 73].

Por otra parte, la distancia mínima de un código de dimensión constante está íntimamente relacionada con su capacidad para detectar y corregir errores. Concretamente, un código de dimensión constante $\mathcal{C}$ detecta hasta $d_S(\mathcal{C})$ errores y puede corregir hasta $d_S(\mathcal{C})/2 - 1$. Por ende, desde su aparición en 2008, y hasta la actualidad, se ha prestado especial atención al estudio de códigos de dimensión constante de máxima distancia. En particular, en [30], los autores introducen la siguiente familia de códigos de dimensión constante de distancia máxima, para dimensiones $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$.

Un códgigo *spread parcial de dimensión* k (or un k-*spread* parcial) de $\mathbb{F}_q^n$ es un subconjunto $\mathcal{C}$ de $\mathcal{G}_q(k, n)$ en que subespacios diferentes se cortan siempre trivialmente. Equivalentemente, $\mathcal{C}$ es un k-*spread* parcial si cada vector no nulo de $\mathbb{F}_q^n$ vive, a lo sumo, en un elemento de $\mathcal{C}$.

El cardinal de un k-*spread* parcial $\mathcal{C}$ de $\mathbb{F}_q^n$ está acotado superiormente por el valor

$$|\mathcal{C}| \leqslant A_q(n, 2k, k) \leqslant \frac{q^n - q^r}{q^k - 1}, \tag{4}$$

donde r es el resto de dividir n entre k. Como caso particular de estos códigos, tenemos los **códigos *spread***, introducidos en [56].

Un *(código) k-spread* $\mathcal{S}$ of $\mathbb{F}_q^n$ es un subconjunto de $\mathcal{G}_q(k,n)$ tal que cada vector no nulo de $\mathbb{F}_q^n$ vive en uno, y solo en uno, de los elementos de $\mathcal{S}$. En otras palabras, un *k-spread* es una partición de $\mathbb{F}_q^n$ en subespacios k-dimensionales.

Los *k-spreads* son objetos clásicos ampliamente estudiados desde el prisma de la Geometría Finita (véase [65]). Su existencia está condicionada por la relación entre k y n. Más concretamente, k debe ser un divisor de n. En esta situación, el cardinal de cualquier *k-spread* $\mathcal{S}$ de $\mathbb{F}_q^n$ es

$$|\mathcal{S}| = A_q(n, 2k, k) = \frac{q^n - 1}{q^k - 1}.$$

En este sentido, los códigos *spread* son óptimos ya que alcanzan la distancia máxima para su dimensión k y tienen el mejor cardinal posible, fijadas su dimensión y distancia. El estudio de esta familia de códigos de dimensión constante ha dado lugar a multitud de trabajos en la última década, basta ver [56, 57, 68, 69]. Como ejemplo de *k-spread* de $\mathbb{F}_q^n$, tenemos la siguiente construcción, presentada por primera vez en el contexto de la codificación de red en [56, Teorema 1]. Dados enteros positivos k y s, consideremos $n = ks$ y la matriz *companion* $P \in \mathrm{GL}(k, q)$ de un polinomio mónico e irreducble de grado k en $\mathbb{F}_q[x]$. El conjunto

$$\mathcal{S}(s, k, P) = \{\mathrm{rowsp}(S) \mid S \in \Sigma\} \subseteq \mathcal{G}_q(k, n) \tag{5}$$

es un *k-spread* de $\mathbb{F}_q^n$, donde Σ es el conjunto de matrices

$$\Sigma = \left\{(A_1|A_2|\dots|A_s) \,\middle|\, A_i \in \mathbb{F}_q[P]\right\} \tag{6}$$

tales que el primer bloque no nulo (por la izquierda) es la matriz identidad I_k. En el ámbito de la Geometría Finita, esta construcción se debe a Segre (véase [65]) y se obtiene por aplicación de técnicas conocidas como **reducción del cuerpo** a la Grassmanniana de rectas de $\mathbb{F}_{q^k}^s$.

Otra familia de códigos de dimensión constante que ha despertado el interés de los expertos es la de los **códigos orbitales**. Estos códigos aparecen por primera vez en [69] como órbitas de la acción transitiva del grupo general lineal $\mathrm{GL}(n, q)$ sobre la Grassmanniana $\mathcal{G}_q(k, n)$ dada por

$$\begin{array}{ccc} \mathcal{G}_q(k,n) \times \mathrm{GL}(n,q) & \longrightarrow & \mathcal{G}_q(k,n) \\ (\mathcal{U}, A) & \longmapsto & \mathcal{U} \cdot A = \mathrm{rowsp}(UA), \end{array} \tag{7}$$

donde $U \in \mathbb{F}_q^{k \times n}$ es cualquier matriz generadora del subespacio $\mathcal{U}$.

> Dados $\mathcal{U} \in \mathcal{G}_q(k,n)$ y un subgrupo $\mathbf{H}$ de $\mathrm{GL}(n,q)$, llamamos *código orbital generado por* $\mathcal{U}$ *bajo la acción de* $\mathbf{H}$ a la órbita
>
> $$\mathrm{Orb}_{\mathbf{H}}(\mathcal{U}) = \{\mathcal{U} \cdot A \mid A \in \mathbf{H}\} \subseteq \mathcal{G}_q(k,n).$$
>
> El *estabilizador de* $\mathcal{U}$ *(en* $\mathbf{H}$*)* es el subgrupo de $\mathbf{H}$ dado por
>
> $$\mathrm{Stab}_{\mathbf{H}}(\mathcal{U}) = \{A \in \mathbf{H} \mid \mathcal{U} \cdot A = \mathcal{U}\}.$$

En [69] se prueba que la estructura orbital de un código simplifica en gran medida el cálculo de sus parámetros y también es útil a la hora de encontrar algoritmos de decodificación eficientes. Además, en el mismo artículo, los autores exhiben construcciones orbitales con parámetros óptimos, motivando así, más si cabe, el uso de códigos orbitales. En especial, prueban que el *k-spread* $\mathcal{S}(s,k,P)$ dado en la expresión (5) es la órbita del subespacio $\mathcal{U} = \mathrm{rowsp}(I_k \mid 0_{k\times(n-k)})$ bajo la acción del subgrupo generado por el siguiente conjunto de matrices por bloques

$$\left(\begin{array}{c|c|c|c} I_k & A_1 & \cdots & A_{s-1} \\ \hline 0 & I_k & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & I_k \end{array}\right), \left(\begin{array}{c|c|c|c|c} 0 & I_k & A_1 & \cdots & A_{s-2} \\ \hline I_k & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & I_k & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & I_k \end{array}\right), \ldots, \left(\begin{array}{c|c|c|c} 0 & \ldots & 0 & I_k \\ \hline 0 & \ldots & I_k & 0 \\ \hline \vdots & \iddots & \vdots & \vdots \\ \hline I_k & \ldots & 0 & 0 \end{array}\right) \tag{8}$$

con $A_i \in \mathbb{F}_q[P]$, para todo $1 \leqslant i \leqslant s-1$.

Por otra parte, en [68] se hace un estudio específico de *códigos orbitales cíclicos*: aquellos obtenidos como órbitas de grupos cíclicos de $\mathrm{GL}(n,q)$. En dicho trabajo, los autores utilizan el isomorfismo $\mathbb{F}_q$-lineal entre el cuerpo $\mathbb{F}_{q^n}$ y el espacio vectorial $\mathbb{F}_q^n$ y prueban que, para cada divisor k de n, la órbita

$$\mathcal{S} = \mathrm{Orb}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^k}) = \{\mathbb{F}_{q^k}\alpha \mid \alpha \in \mathbb{F}_{q^n}^*\} \tag{9}$$

es un *k-spread* del cuerpo $\mathbb{F}_{q^n}$, con estabilizador $\mathrm{Stab}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^k}) = \mathbb{F}_{q^k}^*$. En este caso, se utiliza la acción natural del grupo multiplicativo $\mathbb{F}_{q^n}^*$ sobre los elementos del cuerpo $\mathbb{F}_{q^n}$. Este mismo enfoque aparece de nuevo en [25], donde Gluesing-Luerssen *et al.* presentan la familia de **códigos** β**-cíclicos**, es decir, aquellos obtenidos como órbitas bajo la acción de subgrupos $\langle\beta\rangle$ de $\mathbb{F}_{q^n}^*$. Más concretamente, dado un $\mathbb{F}_q$-espacio vectorial $\mathcal{U} \subseteq \mathbb{F}_{q^n}$, el *código (orbital)* β*-cíclico* generado por $\mathcal{U}$ es la órbita

$$\mathrm{Orb}_\beta(\mathcal{U}) = \{\mathcal{U}\beta^i \mid 1 \leqslant i \leqslant |\beta|\},$$

donde $\mathcal{U}\beta^i = \{u\beta^i \mid u \in \mathcal{U}\}$. En ese mismo trabajo, los autores estudian los posibles parámetros de los códigos β-cíclicos en función de su **mejor amigo**, es decir, el mayor subcuerpo de $\mathbb{F}_q^n$ sobre el que los elementos de la órbita son espacios vectoriales.

Códigos *flag*: el estado del arte

Cuando trabajamos con códigos de dimensión constante, cada palabra código (un subespacio) es enviada haciendo un solo uso del canal. Sin embargo, esta operación puede repetirse tantas veces como sea preciso. En tal caso, hablamos de **códigos (de subespacio) multi-disparo**, cuyo estudio se aborda por primera vez en [60]. En este contexto, las palabras código son elementos de $\mathcal{P}_q(n)^r$, es decir, sucesiones de r subespacios de $\mathbb{F}_q^n$. Estos códigos son una alternativa muy interesante a los códigos de subespacio cuando queremos obtener mejores parámetros pero no podemos modificar los valores de n o q. Por otra parte, trabajar con códigos multi-disparo en $\mathcal{P}_q(n)^r$ es más sencillo que hacerlo con sus códigos de subespacio equivalentes de $\mathbb{F}_q^{nr}$.

Como ejemplo de códigos multi-disparo, tenemos la familia de **códigos *flag***, cuyo uso en el ámbito de la codificación de red fue introducido por Liebhold *et al.* en sus trabajos [52, 53, 54]. A continuación, recordaremos algunas definiciones básicas en esta línea de investigación.

Dados enteros $1 \leqslant t_1 < \cdots < t_r < n$, llamamos *flag de tipo* $(t_1, \ldots, t_r)$ en $\mathbb{F}_q^n$ a toda sucesión $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ de $\mathbb{F}_q$-subespacios vectoriales de $\mathbb{F}_q^n$ tales que

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n.$$

y $\dim(\mathcal{F}_i) = t_i$, para todo $1 \leqslant i \leqslant r$. La *variedad de flags de tipo* $(t_1, \ldots, t_r)$ es el conjunto $\mathcal{F}_q((t_1, \ldots, t_r), n)$ de todos los *flags* de ese tipo. Llamamos *tipo completo* al vector $(1, \ldots, n-1)$ y nos referimos a los *flags* de este tipo como *flags completos*.

La variedad de *flags* también tiene estructura de espacio métrico, de hecho, en [54], los autores analizan varias distancias para *flags* y concluyen que la siguiente es la más apropiada para cuantificar errores y borrados en el contexto de la codificacción de red. Dados dos *flags* $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ y $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ de tipo $(t_1, \ldots, t_r)$, su *distancia de flags* es

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i). \tag{10}$$

Esta métrica generaliza la distancia de subespacio definida en (2) a la variedad de *flags*. Por analogía a los códigos de dimensión constante, definimos los códigos *flag* (de tipo constante) de la siguiente forma.

Dada una sucesión de enteros positivos $1 \leqslant t_1 < \cdots < t_r < n$, llamamos *código flag* de tipo $(t_1, \ldots, t_r)$ en $\mathbb{F}_q^n$ a todo subconjunto no vacío $\mathcal{C}$ de $\mathcal{F}_q((t_1, \ldots, t_r), n)$. Su *distancia (de flags) mínima* es

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\} \tag{11}$$

siempre que $|\mathcal{C}| \geqslant 2$. En otro caso, convenimos $d_f(\mathcal{C}) = 0$.

El grupo general lineal $\mathrm{GL}(n, q)$ también actúa transitivamente sobre la variedad de *flags* $\mathcal{F}_q((t_1, \ldots, t_r), n)$ vía

$$\mathcal{F} \cdot A := (\mathcal{F}_1 \cdot A, \ldots, \mathcal{F}_r \cdot A), \tag{12}$$

donde $\mathcal{F}_i \cdot A$ es el subespacio definido como en (7). En consecuencia, para cualquier *flag* $\mathcal{F} \in \mathcal{F}_q((t_1, \ldots, t_r), n)$, se cumple $\mathcal{F}_q((t_1, \ldots, t_r), n) = \mathrm{Orb}_{\mathrm{GL}(n,q)}(\mathcal{F})$. El estabilizador asociado a esta órbita es un subgrupo de matrices triangulares inferiores por bloques, donde el tamaño de dichos bloques está determinado por los enteros positivos $t_1, t_2 - t_1, \ldots, t_r - t_{r-1}$ y $n - t_r$. Esta acción permite obtener códigos *flag* como órbitas bajo la acción de subgrupos de $\mathrm{GL}(n, q)$.

Dados un *flag* $\mathcal{F} \in \mathcal{F}_q((t_1, \ldots, t_r), n)$ y un subgrupo $\mathbf{H}$ de $\mathrm{GL}(n, q)$. El *código flag orbital* generado por $\mathcal{F}$ bajo la acción de $\mathbf{H}$ es

$$\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) = \{\mathcal{F} \cdot A \mid A \in \mathbf{H}\}.$$

Su *estabilizador* asociado es el subgrupo de $\mathbf{H}$

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \{A \in \mathbf{H} \mid \mathcal{F} \cdot A = \mathcal{F}\}.$$

Este es el enfoque utilizado en [54], donde el lector puede encontrar las primeras construcciones de códigos *flag* orbitales, todas ellas con estabilizador trivial, y un algoritmo de decodificación sobre el canal de borrado. En el mismo trabajo, los autores también generalizan la noción de código matricial (dotado de la métrica de rango) al contexto de los códigos *flag*.

Tal y como ocurre con los códigos de dimensión constante, determinar el valor $A_q^f(n, d, (t_1, \ldots, t_r))$, es decir, el máximo cardinal posible para códigos *flag* en $\mathcal{F}_q((t_1, \ldots, t_r), n)$ con distancia d, es un problema de gran interés. Esta cuestión ha sido recientemente abordada por Kurz en su trabajo [47], donde desarrolla técnicas para obtener cotas inferiores y superiores para $A_q^f(n, d, (1, \ldots, n-1))$. En el mismo trabajo, podemos encontrar una lista exhaustiva de dichas cotas para valores pequeños de n, así como algunos ejemplos fuera del tipo completo.

Nuestras contribuciones a la teoría de códigos *flag*

A lo largo de esta sección, detallamos los avances en la teoría de códigos *flag* derivados de los trabajos presentados en esta tesis. Algunos de ellos ya han sido publicados (Capítulos 1-5). Otros se encuentran en proceso de revisión por parte de las revistas correspondientes (Capítulos 6-9). A continuación, presentamos algunos de los resultados más relevantes que el lector puede encontrar en dichos trabajos. Con el objetivo de mostrar de forma clara las conexiones entre ellos, seguiremos este orden:

Capítulo 1 → Capítulo 2 → Capítulo 3 → Capítulo 6 → Capítulo 4 → Capítulo 7 → Capítulo 5 → Capítulo 8 → Capítulo 9.

Empezaremos, como si de un cuento se tratara, por el principio. El origen de esta tesis, y el verdadero *leitmotiv* de nuestros trabajos, es la relación existente entre un código *flag* y sus **códigos proyectados**: una familia de códigos de dimensión constante que podemos asociar de forma natural al código *flag* de partida. El concepto de código proyectado (de un código *flag*) aparece por primera vez en el Capítulo 1 y juega un papel protagonista en el resto nuestros trabajos. A continuación, daremos la definición formal de estos códigos.

Consideremos una sucesión de enteros $1 \leqslant t_1 < \cdots < t_r < n$. Para cada índice $1 \leqslant i \leqslant r$, utilizaremos la proyección

$$p_i : \mathcal{F}_q((t_1, \ldots, t_r), n) \to \mathcal{G}_q(t_i, n) \tag{13}$$

dada por $p_i((\mathcal{F}_1, \ldots, \mathcal{F}_r)) = \mathcal{F}_i$, para definir el i-ésimo código proyectado de la siguiente forma.

(Capítulo 1, Definición 3.6) Sea $\mathcal{C}$ un código *flag* de tipo $(t_1, \ldots, t_r)$ en $\mathbb{F}_q^n$. Para cada $1 \leqslant i \leqslant r$, el *$i$-ésimo código projectado de $\mathcal{C}$* es el código de dimensión constante

$$\mathcal{C}_i = p_i(\mathcal{C}) = \{p_i(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\} \subseteq \mathcal{G}_q(t_i, n). \tag{14}$$

Notemos que, para cada $1 \leqslant i \leqslant r$, la desigualdad $|\mathcal{C}_i| \leqslant |\mathcal{C}|$ se cumple trivialmente. En caso de darse la igualdad para cada valor $1 \leqslant i \leqslant r$, tenemos la siguiente familia de códigos *flag*.

(Capítulo 1, Definición 3.10) Decimos que un código *flag* $\mathcal{C}$ de tipo $(t_1, \ldots, t_r)$ en $\mathbb{F}_q^n$ es *disjunto* si

$$|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|. \tag{15}$$

En otra palabras, si para cada $1 \leqslant i \leqslant r$, la proyección p_i restringida a $\mathcal{C}$ es inyectiva.

Desde el punto de vista geométrico, un código *flag* es disjunto si *flags* diferentes en él tienen todos sus subespacios distintos. Estos dos conceptos –el de códigos proyectados y la *disyunción* de códigos *flag*– fueron introducidos en nuestro primer trabajo [8] (Capítulo 1), como herramientas cruciales para el estudio de códigos *flag* de máxima distancia. Dado un código *flag* $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$, su distancia mínima es un entero par $d_f(\mathcal{C})$ que satisface

$$0 \leqslant d_f(\mathcal{C}) \leqslant \sum_{2t_i \leqslant n} 2t_i + \sum_{2t_i > n} 2(n - t_i). \tag{16}$$

En el caso de que $d_f(\mathcal{C})$ alcance esta cota superior, decimos que $\mathcal{C}$ es un *código flag de distancia óptima*. El siguiente resultado caracteriza a esta familia de códigos en función de sus códigos proyectados.

> **(Capítulo 1, Teorema 3.11)** Un código *flag* es de distancia óptima si, y solo si, es disjunto y todos sus códigos proyectados alcanzan la distancia máxima para las dimensiones correspondientes.

Notemos que, en particular, para dimensiones hasta $\lfloor \frac{n}{2} \rfloor$, los códigos proyectados de un código *flag* de distancia óptima son *spreads* parciales, todos ellos del mismo cardinal. Por ello, y motivados por las buenas propiedades de los códigos *spread*, nos planteamos la posibilidad de construir códigos *flag* de distancia óptima con un *spread* como código proyectado. En tal caso, debido a la condición de disyunción, solo un *spread* puede aparecer como proyectado. Este tema será ampliamente abordado en el Capítulo 1. Además, en el Teorema 3.12, probamos que, si un divisor k de n forma parte del vector tipo de un código *flag* de distancia óptima $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$, entonces

$$|\mathcal{C}| \leqslant \frac{q^n - 1}{q^k - 1}$$

y la igualdad se alcanza si, y solo si, el código proyectado de dimensión k es un k-*spread* de $\mathbb{F}_q^n$. En cierto sentido, los códigos *flag* de distancia óptima con un *spread* entre sus códigos proyectados heredan las buenas propiedades del *spread*. Esta idea aparece constantemente a lo largo de los Capítulos 1-3 y 6, donde, usando diferentes enfoques, construimos códigos *flag* de distancia óptima con un *spread* como código proyectado para cada situación posible. En concreto, en el Capítulo 1, atacamos este problema para el tipo completo y concluimos que, en caso de existir códigos *flag* de distancia óptima en $\mathbb{F}_q^n$ con un k-*spread* como proyectado, debe ocurrir $n = 2k$ o $k = 1$ y $n = 3$ (Capítulo 1, Proposición 4.1) y probamos su existencia para el caso $n = 2k$, para cualquier elección del entero positivo k, al exhibir una construcción sistemática con el k-*spread* $\mathcal{S}(2, k, P)$ definido en (5) como código proyectado. Los detalles de esta construcción pueden encontrarse en el Capítulo 1 (Teorema 4.5).

El caso $k = 1$ y $n = 3$ forma parte del estudio que realizamos en el Capítulo 2, donde tratamos el problema general de construir códigos *flag* de distancia óptima en $\mathbb{F}_q^n$ con un k-*spread* como código proyectado, ahora sí, para cualquier divisor k de n. En primer lugar, probamos que tan solo los vectores tipo $(t_1, \ldots, t_r)$ satisfaciendo

$$k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\}. \tag{17}$$

son *admisibles* para nuestro objetivo. Esta condición es, hasta cierto punto, razonable. Los códigos k-*spread* alcanzan la maxima distancia y tienen el mejor cardinal posible. ¿El precio a pagar para poder construirlos? Su dimensión k debe ser un divisor de n. En nuestro caso, los códigos *flag* de distancia óptima con un k-*spread* como código proyectado tienen también parámetros óptimos. A cambio, la condición anterior sobre el vector tipo es necesaria. El resto del trabajo está dedicado a la construcción sistemática de estos códigos, para cada elección de los parámetros. Esta construcción se presenta de forma gradual y se apoya en dos pilares fundamentales: el primero es la existencia de **emparejamientos perfectos** en grafos bipartitos regulares, que nos permite dar códigos *flag* de distancia óptima de tipo $(1, n-1)$ con el *spread* de rectas de $\mathbb{F}_q^n$ como código proyectado (Capítulo 2, Teorema 3.6). Por otra parte, utilizando técnicas de **reducción del cuerpo** obtenemos una nueva construcción de tipo $(k, n-k)$ con un k-*spread* Desarguesiano como código proyectado (Teorema 3.8), para cualquier valor de k. A continuación, extendemos esta construcción al vector admisible completo $(1, \ldots, k, n-k, \ldots, n-1)$ en el Teorema 3.10 del mismo trabajo. A partir de esta última, podemos obtener construcciones para cualquier tipo admisible simplemente eliminando aquellas dimensiones del vector tipo admisible completo que no queramos conservar. La traducción de nuestro problema en un problema clásico de Teoría de Grafos –el de encontrar emparejamientos perfectos de grafos bipartitos y regulares– nos permite salvar el salto entre las dimensiones k y $n-k$ y obtener *flags* adecuados para nuestros propósitos.

Por otra parte, en los Capítulos 3 y 6, consideramos la posibilidad de construir códigos *flag* orbitales de distancia óptima con un k-*spread* como proyectado, bajo la acción de algún subgrupo de $\mathrm{GL}(n, q)$. Esta pregunta surge de forma natural, teniendo en cuenta que los códigos *spread* han sido tratados desde el punto de vista orbital. En el Capítulo 3, nos centramos en esta cuestión para el tipo completo con $n = 2k$. Nuestro punto de partida es el k-*spread* $\mathcal{S}(2, k, P)$ dado en (5). Recordemos que, en [69], se prueba que $\mathcal{S}(2, k, P)$ puede ser visto como la órbita del subespacio $\mathcal{U} = \mathrm{rowsp}(I_k \mid 0_{k\times k})$ bajo la acción del subgrupo

$$\mathbf{G} = \left\langle \left(\begin{array}{c|c} 0_{k\times k} & I_k \\ \hline I_k & 0_{k\times k} \end{array} \right), \left(\begin{array}{c|c} I_k & A \\ \hline 0_{k\times k} & I_k \end{array} \right) \;\middle|\; A \in \mathbb{F}_q[P] \right\rangle \tag{18}$$

de $\mathrm{GL}(n, q)$ dado en (8). En nuestro caso, empezamos el Capítulo 3 probando que $\mathbf{G}$ no proporciona códigos *flag* de distancia óptima. De hecho, sus órbitas no son siquiera códigos *flag* disjuntos. Por ello, estudiamos condiciones para asegurar la disyunción de códigos orbitales bajo la acción de subgrupos arbitrarios de $\mathrm{GL}(n, q)$ y concluimos que, dados un *flag* $\mathcal{F} \in \mathcal{F}_q((t_1, \dots, t_r), n)$ y un subgrupo $\mathbf{H}$ de $\mathrm{GL}(n, q)$, la órbita $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ es un código *flag* disjunto si, y solo si, se cumple la relación

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_1) = \cdots = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_r). \tag{19}$$

Además, en el mismo capítulo, el lector puede encontrar el siguiente resultado.

> **(Capítulo 3, Proposición 3.7)** Sea $\mathcal{F}$ un *flag* completo en $\mathbb{F}_q^{2k}$ y $\mathbf{H}$, un subgrupo de $\mathrm{GL}(2k, q)$ tal que el código orbital $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_k)$ es de máxima distancia. Si $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_k) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$ para todo $1 \leqslant i \leqslant 2k-1$, entonces $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ es un código *flag* de distancia óptima de tamaño $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_k)|$.

En virtud de este resultado, para obtener códigos *flag* de distancia óptima bajo la acción de cierto subgrupo $\mathbf{H}$ de $\mathbf{G}$ basta que dicho subgrupo proporcione estabilizador trivial al actuar sobre $\mathcal{U}$ (o cualquier otro subespacio del k-*spread* $\mathcal{S}(2, k, P)$). En tal caso, el mejor cardinal posible solo podría alcanzarse si $\mathbf{H}$ fuese de orden $\frac{q^n-1}{q^k-1}$. Con el objetivo de encontrar un subgrupo $\mathbf{H}$ de $\mathbf{G}$ adecuado, nos sumergimos en la estructura de grupo de $\mathbf{G}$ y probamos que contiene una copia isomorfa del grupo special lineal $\mathrm{SL}(2, q^k)$ (Capítulo 3, Proposición 4.2). En nuestro caso, seleccionamos un grupo $\mathbf{H}$ de $\mathbf{G}$ isomorfo a un subgrupo de Singer de $\mathrm{SL}(2, q^k)$ y lo hacemos actuar sobre *flags*. Los *subgrupos de Singer* de $\mathrm{GL}(n, q)$ son grupos cíclicos del mayor orden posible: $q^n - 1$. Sabemos que estos grupos actúan transitivamente sobre las Grassmannianas de rectas e hiperplanos de $\mathbb{F}_q^n$. Además, en [18], se prueba que, para cada divisor k de n, existe exactamente una órbita de bajo la acción de un subgrupo de Singer que es un k-*spread* de $\mathbb{F}_q^n$. Utilizando el grupo $\mathbf{H}$, obtenemos construcciones de códigos *flag* completos de distancia óptima en $\mathbb{F}_q^{2k}$ con el k-*spread* $\mathcal{S}(2, k, P)$ como código proyectado para valores pares de q (Capítulo 3, Teorema 4.14). Si q es impar, alcanzamos también el máximo cardinal al considerar la unión de dos órbitas distintas, ambas bajo la acción de $\mathbf{H}$ (Capítulo 3, Proposición 4.15).

La continuación natural de este trabajo puede encontrarse en el Capítulo 6, donde encontramos dos partes muy diferenciadas. La primera de ellas se centra en el estudio general de códigos *flag* de distancia óptima. En el Teorema 4.8 de este trabajo damos una nueva caracterización de los códigos *flag* de distancia óptima en términos de, a lo sumo, dos códigos proyectados: aquellos con las dimensiones más próximas a $\frac{n}{2}$, tanto por la izquierda como por la derecha. Este resultado se aplica a lo largo de la segunda parte para construir códigos *flag* orbitales de distancia óptima de tipo $(1, \dots, k, n-k, \dots, n-1)$ en $\mathbb{F}_q^n$ con un k-*spread* como código proyectado, para cada divisor arbitrario k de n. En este caso, en virtud del Teorema 4.8, las dimensiones k y $n-k$ son las protagonistas de la construcción, en la que utilizamos la acción de un subgrupo de Singer de $\mathrm{GL}(n, q)$ sobre la variedad de *flags* de este tipo. Además, la unicidad de los subgrupos de un grupo cíclico nos permite caracterizar qué subgrupos (del subgrupo de Singer de partida) proporcionan códigos *flag* de distancia óptima como sus órbitas, en términos de sus órdenes (Capítulo 6, Teorema 5.1). Además, para aquellos casos en que el código *flag* orbital obtenido no alcanza el máximo cardinal posible, es decir, $\frac{q^n-1}{q^k-1}$, damos una nueva construcción como unión de distintas órbitas (bajo la acción de un mismo subgrupo) de tamaño máximo en función del orden del subgrupo que actúa (Capítulo 6, Teorema 5.2).

Recordemos que, por tener un k-*spread* como código proyectado, esta construcción orbital de código *flag* de distancia óptima presenta la restricción del vector tipo dada en (17). Dicha restricción desaparece en los casos $k = 1$ y $n = 3$ o $n = 2k$, en los que somos capaces de dar construcciones orbitales de máxima distancia de tipo completo y, a través de un proceso de agujereado adecuado, de cualquier otro tipo. El caso $n = 2k$ también queda cubierto con la construcción que presentamos en el Capítulo 3. No obstante, hasta el momento, no se conocía ninguna construcción sistemática de códigos *flag* orbitales de distancia óptima con máximo cardinal para valores impares de n. La parte final del Capítulo 6 está dedicada a este problema. Concretamente, si $n = 2k + 1$, utilizamos la acción de un subgrupo $\mathbf{G}$ de $\mathrm{GL}(n, q)$ isomorfo a un subgrupo de Singer de $\mathrm{GL}(k + 1, q)$ y caracterizamos aquellos *flags* completos $\mathcal{F}$ que proporcionan órbitas $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})$ de máxima distancia. Todas ellas contienen exactamente $q^{k+1} - 1$ *flags* (Teorema 5.7). No obstante, en [47, Proposición 2.4] se prueba que el mejor cardinal para un código *flag* completo de estas características es $q^{k+1} + 1$. Por ello, dedicamos el resto del trabajo a la caracterización de aquellos *flags* completos $\mathcal{F}', \mathcal{F}''$ que hacen que $\mathrm{Orb}(\mathcal{F}) \cup \{\mathcal{F}', \mathcal{F}''\}$ siga teniendo la mejor distancia posible y, en este caso, también tamaño máximo (Capítulo 6, Teorema 5.11).

En [25] se prueba que que la acción de los subgrupos de Singer de $\mathrm{GL}(n, q)$ sobre los $\mathbb{F}_q$-subespacios vectoriales de $\mathbb{F}_q^n$ puede interpretarse como la acción del grupo multiplicativo $\mathbb{F}_{q^n}^*$ sobre los $\mathbb{F}_q$-subespacios del cuerpo $\mathbb{F}_{q^n}$. Este es el lenguaje que usamos en el Capítulo 4, donde trabajamos con *flags* en el cuerpo $\mathbb{F}_{q^n}$ y estudiamos propiedades teóricas de los códigos *flag* construidos como órbitas bajo la acción de subgrupos $\langle\beta\rangle$ del grupo de Singer $\mathbb{F}_{q^n}^*$, a los que llamamos *códigos flag β-cíclicos*. En nuestro estudio, generalizamos algunos conceptos presentados en [25] al contexto de los códigos *flag* como, por ejemplo:

> Sea $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ un *flag* en $\mathbb{F}_{q^n}$. Llamamos *mejor amigo del flag $\mathcal{F}$* al mayor subcuerpo $\mathbb{F}_{q^m}$ de $\mathbb{F}_{q^n}$ sobre el que todos los subespacios del *flag* tienen estructura de espacio vectorial.

Tal y como ocurre en el caso de los códigos de subespacio, el mejor amigo de un *flag* está íntimamente ligado a su estabilizador bajo la acción de $\mathbb{F}_{q^n}^*$. Más concretamente, si $\mathbb{F}_{q^m}$ es el mejor amigo del *flag* $\mathcal{F}$, entonces $\mathrm{Stab}_{\mathbb{F}_{q^n}^*}(\mathcal{F}) = \mathbb{F}_{q^m}^*$. Este hecho permite obtener el cardinal y cotas para la distancia mínima de los códigos β-cíclicos generados por $\mathcal{F}$. También dedicamos parte de este trabajo al estudio de familias de códigos *flag* orbitales β-cíclicos con valores extremos de la distancia (una vez fijado el mejor amigo). En primer lugar, nos centramos en los *códigos flag β-cíclicos de Galois*, es decir, aquellos generados por *flags* dados por torres de cuerpos encajados, a los que llamamos *flags de Galois*, bajo la acción de subgrupos de $\mathbb{F}_{q^n}^*$. Para ser más exactos, dada una sucesión de divisores $1 \leqslant t_1 < t_2 < \cdots < t_r$ de n tales que cada t_i divide a t_{i+1}, el *flag* de Galois de tipo $(t_1, \ldots, t_r)$ es $\mathcal{F} = (\mathbb{F}_{q^{t_1}}, \ldots, \mathbb{F}_{q^{t_r}})$. En el Capítulo 4 (Teorema 4.14), probamos que el conjunto de distancias admisibles para $\mathrm{Orb}_\beta(\mathcal{F})$ es

$$\{2t_1, 2(t_1 + t_2), \ldots, 2(t_1 + \cdots + t_r)\} \tag{20}$$

y caracterizamos los subgrupos $\langle\beta\rangle$ de $\mathbb{F}_{q^n}^*$ que proporcionan cada uno de estos valores de la distancia. En particular, si el grupo que actúa es $\mathbb{F}_{q^n}^*$, obtenemos la menor distancia posible para códigos con $\mathbb{F}_{q^{t_1}}$ como mejor amigo: $2t_1$. Sin embargo, este caso nos resulta especialmente interesante pues encierra una engranaje de *spreads* perfectamente encajados. Precisamente, para cada $1 \leqslant i \leqslant r$, el i-ésimo proyectado es el t_i-*spread* $\mathrm{Orb}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^{t_i}})$ dado en (9) y todos estos *spreads* bailan al son del grupo $\mathbb{F}_{q^n}^*$. Además, establecemos una correspondencia entre el conjunto de distancias en (20) y los subgrupos $\langle\beta\rangle$ de $\mathbb{F}_{q^n}^*$. En concreto, también obtenemos construcciones de distancia óptima $2(t_1 + \cdots + t_r)$. Conectamos este hecho con el estudio general de códigos orbitales cíclicos de distancia óptima con mejor amigo prefijado. En primer lugar, caracterizamos los códigos *flag* orbitales β-cíclicos de distancia óptima cuando β es un elemento primitivo del cuerpo $\mathbb{F}_{q^n}^*$ (Capítulo 4, Corolario 4.23) y damos condiciones necesarias para las dimensiones del vector tipo cuando β no es necesariamente primitivo (Capítulo 4, Teorema 4.21). Notemos que las construcciones dadas en los Capítulos 3 y 6 pueden interpretarse en este escenario, identificando los grupos que allí intervienen con (un subgrupo de) $\mathbb{F}_{q^n}^*$. La condición de tener un k-*spread* como código proyectado se traduce en elegir un *flag* generador en $\mathbb{F}_{q^n}$ con el subcuerpo $\mathbb{F}_{q^k}$ entre sus subespacios y con el cuerpo base $\mathbb{F}_q$ como mejor amigo.

La idea de utilizar subcuerpos de $\mathbb{F}_{q^n}$ como subespacios del *flag* generador es estudiada en mayor profundidad en el Capítulo 7. Allí, introducimos los *flags de Galois generalizados*: *flags* en $\mathbb{F}_{q^n}$ con subcuerpos en al menos uno pero no todos sus subespacios. La cadena maximal de cuerpos encajados en un *flag* de estas características es su *flag de Galois subyacente*. En este capítulo estudiamos cómo la presencia de ciertos cuerpos distribuidos a lo largo del *flag* semilla afecta al comportamiento de los códigos orbitales que este genera. En particular, probamos que el *flag* de Galois subyacente determina un conjunto de valores potencialmente alcanzables para la distancia mínima (Capítulo 7, Teorema 3.31 y Definición 3.32). Los valores de la distancia fuera de este conjunto son automáticamente descartados de nuestro estudio. Por similitud con los códigos *flag* de Galois, y motivados por el Teorema 4.14 del Capítulo 4, nos preguntamos si, dado un *flag* de Galois generalizado $\mathcal{F}$, todos los valores potencialmente alcanzables de la distancia son realmente alcanzables para algún código $\mathrm{Orb}_\beta(\mathcal{F})$, con $\beta \in \mathbb{F}_{q^n}^*$.

Sin perder esta cuestión de vista, en la segunda parte de este capítulo, presentamos una construcción sistemática de códigos *flag* de Galois generalizados con código de Galois subyacente prefijado. En nuestro caso, utilizamos *flag* generadores escritos en una forma regular muy concreta, lo que nos permite controlar los parámetros y algunas propiedades estructurales del código. Además, cuando consideramos órbitas bajo la acción de $\mathbb{F}_{q^n}^*$, obtenemos construcciones que pueden ser fácilmente decodificadas haciendo uso de las técnicas desarrolladas en el Capítulo 5 y determinamos los valores exactos de los parámetros de nuestra construcción. Eventualmente, estas construcciones son de distancia óptima (Corolarios 4.4 y 4.16). Por otra parte, cuando hacemos actuar a un subgrupo propio $\langle\beta\rangle$ de $\mathbb{F}_{q^n}^*$, perdemos parte del control pero todavía somos capaces de determinar un reducido abanico de posibilidades para la distancia mínima (Capítulo 7, Teorema 4.15). Estas construcciones, además de constituir una variada batería de ejemplos de códigos de Galois generalizados, nos permiten asegurar que no todos los valores potencialmente alcanzables de la distancia (fijado el *flag* de Galois subyacente) pueden ser obtenidos como la distancia mínima de algún código *flag* β-cíclico de Galois generalizado.

Como dijimos al principio de la introducción, establecer conexiones entre los códigos *flag* y sus códigos proyectados se ha convertido el hilo conductor de esta tesis. Hasta ahora nos hemos centrado principalmente en los códigos *flag* de distancia óptima que, como se prueba en el Capítulo 1, deben ser, en particular, disjuntos. El concepto de *disyunción* para códigos *flag* se estudia con mayor detalle en los Capítulos 5 y 8, donde introducimos la noción de *consistencia* para códigos *flag* (respecto a sus proyectados) y la familia de *códigos flag M-disjuntos*, respectivamente. A continuación exponemos las ideas principales de estos trabajos.

En primer lugar, en el Capítulo 5, partimos de un código *flag* disjunto $\mathcal{C} \subseteq \mathcal{F}_q((t_1,\ldots,t_r),n)$. En otras palabras, un código $\mathcal{C}$ satisfaciendo $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$. Esta condición no es más que una relación de consistencia entre los cardinales del código *flag* y los de sus proyectados. Por ello, empleamos también el término de *cardinal-consistente* para referirnos a los códigos *flag* disjuntos. Por otra parte, también presentamos la siguiente familia de códigos *flag*, en la que la distancia mínima del código *flag* es, en cierto modo, consistente con la de sus códigos proyectados.

Decimos que un código *flag* $\mathcal{C} \subseteq \mathcal{F}_q((t_1,\ldots,t_r),n)$ es *distancia-consistente* si, para cada par de *flags* $\mathcal{F},\mathcal{F}' \in \mathcal{C}$, se cumple

$$d_f(\mathcal{C}) = d_f(\mathcal{F},\mathcal{F}') \iff d_S(\mathcal{C}_i) = d_S(\mathcal{F}_i,\mathcal{F}'_i), \ \forall i = 1,\ldots,r. \tag{21}$$

Esta condición es equivalente a decir que los *flags* más cercanos en un código distancia-consistente $\mathcal{C}$ están dados por sucesiones de subespacios que determinan las distancias mínimas de todos los proyectados. En otras palabras, la idea de "proximidad" en el código *flag* es consistente con la de sus códigos proyectados. Además, y como consecuencia directa de (21), si $\mathcal{C}$ es distancia-consistente, en particular, verifica $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$. No obstante, esta condición no caracteriza a la familia de códigos distancia-consistentes. Con el objetivo de poder controlar perfectamente los parámetros de los códigos *flag* en términos de los de sus códigos proyectados, presentamos también la clase de códigos *flag* consistentes como aquellos que son a la vez consistentes para el cardinal (disjuntos) y para la distancia. Además, los caracterizamos a través del siguiente resultado:

(Capítulo 5, Teorema 1) Un código *flag* $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ es consistente si, y solo si, se verifican las siguientes afirmaciones:

(1) $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$ y

(2) $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$.

En particular, los códigos *flag* de distancia óptima son ejemplo de códigos consistentes en los que los proyectados alcanzan la distancia máxima para sus respectivas dimensiones (Capítulo 5, Corolario 3).

Además del cardinal y la distancia mínima, probamos que algunas propiedades estructurales también se transfieren de un código *flag* consistente a sus códigos proyectados y viceversa, como explicamos a continuación. Para ello, notemos que trabajar con códigos *flag* utilizando sus códigos proyectados nos permite, en general, traer algunos conceptos de la teoría de códigos de subespacio al escenario de los códigos *flag* de dos formas, en principio, distintas. Por ejemplo, en el Capítulo 5, generalizamos los conceptos de *código equidistante* y *código girasol*, previamente estudiados en [21, 29] en el ámbito de los códigos de subespacio.

(Capítulo 5, Definiciones 6-9) Decimos que un código *flag* $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ es:

- *Equidistante* si, para cada pareja de *flags* distintos $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, se verifica $d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}')$.
- *Equidistante-proyectado* si todos sus códigos proyectados son códigos de dimension constante equidistantes.
- Un *girasol* si existe una sucesión de subespacios encajados $C_1 \subsetneq \cdots \subsetneq C_r$ tales que, para cada par de *flags* distintos $\mathcal{F}, \mathcal{F}'$ en $\mathcal{C}$ y todo $1 \leqslant i \leqslant r$, tenemos $\mathcal{F}_i \cap \mathcal{F}'_i = C_i$. En tal caso, la sucesión $C = (C_1, \ldots, C_r)$ se llama *el centro del girasol.*
- Un *girasol-proyectado* si cada código proyectado $\mathcal{C}_i$ de $\mathcal{C}$ es un girasol de $\mathcal{G}_q(t_i, n)$.

En general, no hay una relación directa entre las propiedades de ser equidistante (resp. un girasol) y equidistante-proyectado (resp. girasol-proyectado). Sin embargo, bajo la condición de consistencia para códigos *flag*, probamos que son equivalentes, es decir, un código *flag* consistente es equidistante (resp. un girasol) si, y solo si, todos sus proyectados son equidistantes (resp. girasoles) de las dimensiones correspondientes (Capítulo 5, Teoremas 2 y 3). Por último, y también en el Capítulo 5, mostramos que los códigos consistentes presentan grandes bendiciones a la hora de ser decodificados y damos un algoritmo de decodificación (Capítulo 5, Algoritmo 1) válido para cualquier código *flag* consistente. En particular, dicho algoritmo puede aplicarse a la construcción presentada en el Teorema 4.2 del Capítulo 7 o a cualquiera de los códigos *flag* de distancia óptima construidos en los Capítulos 1-4, 6 y 7.

Cuando trabajamos con códigos *flag* consistentes –y, en particular, con códigos de distancia óptima– la distancia mínima se calcula siempre de la misma forma: sumando las distancias de los códigos proyectados. Este comportamiento es característico de esta familia de códigos pero, en general, cada valor de la distancia de *flags* puede obtenerse como suma de distintas combinaciones de distancias de subespacio. Por esta razón, en el Capítulo 8, introducimos diferentes versiones del concepto de *vector distancia*.

(Capítulo 8, Definiciones 3.1 y 3.6) El *vector distancia* asociado al par de *flags* $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q((t_1, \dots, t_r), n)$ es

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') = (d_S(\mathcal{F}_1, \mathcal{F}'_1), \dots, d_S(\mathcal{F}_r, \mathcal{F}'_r)) \in 2\mathbb{Z}^r. \qquad (22)$$

El *conjunto de vectores distancia* de un código *flag* $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$ contiene a todos los vectores distancia asociados a pares de *flags* en $\mathcal{C}$ para los que se alcanza la distancia mínima del código.

En particular, los códigos *flag* consistentes se caracterizan por ser disjuntos y tener a $(d_S(\mathcal{C}_1), \dots, d_S(\mathcal{C}_r))$ como único vector distancia. Por el contrario, un código *flag* arbitrario puede tener un conjunto de vectores distancia con más de un elemento.

El Teorema 3.9 de este Capítulo 8 permite caracterizar a los vectores distancia (asociados a cierto valor de la distancia de *flags*) en términos de sus componentes y una serie de relaciones entre ellas. En otras palabras, determinamos todos los vectores de $2\mathbb{Z}^r$ que realmente representan configuraciones válidas de la distancia entre *flags*. Además, parte del mismo trabajo está dedicado al estudio de las distancias alcanzables al fijar cierta componente en un vector distancia. Prestamos especial atención a aquellas distancias asociadas a vectores distancia con una componente nula, es decir, aquellas entre *flags* con, el menos, un subespacio común. A continuación, extendemos este análisis y consideramos vectores distancia de longitud r con $1 \leqslant M \leqslant r$ componentes nulas, dicho de otro modo, investigamos distancias entre *flags* que comparten, al menos, M subespacios.

En la segunda parte del Capítulo 8, utilizamos nuestro estudio previos para obtener propiedades de códigos *flag* con una distancia mínima prefijada. En nuestro camino, generalizamos el concepto de disyunción de códigos *flag* y el de códigos proyectados, haciendo uso de la siguiente familia de proyecciones. Dado un vector tipo $(t_1, \dots, t_r)$, un entero $1 \leqslant M \leqslant r$ e índices ordenados $1 \leqslant i_1 < \cdots < i_M \leqslant r$, consideramos la proyección

$$\begin{array}{rccc} p_{(i_1,\dots,i_M)} & : & \mathcal{F}_q((t_1,\dots,t_r),n) & \longrightarrow & \mathcal{F}_q((t_{i_1},\dots,t_{i_M}),n) \\ & & (\mathcal{F}_1,\dots,\mathcal{F}_r) & \longmapsto & (\mathcal{F}_{t_{i_1}},\dots,\mathcal{F}_{t_{i_M}}). \end{array} \tag{23}$$

(Capítulo 8, Definición 5.2) Sea $\mathcal{C}$ un código *flag* en $\mathcal{F}_q((t_1,\dots,t_r),n)$. Su $(i_1,\dots,i_M)$-*código proyectado* es el código *flag*

$$p_{(i_1,\dots,i_M)}(\mathcal{C}) = \{\ p_{(i_1,\dots,i_M)}(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\ \} \tag{24}$$

de tipo $(t_{i_1},\dots,t_{i_M})$ en $\mathbb{F}_q^n$.

Notemos que, para cualquier elección de índices $1 \leqslant i_1 < \cdots < i_M \leqslant r$, se cumple trivialmente $|p_{(i_1,\dots,i_M)}(\mathcal{C})| \leqslant |\mathcal{C}|$. Aquellos códigos *flag* para los que se alcanza la igualdad son de especial interés para nuestro estudio.

(Capítulo 8, Definición 5.3) Un código *flag* $\mathcal{C}$ se dice $(i_1,\dots,i_M)$-*disjunto* si la proyección $p_{(i_1,\dots,i_M)}$ restringida a $\mathcal{C}$ es inyectiva, es decir, si $|\mathcal{C}| = |p_{(i_1,\dots,i_M)}(\mathcal{C})|$. Si esto ocurre para cada elección de M índices $1 \leqslant i_1 < \cdots < i_M \leqslant r$, entonces decimos que $\mathcal{C}$ es M-*disjunto*.

En estos términos, los códigos *flag* disjuntos son ahora 1-disjuntos. Por otra parte, la clase de códigos *flag* M-disjuntos conecta perfectamente con nuestro estudio previo de la distancia, ya que ningún vector distancia de un código *flag* M-disjunto puede contar con M componentes nulas. Esta propiedad nos permite dar una condición suficiente sobre la distancia mínima de un código para asegurar cierto grado de disyunción (Capítulo 8, Teorema 5.9). Este resultado nos permite, por otra parte, obtener de forma muy natural nuevas cotas superiores para el valor $A_q^f(n,d,(t_1,\dots,t_r))$, esto es, el máximo cardinal posible para un código *flag* en $\mathcal{F}_q((t_1,\dots,t_r),n)$ con distancia mínima d, en función de cuántos subespacios pueden compartir dos *flags* distintos sin bajar de la distancia d. La Sección 6 del Capítulo 8 esta dedicada a esta cuestión. Allí obtenemos resultados para cualquier valor de los parámetros q, n, d y el vector tipo.

Este estudio de la distancia de *flags* utiliza un enfoque algebraico, donde los vectores distancia y sus propiedades nos permiten obtener información relevante sobre los códigos *flag*. Por otra parte, en el Capítulo 9 presentamos un estudio de la distancia entre *flags* completos, desde un punto de vista combinatorio. En este caso, por razones técnicas, utilizamos la métrica equivalente inducida por la *distancia de inyección (de subespacios)* dada por:

$$d_I(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{n-1} (i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i)) = \frac{1}{2} d_f(\mathcal{F}, \mathcal{F}').$$

En este trabajo, introducimos la idea de *camino de distancia (en el soporte de distancia)* como una representación gráfica de los vectores distancia.

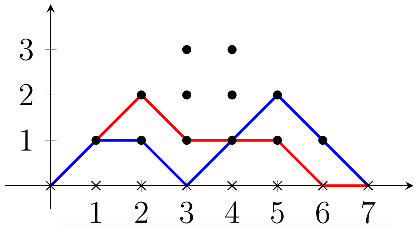

Figure 2: Caminos de distancia en el soporte de distancia para $n = 7$.

Utilizando esta nueva perspectiva, el cálculo de distancias de *flag* se reduce a contar puntos circulares del soporte de distancia en un camino de distancia o por debajo de él. Análogamente, el valor complementario de la distancia, al que llamamos *codistancia*, coincide con el número de puntos que quedan sobre un camino. Por otra parte, tras enriquecer el soporte de distancia, añadiendo una red auxiliar de puntos rojos, y efectuar una rotación, obtenemos un diagrama de Ferrers: el *diagrama de Ferrers marco* asociado a la variedad de *flags* completos en $\mathbb{F}_q^n$.

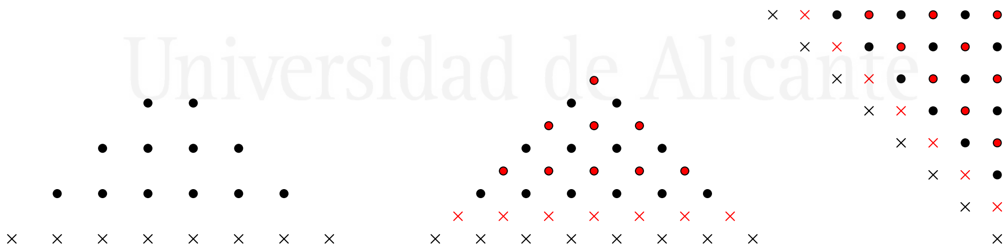

Figure 3: Soporte, soporte enriquecido y diagrama de Ferrers marco para $n = 7$.

Haciendo uso de este diagrama de Ferrers, establecemos una correspondencia biyectiva entre el conjunto de caminos de distancia y determinados elementos propios de la Teoría de Particiones, relacionados con el número de puntos circulares negros contenidos en *subdiagramas de Ferrers* adecuados (Capítulo 9, Teorema 4.28). Este punto de vista combinatorio nos permite establecer conexiones entre los parámetros de códigos *flag* completos y los de sus códigos proyectados. Más concretamente, fijado un código *flag* completo $\mathcal{C}$ en $\mathbb{F}_q^n$, le asociamos un conjunto de diagramas de Ferrers y utilizamos sus *rectángulos de Durfee* para obtener información sobre el cardinal y la distancia de cada uno de sus proyectados $\mathcal{C}_i$ y viceversa (Capítulo 9, Teoremas 5.2, 5.6-5.8, 5.11 y Corolario 5.10). Estas nuevas herramientas nos permiten, además, reinterpretar algunos resultados conocidos en términos combinatorios. Prueba de ello es el Teorema 5.17 del Capítulo 9, donde caracterizamos de nuevo los códigos *flag* de distancia óptima en función de sus conjuntos de caminos de distancia y/o de subdiagramas de Ferrers asociados. Este resultado, el último teorema de esta tesis pero también el primero, cierra nuestro estudio sobre códigos *flag*.

Las relaciones descritas a lo largo de esta sección introductoria quedan recogidas de forma esquemática en el Figura 7 de la página 53 de esta memoria.

Part I

Introduction

Flag codes in the network coding setting were introduced by Liebhold, Nebe and Vazquez-Castro a few years ago as a generalization of constant dimension codes. The seminal work in this research line is [54] and, since then, other works in this respect have appeared (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 23, 47, 52, 53, 59]).

This thesis is devoted to the study of different but interconnected aspects related to flag codes. It is presented as the compendium of the following works, that the reader can find, in this order, in Chapters 1-9.

- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes from planar spreads in Network Coding*, Finite Fields and their Applications, Vol. 68 (2020), 101745.
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum distance flag codes from spreads via perfect matchings in graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An orbital construction of optimum distance flag codes*, Finite Fields and their Applications, Vol. 73 (2021), 101861.
- C. Alonso-González and M. A. Navarro-Pérez, *Cyclic orbit flag codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2331-2356.
- C. Alonso-González and M. A. Navarro-Pérez, *Consistent flag codes*, Mathematics, Vol. 8(12) (2020), 2234.
- M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes of maximum distance and constructions using Singer groups*, https://arxiv.org/abs/2109.00270 (preprint)
- C. Alonso-González and M. A. Navarro-Pérez, *On generalized Galois cyclic orbit flag codes*, https://arxiv.org/abs/2111.09615 (preprint).
- C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes: distance vectors and cardinality bounds*, https://arxiv.org/abs/2111.00910 (preprint).
- C. Alonso-González and M. A. Navarro-Pérez, *A combinatorial approach to flag codes*, https://arxiv.org/abs/2111.15388 (preprint).

We dedicate the first part of the thesis to explain the main contributions in these works as well as to exhibit the interrelationship among them. To this end, we start with some preliminaries about constant dimension codes and we also comment the state of the art in the flag codes framework.

Constant dimension codes

Network coding was introduced in [1] as a new method to send information through channels modelled as directed acyclic multigraphs with (possibly) several senders and receivers, where intermediate nodes are allowed to perform linear combinations of the received packets (vectors), instead of simply routing them. In [1], it was proved that the use of linear combinations at every intermediate node improves the communication rate. The *butterfly* network, represented in the next figure, is a well-known example of this:

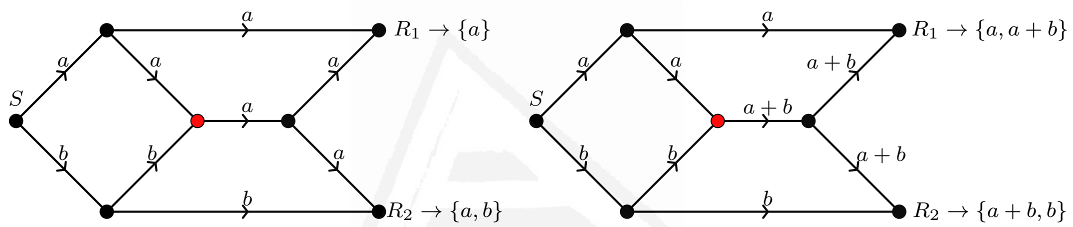

Figure 4: *Butterfly* network

Observe that in the first case, where intermediate nodes just resend packets, two uses of the channel are needed so that both receivers R_1 and R_2 get messages a and b since, despite the fact that the "bottleneck" node (marked in red) receives both a and b, it can just send one of these messages. On the other hand, if we allow coding at intermediate nodes, both receivers are able to recover a and b in just a channel use.

On the negative side, whenever errors occur and a packet (a vector) is affected, every linear combination involving it is corrupted as well. This fact makes network coding especially vulnerable to error propagation and opens the door to construct suitable error-correcting codes for this new scenario. Moreover, in view of the previous example, one can see that the receivers need to know how the network works to be able to recover vectors a and b from the received ones. In order to overcome this problem, in [45], Koetter and Kschischang suggest the use of vector subspaces (instead of vectors) as codewords. Notice that, since vector subspaces are invariant under linear combinations, neither the sender(s) nor the receiver(s) need information about the performed linear combinations of the network. That paper represents the first algebraic approach to network coding through *non-coherent networks*, those where intermediate nodes compute and send random linear combinations and the topology of the network is unknown.

There, the authors introduce the class of *subspace codes* and define a suitable metric to count errors and erasures produced through the communication process. Let us precise these ideas. To do so, and for the rest of the thesis, we will consider the following notation:

- Let q be a prime power,
- $\mathbb{F}_q$ denotes the *finite field* with q elements and
- k and n are positive integers such that $1 \leqslant k < n$.
- $\mathbb{F}_q^n$ represents the n-dimensional vector space over the field $\mathbb{F}_q$.
- The *Grassmann variety* (or just the *Grassmannian*) $\mathcal{G}_q(k, n)$ is the set of k-dimensional $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_q^n$.
- The *projective geometry* of $\mathbb{F}_q^n$ is the set $\mathcal{P}_q(n)$ containing all the vector subspaces of $\mathbb{F}_q^n$.

The projective geometry $\mathcal{P}_q(n)$ can be seen as a metric space endowed with the *subspace distance* given by

$$d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}), \tag{25}$$

for every $\mathcal{U}, \mathcal{V} \in \mathcal{P}_q(n)$. In particular, if both subspaces have the same dimension, say $\dim(\mathcal{U}) = \dim(\mathcal{V}) = k$, the previous expression becomes

$$d_S(\mathcal{U}, \mathcal{V}) = 2\big(k - \dim(\mathcal{U} \cap \mathcal{V})\big). \tag{26}$$

This metric allows to define error-correcting codes as follows (see [45]).

A *subspace code* of length n is a nonempty subset $\mathcal{C} \subseteq \mathcal{P}_q(n)$. If every codeword (subspace) in $\mathcal{C}$ has the same dimension, say $1 \leqslant k < n$, then $\mathcal{C}$ is said to be a *constant dimension* code (in $\mathcal{G}_q(k, n)$).

Given a constant dimension code $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$, its *minimum distance* is given by

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U}, \mathcal{V}) \mid \mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\},$$

whenever $|\mathcal{C}| \geqslant 2$. In case that $|\mathcal{C}| = 1$, we just put $d_S(\mathcal{C}) = 0$. In general, according to (26), we have that $d_S(\mathcal{C})$ is an even integer such that

$$0 \leqslant d_S(\mathcal{C}) \leqslant \begin{cases} 2k & \text{if} \quad 2k \leqslant n, \\ 2(n-k) & \text{if} \quad 2k \geqslant n. \end{cases} \tag{27}$$

The cardinality of a constant dimension code is related to the amount of different messages it can encode. As a consequence, given $1 \leqslant k < n$ and a value of the distance d satisfying these bounds, we are interested in obtaining constructions of constant dimension codes in $\mathcal{G}_q(k,n)$ having minimum distance d and large cardinality. The value $A_q(n,d,k)$ represents the largest possible size for such codes. However, in many cases, these values are not even known. Determining the exact value of $A_q(n,d,k)$ or giving lower and upper bounds for it is a interesting problem that has been tackled in many papers in recent years (see [22, 35, 41, 43, 48, 66, 67, 70, 71, 73], for instance).

As for general error-correcting codes, the minimum distance of a constant dimension code is clearly related to its capability to detect and correct errors. More precisely, a constant dimension code $\mathcal{C}$ detects up to $d_S(\mathcal{C})-1$ errors and corrects up to $d_S(\mathcal{C})/2-1$. As a result, special attention has been given to those codes in $\mathcal{G}_q(k,n)$ attaining the maximum possible (minimum) distance.

More precisely, in [30], the following family of constant dimension codes of maximum distance is well studied, for $k \leqslant \lfloor \frac{n}{2} \rfloor$.

> A *partial spread code of dimension* k (or partial k-spread, for short) of $\mathbb{F}_q^n$ is a subset $\mathcal{C}$ of $\mathcal{G}_q(k,n)$ in which different subspaces pairwise intersect trivially. Equivalently, $\mathcal{C} \subseteq \mathcal{G}_q(k,n)$ is a partial k-spread if every nonzero vector of $\mathbb{F}_q^n$ lies in, at most, a subspace in $\mathcal{C}$.

It is well-known that the cardinality of any partial k-spread of $\mathbb{F}_q^n$ is upper bounded by

$$A_q(n,2k,k) \leqslant \frac{q^n-q^r}{q^k-1}, \tag{28}$$

where r is the remainder obtained dividing n by k. As a special case of these codes, one has the class of **spread codes**, first introduced in [56] as follows.

> A *k-spread (code)* $\mathcal{S}$ of $\mathbb{F}_q^n$ is a subset of $\mathcal{G}_q(k,n)$ in which every nonzero vector of $\mathbb{F}_q^n$ lies in one, and only one, subspace in $\mathcal{S}$. In other words, spreads are partitions of $\mathbb{F}_q^n$ into k-dimensional vector subspaces.

Spreads are classical objects coming from Finite Geometry (see [65]). It is well-known that k-spreads of $\mathbb{F}_q^n$ exist if, and only if, k is a divisor of n and in this case, their cardinality is exactly

$$|\mathcal{S}| = A_q(n,2k,k) = \frac{q^n-1}{q^k-1}.$$

It turns out that spreads are optimal codes in the following sense: they attain the best distance for their dimension and have the largest possible size among codes with their dimension and distance.

The study of spread codes has lead to many papers in the last decade (see, for instance, [56, 57, 68, 69]). If $n=ks$ and $P \in \mathrm{GL}(k,q)$ is the companion matrix of a monic irreducible polynomial of degree k in $\mathbb{F}_q[x]$, the following set is a well-known example of k-spread:

$$\mathcal{S}(s,k,P) = \{\mathrm{rowsp}(S) \mid S \in \Sigma\} \subseteq \mathcal{G}_q(k,n), \tag{29}$$

where

$$\Sigma = \{(A_1|A_2|\dots|A_s) \bigm| A_i \in \mathbb{F}_q[P]\} \tag{30}$$

and the first non-zero block from the left is equal to I_k. This construction is originally due to Segre (see [65]) and it is obtained by applying field reduction to the Grassmannian of lines of $\mathbb{F}_{q^k}^s$. In the network coding setting, this construction was presented in [56, Theorem 1].

Another interesting family of constant dimension codes is the one of **orbit codes**. They were presented in [69]. In that paper, the authors introduce those constant dimension codes arising as orbits under the transitive action of the general linear group $\mathrm{GL}(n,q)$ on the Grassmannian $\mathcal{G}_q(k,n)$ defined as

$$\begin{array}{ccc} \mathcal{G}_q(k,n) \times \mathrm{GL}(n,q) & \longrightarrow & \mathcal{G}_q(k,n) \\ (\mathcal{U}, A) & \longmapsto & \mathcal{U} \cdot A = \mathrm{rowsp}(UA), \end{array} \tag{31}$$

where $U \in \mathbb{F}_q^{k\times n}$ is any generator matrix of $\mathcal{U}$.

Given $\mathcal{U} \in \mathcal{G}_q(k,n)$ and a subgroup $\mathbf{H}$ of $\mathrm{GL}(n,q)$, the set

$$\mathrm{Orb}_{\mathbf{H}}(\mathcal{U}) = \{\mathcal{U} \cdot A \mid A \in \mathbf{H}\} \subseteq \mathcal{G}_q(k,n)$$

is called the *orbit code generated by* $\mathcal{U}$ *under the action of* $\mathbf{H}$. The *stabilizer* of $\mathcal{U}$ (w.r.t. $\mathbf{H}$) is the subgroup of $\mathbf{H}$ given by

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{U}) = \{A \in \mathbf{H} \mid \mathcal{U} \cdot A = \mathcal{U}\}.$$

The parameters of orbit codes can be computed in an easier manner (than the ones of general constant dimension codes). Moreover, the orbital structure has been exploited to provide efficient decoding algorithms (see [68]). The authors also motivate the use of orbit codes by exhibiting constructions with optimal parameters. More precisely, in [69], they show that the k-spread $\mathcal{S}(s,k,P)$ in (29) can be expressed as the orbit generated by $\mathcal{U} = \mathrm{rowsp}(I_k \mid 0_{k\times(n-k)})$ under the action of the subgroup of $\mathrm{GL}(n,q)$ generated by the set of block matrices

$$\left(\begin{array}{c|c|c|c} I_k & A_1 & \cdots & A_{s-1} \\ \hline 0 & I_k & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & I_k \end{array}\right), \left(\begin{array}{c|c|c|c|c} 0 & I_k & A_1 & \cdots & A_{s-2} \\ \hline I_k & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & I_k & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & I_k \end{array}\right), \dots, \left(\begin{array}{c|c|c|c} 0 & \dots & 0 & I_k \\ \hline 0 & \dots & I_k & 0 \\ \hline \vdots & \iddots & \vdots & \vdots \\ \hline I_k & \dots & 0 & 0 \end{array}\right) \tag{32}$$

where $A_i \in \mathbb{F}_q[P]$, for $1 \leqslant i \leqslant s-1$.

When the acting group is cyclic, we speak about *cyclic orbit codes*. The seminal work in this line is [68]. In that work, another construction of k-spread appears. In this case, the authors use the fact that the extension field $\mathbb{F}_{q^n}$ is isomorphic to $\mathbb{F}_q^n$ (as $\mathbb{F}_q$-vector spaces) and show that if k divides n, then the orbit

$$\mathcal{S} = \mathrm{Orb}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^k}) = \{\mathbb{F}_{q^k}\alpha \mid \alpha \in \mathbb{F}_{q^n}^*\} \tag{33}$$

is a k-spread of the field $\mathbb{F}_{q^n}$, with $\mathrm{Stab}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^k}) = \mathbb{F}_{q^k}^*$. Following this approach, in [25], Gluesing-Luerssen *et al.* present the class of **β-cyclic codes** by using the natural multiplicative action of subgroups $\langle\beta\rangle$ of $\mathbb{F}_{q^n}^*$. More precisely, given a k-dimensional $\mathbb{F}_q$-vector subspace $\mathcal{U}$ of the extension field $\mathbb{F}_{q^n}$ and a nonzero element $\beta \in \mathbb{F}_{q^n}^*$ with multiplicative order $|\beta|$, the *β-cyclic (orbit) code* generated by $\mathcal{U}$ is the set

$$\mathrm{Orb}_\beta(\mathcal{U}) = \{\mathcal{U}\beta^i \mid 1 \leqslant i \leqslant |\beta|\},$$

where $\mathcal{U}\beta^i = \{u\beta^i \mid u \in \mathcal{U}\}$. In the same work, the authors study the possible parameters of β-cyclic codes depending on largest field over which the generating subspace is a vector space, that is, its **best friend**.

Flag codes: state of the art

Constant dimension codes in $\mathcal{G}_q(k,n)$ are examples of codes in which every codeword (a subspace) requires a single use of the channel to be sent. However, the subspace channel can be used many times, giving rise to the so-called **multishot (subspace) codes**, introduced in [60]. In this setting, codewords are elements of $\mathcal{P}_q(n)^r$, i.e., sequences of r subspaces of $\mathbb{F}_q^n$. The authors suggest using this class of codes as an interesting alternative to subspace codes if we are interested in obtaining better values of the distance or the cardinality in case that neither the field size q nor the length n could be modified. Moreover, working with multishot codes in $\mathcal{P}_q(n)^r$ can be easier than doing it with their equivalent subspace codes of $\mathbb{F}_q^{nr}$.

As a particular case of multishot codes, we have the class of **flag codes**. These codes were introduced by Liebhold *et al.* in their pioneering works [52, 53, 54]. Let us recall the basic definitions with this respect.

Given integers $1 \leqslant t_1 < \cdots < t_r < n$, a *flag of type* $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a sequence $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ of $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_q^n$ such that

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n$$

and $\dim(\mathcal{F}_i) = t_i$, for $1 \leqslant i \leqslant r$. The *flag variety of type* $(t_1, \ldots, t_r)$ is the set $\mathcal{F}_q((t_1, \ldots, t_r), n)$ containing all the flags of this type. If the type vector is $(1, \ldots, n-1)$ we speak about *full flags* and the *full flag variety*.

The flag variety can be seen as a metric space. In [54], the authors analyze several distances and conclude that the following one is the most appropriate to measure errors and erasures in the network coding setting. Given flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ in $\mathcal{F}_q((t_1, \ldots, t_r), n)$, their *flag distance* is given by

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i). \tag{34}$$

Observe that it generalizes the subspace distance given in (26) to the flag scenario. By analogy to constant dimension codes, **flag codes** are defined as follows.

Given integers $1 \leqslant t_1 < \cdots < t_r < n$, a *flag code* of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a nonempty subset $\mathcal{C}$ of $\mathcal{F}_q((t_1, \ldots, t_r), n)$. Its *minimum (flag) distance* is the value

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\} \tag{35}$$

whenever $|\mathcal{C}| \geqslant 2$. For trivial codes with $|\mathcal{C}| = 1$, we put $d_f(\mathcal{C}) = 0$.

In [54], the authors work with the well-known transitive action of the general linear group $\mathrm{GL}(n, q)$ on the flag variety $\mathcal{F}_q((t_1, \ldots, t_r), n)$ given by

$$\mathcal{F} \cdot A := (\mathcal{F}_1 \cdot A, \ldots, \mathcal{F}_r \cdot A), \tag{36}$$

where $\mathcal{F}_i \cdot A$ is defined in (31). This means that the whole flag variety can be expressed as the orbit $\mathrm{Orb}_{\mathrm{GL}(n,q)}(\mathcal{F})$ generated by any flag $\mathcal{F} \in \mathcal{F}_q((t_1, \ldots, t_r), n)$. The stabilizer of this orbit is given by the subgroup of lower block triangular matrices, with blocks sizes determined by the positive integers $t_1, t_2 - t_1, \ldots, t_r - t_{r-1}$ and $n - t_r$. In this context, one can also study flag codes arising as orbits of this action.

Given a flag $\mathcal{F} \in \mathcal{F}_q((t_1, \dots, t_r), n)$ and a subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$, the *orbit flag code* generated by $\mathcal{F}$ under the action of $\mathbf{H}$ is

$$\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) = \{\mathcal{F} \cdot A \mid A \in \mathbf{H}\}.$$

Its associated *stabilizer* is the subgroup of $\mathbf{H}$ given as

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \{A \in \mathbf{H} \mid \mathcal{F} \cdot A = \mathcal{F}\}.$$

The reader can find some interesting orbit constructions of flag codes in [54], all of them with trivial stabilizer. Moreover, the authors bring matrix codes (endowed with the rank metric) into the flag code setting and also present a decoding algorithm for one of their constructions.

As for constant dimension codes, determining the value of $A_q^f(n, d, (t_1, \dots, t_r))$, i.e, the maximum possible size for flag codes in $\mathcal{F}_q((t_1, \dots, t_r), n)$ with minimum distance equal to d, is an interesting problem which has been recently tackled in [47]. There, the author develops techniques to obtain upper bounds for $A_q^f(n, d, (1, \dots, n-1))$ and give an exhaustive list of bounds for small values of n. Out of the full type case, some examples are also given.

Flag codes: our contributions

This part of the section is devoted to explain in detail the advances in the theory of flag codes introduced in our published papers (Chapters 1-5), preprints and forthcoming works (Chapters 6-9). We do so by collecting the most relevant results in the subsequent sections. In order to relate them and exhibit properly the connections among our works, we follow this order:

Chapter 1 → Chapter 2 → Chapter 3 → Chapter 6 → Chapter 4 → Chapter 7 → Chapter 5 → Chapter 8 → Chapter 9.

Let us start from the very beginning. The origin of this thesis, and a recurrent *leitmotiv* in our works, is the relationship of a given flag code with a special family of constant dimension codes that can be naturally associated to it: the **projected codes**, first introduced in Chapter 1 and used through the rest of chapters. Let us precise this concept. To do so, given positive integers $t_1 < \cdots < t_r < n$ and $1 \leqslant i \leqslant r$, we consider the projection

$$p_i : \mathcal{F}_q((t_1, \dots, t_r), n) \to \mathcal{G}_q(t_i, n) \tag{37}$$

defined as $p_i((\mathcal{F}_1, \dots, \mathcal{F}_r)) = \mathcal{F}_i$. Using these maps we define the projected codes of a flag code as follows.

(Chapter 1, Definition 3.6) Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. For every $1 \leqslant i \leqslant r$, its *$i$-th projected code* is the constant dimension code

$$\mathcal{C}_i = p_i(\mathcal{C}) = \{p_i(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\} \subseteq \mathcal{G}_q(t_i, n). \tag{38}$$

Observe that, for every $1 \leqslant i \leqslant r$, we clearly have $|\mathcal{C}_i| \leqslant |\mathcal{C}|$. If the last inequality holds with equality for every projected code, we have the next special class of flag codes.

(Chapter 1, Definition 3.10) A flag code $\mathcal{C}$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is said to be *disjoint* if it holds

$$|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r| \tag{39}$$

or, equivalently, if for every $1 \leqslant i \leqslant r$, the projection map p_i is injective when restricted to $\mathcal{C}$.

Observe that, from a geometrical point of view, disjoint flag codes are those where different flags have all their subspaces different. These two concepts – the one of projected code and the notion of *disjointness* for flag codes– were introduced in our first work, [8] (see Chapter 1), in order to study flag codes with maximum distance. More precisely, given a flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$, its minimum distance is an even integer $d_f(\mathcal{C})$ satisfying

$$0 \leqslant d_f(\mathcal{C}) \leqslant \sum_{2t_i \leqslant n} 2t_i + \sum_{2t_i > n} 2(n - t_i). \tag{40}$$

In case $d_f(\mathcal{C})$ attains this upper bound, we say that $\mathcal{C}$ is an *optimum distance flag code*. One of the most relevant results in the same paper is the next characterization for optimum distance flag codes.

(Chapter 1, Theorem 3.11) A flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ is an optimum distance flag code if, and only if, it is disjoint and every projected code $\mathcal{C}_i$ attains the maximum distance for dimension t_i.

At this point, motivated by this result and also by the optimality of spread codes among constant dimension codes with maximum distance, we consider the possibility of constructing optimum distance flag codes having a spread as a projected code. In such a case, by means of the disjointness condition, only one spread can appear as the projected code of an optimum distance flag code. Moreover, in Chapter 1 (see Theorem 3.12), we show that, whenever a divisor k of n appears in the type vector of an optimum distance flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$, then its cardinality must satisfy

$$|\mathcal{C}| \leqslant \frac{q^n - 1}{q^k - 1}$$

and the equality holds, if and only if, $\mathcal{C}$ has a k-spread as its projected code of dimension k. In other words, the excellent properties satisfied by spread codes are, in some sense, transferred to optimum distance flag codes when a k-spread arises as their projected code. This idea is also present in Chapters 1-3 and 6, where, for distinct situations and using different approaches, we provide constructions of optimum distance flag codes with a spread as a projected code. More precisely, in Chapter 1, we address the construction of such codes for the full type vector. We conclude that, if optimum distance full flag codes with a k-spread as a projected code exist, then it holds either $n = 2k$ or $k = 1$ and $n = 3$ (Chapter 1, Proposition 4.1). In the same work, we prove the existence of such codes in case that $n = 2k$, for every value of k, by giving a systematic construction of them with the k-spread $\mathcal{S}(2, k, P)$, defined in (29), as a projected code (see Chapter 1, Theorem 4.5), together with a decoding algorithm over the erasure channel.

The remaining situation, that is, $k = 1$ and $n = 3$, becomes a particular case of the study provided in Chapter 2, where we tackle the general problem of constructing optimum distance flag codes on $\mathbb{F}_q^n$ with a k-spread as a projected code for an arbitrary divisor k of n. To this end, we show (see Chapter 2, Theorem 3.3) that such codes could only be constructed for a set of *admissible type vectors*, i.e., those $(t_1, \dots, t_r)$ satisfying

$$k \in \{t_1, \dots, t_r\} \subseteq \{1, \dots, k, n-k, \dots, n-1\}. \tag{41}$$

This condition is, up to certain point, expected. Spread codes have the maximum possible distance and give the largest possible size. The price to pay for constructing them is a condition on the dimension: k must divide n. In our case, optimum distance flag codes with a k-spread as their projected code have the maximum possible distance and optimal cardinality. In this case, a condition on the type vector is demanded. The rest of the paper is dedicated to prove the existence of these codes by giving a systematic construction for every choice of the parameters. We do it gradually by using two essential arguments. On the one hand, the existence of **perfect matchings** on bipartite regular graphs allows us to give optimum distance flag codes of type $(1, n-1)$ with the spread of lines of $\mathbb{F}_q^n$ as a projected code (Theorem 3.6 in Chapter 2). On the other hand, we use **field reduction** techniques to suitably translate the previous construction into another one of type $(k, n-k)$ on $\mathbb{F}_q^n$, having a Desarguesian k-spread as its projected code (see Theorem 3.8). We finish that work by extending the last construction to the full admissible type vector, i.e., $(1, \dots, k, n-k, \dots, n-1)$ in Theorem 3.10 and, by suitably removing projected codes, to any other admissible type vector (Theorem 3.12). The key point in this work is the suitable translation of our problem into a classical and well studied question in Graph Theory. More precisely, the existence of perfect matchings on bipartite graphs allows us to link dimensions k and $n-k$, whenever $2k < n$.

In Chapters 3 and 6, we wonder if these optimum distance flag codes with a k-spread as a projected code could be constructed as orbits under the action of some subgroup of the general linear group $\mathrm{GL}(n, q)$ on the corresponding flag variety. This question naturally arises from the existence of some well-known orbit constructions for spreads. As before, we tackle this problem in two different works. We start by constructing optimum distance full flag codes on $\mathbb{F}_q^n$ with a k-spread as a projected code for the case $n = 2k$ in Chapter 3. To do this, our starting point is the planar k-spread $\mathcal{S}(2, k, P)$ given in (29). Recall that, in [69], the authors exhibit this spread as the orbit generated by the subspace $\mathcal{U} = \mathrm{rowsp}(I_k \mid 0_{k\times k})$ under the action of the subgroup

$$\mathbf{G} = \left\langle \left(\begin{array}{c|c} 0_{k\times k} & I_k \\ \hline I_k & 0_{k\times k} \end{array} \right), \left(\begin{array}{c|c} I_k & A \\ \hline 0_{k\times k} & I_k \end{array} \right) \;\middle|\; A \in \mathbb{F}_q[P] \right\rangle \tag{42}$$

of $\mathrm{GL}(n, q)$ (see (32)). We start Chapter 3 by showing that $\mathbf{G}$ does not generate optimum distance flag codes since their orbits are not even disjoint flag codes. Hence, we study conditions for an orbit flag code to be disjoint, obtaining that, in general, if we consider a flag $\mathcal{F} \in \mathcal{F}_q((t_1, \ldots, t_r), n)$ and an arbitrary subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$, the orbit flag code $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ is disjoint if, and only if, we have the equality of stabilizer subgroups

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_1) = \cdots = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_r). \tag{43}$$

Moreover, in Chapter 3, the reader can find the next result.

(Chapter 3, Proposition 3.7) Let $\mathcal{F}$ be a full flag on $\mathbb{F}_q^{2k}$ and $\mathbf{H}$, a subgroup of $\mathrm{GL}(2k, q)$ such that $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_k)$ has maximum distance. If $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_k) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$ for all $1 \leqslant i \leqslant 2k-1$, then $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ is an optimum distance flag code with size $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_k)|$.

Hence, a sufficient condition for a subgroup $\mathbf{H}$ of $\mathbf{G}$ to generate optimum distance full flag codes on $\mathbb{F}_q^{2k}$ is providing trivial stabilizer when acting on $\mathcal{U}$ (or any other subspace in $\mathcal{S}(2, k, P)$). In this situation, the maximum possible size would only be attained if that subgroup had order $\frac{q^n-1}{q^k-1}$. Then, with the goal of extracting a suitable subgroup $\mathbf{H}$ of $\mathbf{G}$ in mind, we study the group structure of $\mathbf{G}$ and show that it always contains an isomorphic copy of the special linear group $\mathrm{SL}(2, q^k)$ (Chapter 3, Proposition 4.2). Moreover, the existence of subgroups of $\mathrm{SL}(2, q^k)$ of the desired order, allows us to choose a subgroup $\mathbf{H}$ of $\mathbf{G}$, isomorphic to a Singer subgroup of $\mathrm{SL}(2, q^k)$, as our acting group. *Singer subgroups* of $\mathrm{GL}(n, q)$ are cyclic subgroups with the largest possible order, that is, $q^n - 1$. It is well-known that these groups act transitively on the Grassmannians of lines (and also) hyperplanes of $\mathbb{F}_q^n$. Moreover, in [18], the author shows that, if k divides n, there is exactly one orbit of a Singer subgroup of $\mathrm{GL}(n, q)$ that is a k-spread of $\mathbb{F}_q^n$. In our case, using the action of $\mathbf{H}$ allows us to obtain an orbit construction of optimum distance full flag codes on $\mathbb{F}_q^{2k}$ with the k-spread $\mathcal{S}(2, k, P)$ as their projected code whenever q is even (see Theorem 4.14 in Chapter 3). For odd values of q, we get another construction, given as the union of two different orbits under the action of $\mathbf{H}$ (Chapter 3, Proposition 4.15).

The natural continuation of this work is carried in Chapter 6, which contains two differentiated parts. The first one is devoted to study general optimum distance flag codes. There, a new characterization for them is provided in terms of, at most, two projected codes: those with the closest dimensions to the value $\frac{n}{2}$ both from left and right (see Theorem 4.8). We apply this result to the construction of optimum distance orbit flag codes of type $(1, \ldots, k, n-k, \ldots, n-1)$ on $\mathbb{F}_q^n$ with a k-spread as its projected code, for an arbitrary divisor k of n. In this case, the dimensions k and $n-k$ are the ones that play a key role in the construction. Moreover, we consider the action of a Singer subgroup of $\mathrm{GL}(n, q)$ on flags of this type and, by means of the uniqueness of subgroups of cyclic groups, we are able to characterize those subgroups providing optimum distance flag codes as their orbits in terms of their order (see Chapter 6, Theorem 5.1). For those cases in which the maximum cardinality is not obtained by a single orbit, we provide a sufficient condition to attain the largest possible size as the union of different orbits (Chapter 6, Theorem 5.2), just depending on the order of the acting subgroup. Recall that these constructions present the restriction on the type vector given in (41). However, when $k = 1$ and $n = 3$ or $n = 2k$, it allows us to give constructions of full type vector and, by using a suitable puncturing process, of any other type. The case $n = 2k$ was also covered with our work in Chapter 3. On the other hand, systematic constructions of orbit optimum distance full flag codes with the maximum cardinality for odd values of n were not known. In order to fill this gap, for $n = 2k + 1$, we use the action of a subgroup $\mathbf{G}$ of $\mathrm{GL}(n, q)$ isomorphic to a Singer subgroup of $\mathrm{GL}(k + 1, q)$ and a suitable full flag $\mathcal{F}$ to get an orbit construction of optimum distance full flag code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})$ with size $q^{k+1} - 1$ (Theorem 5.7), whereas, by means of [47, Proposition 2.4], the largest possible size in this case is $q^{k+1} + 1$. Thus, we discuss and characterize how to choose two extra full flags $\mathcal{F}', \mathcal{F}''$ so that the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}) \cup \{\mathcal{F}', \mathcal{F}''\}$ still has the maximum distance and optimal size (see Theorem 5.11 in Chapter 6).

Recall that the action of Singer subgroups of $\mathrm{GL}(n, q)$ on $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$ can be translated into the action of the multiplicative group $\mathbb{F}_{q^n}^*$ on $\mathbb{F}_q$-subspaces of the extension field $\mathbb{F}_{q^n}$ (see [25]). This is the approach that we follow in Chapter 4, where we work with flags on the field $\mathbb{F}_{q^n}$ and study theoretical features of those flag codes arising as orbits under the action of subgroups $\langle \beta \rangle$ of the Singer group $\mathbb{F}_{q^n}$, i.e., *β-cyclic orbit codes*. We do so by generalizing some concepts coming from [25] to the flag codes framework. One of them is the following one:

> Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ be a flag on $\mathbb{F}_{q^n}$. The largest subfield $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^n}$ over which all the subspaces of the flag are vector spaces is called *the best friend of $\mathcal{F}$*.

As it happens for subspaces, we prove that the best friend of a flag is closely related to its stabilizer under the action of $\mathbb{F}_{q^n}^*$. More precisely, if $\mathbb{F}_{q^m}$ is the best friend of a flag $\mathcal{F}$, then $\mathrm{Stab}_{\mathbb{F}_{q^n}^*}(\mathcal{F}) = \mathbb{F}_{q^m}^*$. Due to this fact, we use the best friend of a flag in order to give the cardinality and lower and upper bounds for the distance of those β-cyclic orbit flag codes that it generates. Moreover, we devote part of this work to study families of flag codes attaining the extreme values of the distance. We start with the so-called *β-Galois flag codes*, that is, those generated by flags given by sequences of nested subfields of $\mathbb{F}_{q^n}$, i.e., *Galois flags*, under the action of subgroups of $\mathbb{F}_{q^n}^*$. More precisely, if $1 \leqslant t_1 < t_2 < \cdots < t_r$ are divisors of n such that every t_i divides t_{i+1}, the Galois flag of type $(t_1, \ldots, t_r)$ is $\mathcal{F} = (\mathbb{F}_{q^{t_1}}, \ldots, \mathbb{F}_{q^{t_r}})$. In Chapter 4 (Theorem 4.14), we prove that the set of possible distances for the code $\mathrm{Orb}_\beta(\mathcal{F})$ is exactly

$$\{2t_1, 2(t_1 + t_2), \ldots, 2(t_1 + \cdots + t_r)\} \tag{44}$$

and characterize those subgroups $\langle\beta\rangle$ of $\mathbb{F}_{q^n}^*$ that give each of these distance values. In particular, when the acting group is $\mathbb{F}_{q^n}^*$, we obtain the minimum possible distance for codes with $\mathbb{F}_{q^{t_1}}$ as their best friend, which is $2t_1$. However, an interesting structure of nested spreads appears. More precisely, for every $1 \leqslant i \leqslant r$, the i-th projected code is the t_i-spread $\mathrm{Orb}_{\mathbb{F}_{q^n}^*}(\mathbb{F}_{q^{t_i}})$ given in (33) and all these spreads "dance to the tune" of the acting group $\mathbb{F}_{q^n}^*$. Moreover, we are able to establish a correspondence between distance values in (44) and subgroups $\langle\beta\rangle$ of $\mathbb{F}_{q^n}^*$. In particular, we also obtain constructions with distance $2(t_1 + \cdots + t_r)$, i.e., optimum distance flag codes, and connect this fact with the study of general optimum distance flag codes with a prescribed best friend. More precisely, we characterize those optimum distance flag codes arising as β-cyclic orbit flag codes for $\beta \in \mathbb{F}_{q^n}^*$ primitive (Chapter 4, Corollary 4.23) and give necessary conditions for the dimensions that can appear in the type vector for not necessarily primitive elements β (Chapter 4, Theorem 4.21). In particular, the constructions given in Chapters 3 and 6 can be translated into this general scenario by identifying the acting groups used there with (a subgroup of) $\mathbb{F}_{q^n}^*$. Recall that those constructions have a k-spread of $\mathbb{F}_q^n$ as a projected code for some divisor of n. In this framework, that property can be translated by suitably placing the subfield $\mathbb{F}_{q^k}$ as a subspace of the generating flag $\mathcal{F}$ (on the extension field) $\mathbb{F}_{q^n}$, with the ground field $\mathbb{F}_q$ as its best friend.

This idea of using subfields as subspaces of the generating flag is explored in more detail in Chapter 7. There, we introduce *generalized Galois flags* as flags on $\mathbb{F}_{q^n}$ with at least one but not all its subspaces being a subfield of $\mathbb{F}_{q^n}$. The maximal sequence of subfields in a generalized Galois flag is called its *underlying Galois flag*. In this work, we study how the presence of certain subfields affects the behaviour of cyclic orbit flag codes. In particular, we show that the underlying Galois flag determines a set of *potential values* for the distance (see Chapter 7, Theorem 3.31 and Definition 3.32). Distance values out of this set are automatically discarded. This fact makes us wonder how far the structure of the underlying Galois flag determines the properties of the generalized Galois flag code. More precisely, and motivated by Theorem 4.14 in Chapter 4, given a generalized Galois flag $\mathcal{F}$, we wonder if all these potential distance values can actually be attained by some code $\mathrm{Orb}_\beta(\mathcal{F})$, with $\beta \in \mathbb{F}_{q^n}^*$.

In order to put some light on this matter, we provide a systematic construction of generalized Galois flag codes for every prescribed underlying Galois flag. We do so by using a specific generalized Galois flag written in a regular form that allows us to control the parameters of the code. In case of considering $\mathbb{F}_{q^n}^*$ as the acting group, we obtain interesting examples of flag codes that can be easily decoded by using techniques provided in Chapter 5 and we determine the exact parameters of our codes. Moreover, we derive constructions of optimum distance flag codes (Corollaries 4.4 and 4.16). On the other hand, if we take a proper subgroup $\langle\beta\rangle$ of $\mathbb{F}_{q^n}^*$ as the acting group, we lose some control of our codes but still obtain a reduced range of possibilities for the minimum distance (Chapter 7, Theorem 4.15). These constructions, apart from being a miscellaneous battery of examples of generalized Galois flag codes, allow us to ensure that not every potential value of the distance (fixed the underlying Galois flag) can be obtained as the minimum distance of a generalized β-Galois flag code.

Recall that making connections between a flag code and its set of projected codes is the thread that runs through this whole dissertation. Up to now, we have focused mostly on optimum distance flag codes. As proved in Chapter 1, the property of attaining the maximum possible distance requires, in particular, the flag code to be disjoint. This concept of *disjointness* has been deeply studied in Chapters 5 and 8, giving rise to the notions of *consistency* (of a flag code w.r.t. its projected codes) and *M-disjoint flag codes*. Let us explain these works in detail.

First, in Chapter 5, we start from a disjoint flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$, i.e., a flag code with $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$. This condition represents a *cardinality-consistency* relationship for flag codes. On the other hand, we introduce the notion of *distance-consistency* for flag codes.

A flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$ is said to be *distance-consistent* if, for every pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, if holds

$$d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}') \iff d_S(\mathcal{C}_i) = d_f(\mathcal{F}_i, \mathcal{F}'_i), \ \forall i = 1, \dots, r. \tag{45}$$

Observe that this condition is equivalent to say that pairs of closest flags in a distance-consistent flag code are given by nested sequences of closest subspaces in the corresponding projected code. Moreover, as a consequence of (45), for a distance-consistent flag code $\mathcal{C}$, it clearly holds $d_f(\mathcal{C}) = \sum_{i=1}^r d_S(\mathcal{C}_i)$. However, this condition does not characterize this class of flag codes. In order to better control the parameters of flag codes, we introduce the class of *consistent flag codes* as those flag codes being both cardinality-consistent and distance-consistent. For these codes, both cardinality and distance are perfectly determined by the ones of its projected codes. More precisely, we characterize them as follows.

(Chapter 5, Theorem 1) A flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ is consistent if, and only if, the following statements hold:

(1) $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$ and

(2) $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$.

In particular, optimum distance flag codes are consistent flag codes where every projected code, in addition, has the maximum possible distance (Chapter 5, Corollary 3).

Apart from the cardinality and the minimum distance, we show that some structural properties are transferred from a consistent flag code to its projected codes and vice versa. Observe that using the projected codes of a flag code makes us able to generalize some concepts coming from the constant dimension codes framework to flag codes in two possible ways. In Chapter 5 we bring the notions of *equidistant* and *sunflower codes*, studied for subspace codes in [21, 29], to the flag codes setting.

(Chapter 5, Definitions 6, 7, 8 and 9) A flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ is said to be:

- *Equidistant* if, for every pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, it holds

$$d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}').$$

- *Projected-equidistant* if all its projected codes are equidistant constant dimension codes.

- A *sunflower flag code* if there exist nested subspaces $C_1 \subsetneq \cdots \subsetneq C_r$ such that, for every pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ and every $1 \leqslant i \leqslant r$, we have $\mathcal{F}_i \cap \mathcal{F}'_i = C_i$. In this case, the sequence $C = (C_1, \ldots, C_r)$ is called *the center of the sunflower*.

- A *projected-sunflower flag code* if every projected code $\mathcal{C}_i$ of $\mathcal{C}$ is a sunflower in $\mathcal{G}_q(t_i, n)$.

In general, there is not a clear relationship between the properties of being equidistant (resp. sunflower) and projected-equidistant (resp. projected-sunflower). However, under the consistency condition, we prove that they are equivalent, i.e., a consistent flag code is equidistant (resp. a sunflower) if, and only if, their projected codes are equidistant (resp. sunflower) constant dimension codes of the corresponding dimensions (Chapter 5, Theorems 2 and 3). In addition, the consistency condition is also exploited in Chapter 5 to provide a decoding algorithm over the erasure channel (see Algorithm 1), valid for arbitrary consistent flag codes. In particular, it can be applied to the construction given in Theorem 4.2 (Chapter 7) or any optimum distance flag code constructed in Chapters 1-4, 6 and 7.

Observe that when working with consistent flag codes –and, in particular, with optimum distance flag codes– the minimum distance of the flag code is always obtained in the same way, that is, by adding the distances of the projected codes. This behaviour is characteristic of consistent flag codes but, in general, any value of the flag distance might be obtained by many different combinations of subspace distances. This is the reason why, in Chapter 8, we present the notion of *distance vector*.

(Chapter 8, Definitions 3.1 and 3.6) The *distance vector* associated to a pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q((t_1, \dots, t_r), n)$ is

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') = (d_S(\mathcal{F}_1, \mathcal{F}'_1), \dots, d_S(\mathcal{F}_r, \mathcal{F}'_r)) \in 2\mathbb{Z}^r. \tag{46}$$

The *set of distance vectors of a flag code* $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$ contains those distance vectors associated to pairs of flags in $\mathcal{C}$ and giving the minimum distance of the code.

In particular, consistent flag codes can be seen as disjoint flag codes with the singleton set of distance vectors $\{(d_S(\mathcal{C}_1), \dots, d_S(\mathcal{C}_r))\}$ but general flag codes might have a set of distance vectors with more than one element.

In the same work, see Chapter 8 (Theorem 3.9), we also characterize distance vectors in terms of certain conditions satisfied by their components, i.e., we determine those sequences in $2\mathbb{Z}^r$ that truly represent possible realizations of the flag distance. Moreover, we devote part of the paper to determine which values of the flag distance can be obtained by distance vectors with a prescribed component. This can be particularized to the study of attainable values of the flag distance by distance vectors with a null component, that is, flag distances between flags with exactly one common subspace. Later on, we generalized our study to flag distances obtained with distance vectors of length r and $1 \leqslant M \leqslant r$ a components equal to zero, i.e., distances between flags with at least M identical subspaces.

In the second part of Chapter 8, we apply our analysis of the flag distance to derive properties of flag codes with a given minimum distance. In particular, we generalize the class of disjoint flag codes by using the following new family of projected codes. Given a type vector $(t_1, \dots, t_r)$, a positive integer $1 \leqslant M \leqslant r$ and ordered indices $1 \leqslant i_1 < \dots < i_M \leqslant r$, we consider the projection map

$$\begin{array}{rccc} p_{(i_1,\dots,i_M)} : & \mathcal{F}_q((t_1, \dots, t_r), n) & \longrightarrow & \mathcal{F}_q((t_{i_1}, \dots, t_{i_M}), n) \\ & (\mathcal{F}_1, \dots, \mathcal{F}_r) & \longmapsto & (\mathcal{F}_{t_{i_1}}, \dots, \mathcal{F}_{t_{i_M}}). \end{array} \tag{47}$$

(Chapter 8, Definition 5.2) If $\mathcal{C}$ is a flag code in $\mathcal{F}_q((t_1, \dots, t_r), n)$, then its $(i_1, \dots, i_M)$-*projected code* is the flag code

$$p_{(i_1,\dots,i_M)}(\mathcal{C}) = \{\ p_{(i_1,\dots,i_M)}(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\ \}. \tag{48}$$

Observe that, for every choice of indices $1 \leqslant i_1 < \dots < i_M$, it clearly holds $|p_{(i_1,\dots,i_M)}(\mathcal{C})| \leqslant |\mathcal{C}|$. Those flag codes where this inequality holds with equality have special interest for our study.

A flag code $\mathcal{C}$ is $(i_1, \dots, i_M)$-*disjoint* if the projection $p_{(i_1,\dots,i_M)}$ is injective when restricted to $\mathcal{C}$, i.e., if $|\mathcal{C}| = |p_{(i_1,\dots,i_M)}(\mathcal{C})|$. If this happens for every choice of M indices $1 \leqslant i_1 < \dots < i_M \leqslant r$, then we say that $\mathcal{C}$ is an M-*disjoint* flag code.

In these new terms, disjoint flag codes are 1-disjoint. Moreover, notice that distance vectors between different flags in an M-disjoint flag code cannot contain M null components. This fact is used in Chapter 8 in order to give a sufficient condition on the minimum distance of a flag code to ensure certain degree of disjointness (Chapter 8, Theorem 5.9). This result allows us to derive natural upper bounds for $A_q^f(n, d, (t_1, \dots, t_r))$ (the maximum possible size for flag codes in $\mathcal{F}_q((t_1, \dots, t_r), n)$ with minimum distance d) in terms of the maximum number of subspaces that different flags can share without compromising the minimum distance d. We do this for every choice of the parameters q, n, d and the type vector. These results can be found in Section 6 of Chapter 8.

This study of the flag distance parameter is carried from an algebraic point of view. On the other hand, in Chapter 9, the reader can find a combinatorial approach for the full flag distance. In this case, for technical reasons, we use the equivalent metric induced by the *injection distance* for full flags, defined as

$$d_I(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{n-1} (i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i)) = \frac{1}{2} d_f(\mathcal{F}, \mathcal{F}').$$

In this work, we introduce the notion of *distance path in the distance support* as a graphic representation of a distance vector defined as above.

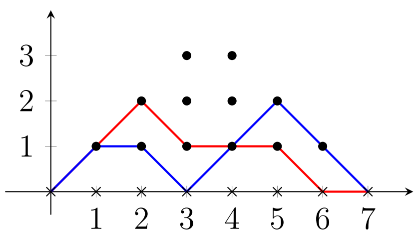

Figure 5: Examples of distance paths in the distance support for $n = 7$.

Using this perspective, we are able to compute flag distances by simply counting the number of circle points (in the distance support) in a given distance path or below it. Similarly, the complementary value of the distance –the *codistance*– coincides with the number of points over a distance path. After a suitable process of enrichment of the distance support and a rotation, we get the *Ferrers diagram frame* associated to the full flag variety on $\mathbb{F}_q^n$.

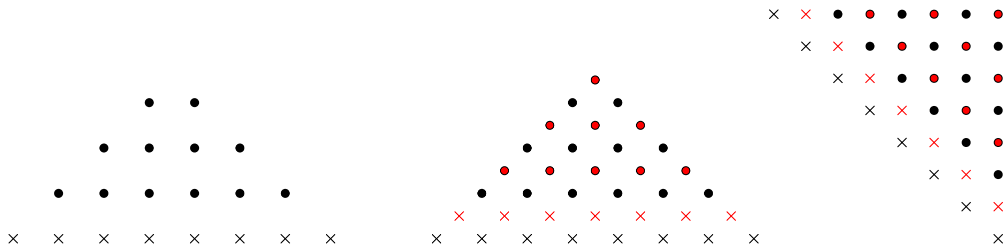

Figure 6: Enrichment and rotation of the distance support for $n = 7$.

We use this new object to establish a one-to-one correspondence between the set of distance paths (associated to a given value of the flag distance) and some elements coming form the Theory of partitions, related to the set of circle black points contained in certain *Ferrers subdiagram* (see Chapter 9, Theorem 4.26). Using this combinatorial viewpoint, we exhibit connections between the parameters of a given full flag code and the ones of its projected codes (of length one). More precisely, given a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ we associate to it a set of Ferrers diagrams and use their *Durfee rectangles* to obtain information about the distance and size of every projected code $\mathcal{C}_i$ and conversely (Chapter 9, Theorems 5.2, 5.6-5.8, 5.11 and Corollary 5.10). These new tools allow us to interpret some known results in combinatorial terms as, for instance, Theorem 5.17 in Chapter 9, which is a characterization of optimum distance flag codes by using the sets of distance paths or Ferrers subdiagrams associated to a flag code. With this result, which is the last theorem in this thesis but also the first one, we close our study on flag codes.

We finish this section with the following bird's-eye view of our work where the reader can see at a glance all the connections described through these pages.

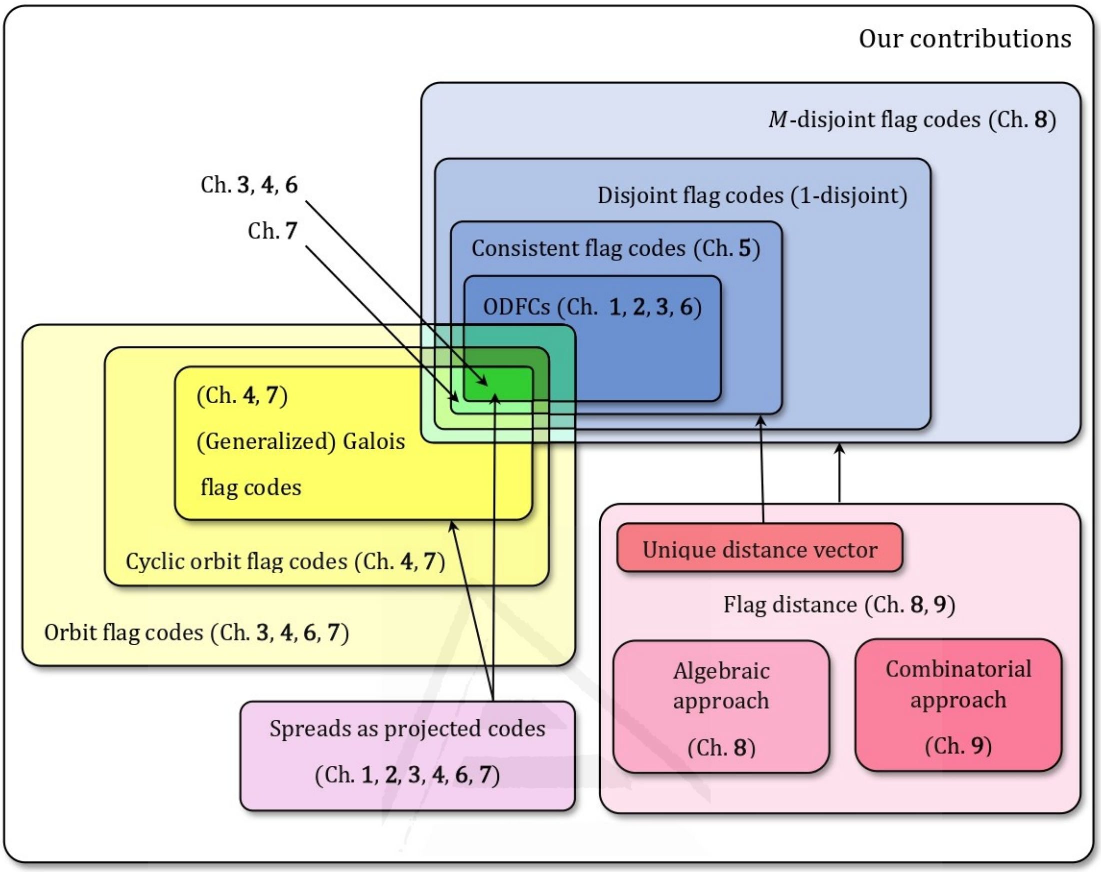

Figure 7: Our contributions to the study of flag codes

Part II

Published papers

CHAPTER 1

FLAG CODES FROM PLANAR SPREADS IN NETWORK CODING

C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà,
Flag Codes from Planar Spreads in Network Coding,
Finite Fields and Their Applications, Vol. 68 (2020), 101745.

Abstract:

In this paper we study a class of multishot network codes given by families of nested subspaces (flags) of a vector space $\mathbb{F}_q^n$, being q a prime power and $\mathbb{F}_q$ the finite field of q elements. In particular, we focus on flag codes having maximum distance (*optimum distance flag codes*). We explore the existence of these codes from spreads, based on the good properties of the latter ones. For $n = 2k$, we show that optimum distance full flag codes with the largest size are exactly those that can be constructed from a planar spread. We give a precise construction of them as well as a decoding algorithm.

https://doi.org/10.1016/j.ffa.2020.101745

CHAPTER 2

FLAG CODES FROM SPREADS VIA PERFECT MATCHINGS IN GRAPHS

C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà,
Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs,
Journal of Algebraic Combinatorics (2021).

Abstract:

In this paper, we study flag codes on the vector space $\mathbb{F}_q^n$, being q a prime power and $\mathbb{F}_q$ the finite field of q elements. More precisely, we focus on flag codes that attain the maximum possible distance (*optimum distance flag codes*) and can be obtained from a spread of $\mathbb{F}_q^n$. We characterize the set of admissible type vectors for this family of flag codes and also provide a construction of them based on well-known results about perfect matchings in graphs. This construction attains both the maximum distance for its type vector and the largest possible cardinality for that distance.

https://doi.org/10.1007/s10801-021-01086-y

CHAPTER 3

AN ORBITAL CONSTRUCTION OF OPTIMUM DISTANCE FLAG CODES

C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà,
An Orbital Construction of Optimum Distance Flag Codes,
Finite Fields and Their Applications, Vol. 73 (2021), 101861.

Abstract:

Flag codes are multishot network codes consisting of sequences of nested subspaces (flags) of a vector space $\mathbb{F}_q^n$, where q is a prime power and $\mathbb{F}_q$, the finite field of size q. In this paper we study the construction on $\mathbb{F}_q^{2k}$ of full flag codes having maximum distance (*optimum distance full flag codes*) that can be endowed with an orbital structure provided by the action of a subgroup of the general linear group. More precisely, starting from a subspace code of dimension k and maximum distance with a given orbital description, we provide sufficient conditions to get an optimum distance full flag code on $\mathbb{F}_q^{2k}$ having an orbital structure directly induced by the previous one. In particular, we exhibit a specific orbital construction with the best possible size from an orbital construction of a planar spread on $\mathbb{F}_q^{2k}$ that strongly depends on the characteristic of the field.

https://doi.org/10.1016/j.ffa.2021.101861

CHAPTER 4

CYCLIC ORBIT FLAG CODES

C. Alonso-González and M. A. Navarro-Pérez,
Cyclic Orbit Flag Codes,
Designs, Codes and Cryptography, Vol. 89 (2021), 2331–2356.

Abstract:

In network coding, a flag code is a set of sequences of nested subspaces of $\mathbb{F}_q^n$, being $\mathbb{F}_q$ the finite field with q elements. Flag codes defined as orbits of a cyclic subgroup of the general linear group acting on flags of $\mathbb{F}_q^n$ are called *cyclic orbit flag codes*. Inspired by the ideas in [25], we determine the cardinality of a cyclic orbit flag code and provide bounds for its distance with the help of the largest subfield over which all the subspaces of a flag are vector spaces (the *best friend* of the flag). Special attention is paid to two specific families of cyclic orbit flag codes attaining the extreme possible values of the distance: *Galois cyclic orbit flag codes* and *optimum distance cyclic orbit flag codes*. We study in detail both classes of codes and analyze the parameters of the respective subcodes that still have a cyclic orbital structure.

https://doi.org/10.1007/s10623-021-00920-5

CHAPTER 5

CONSISTENT FLAG CODES

C. Alonso-González and M. A. Navarro-Pérez,
Consistent Flag Codes,
Mathematics, Vol. 8(12) (2020), 2234.

Abstract:

In this paper we study flag codes on $\mathbb{F}_q^n$, being $\mathbb{F}_q$ the finite field with q elements. Special attention is given to the connection between the parameters and properties of a flag code and the ones of a family of constant dimension codes naturally associated to it (*the projected codes*). More precisely, we focus on *consistent flag codes*, that is, flag codes whose distance and size are completely determined by their projected codes. We explore some aspects of this family of codes and present examples of them by generalizing the concepts of *equidistant* and *sunflower* subspace code to the flag codes setting. Finally, we present a decoding algorithm for consistent flag codes that fully exploits the consistency condition.

https://doi.org/10.3390/math8122234

Part III

Other works

CHAPTER 6

FLAG CODES OF MAXIMUM DISTANCE AND CONSTRUCTIONS USING SINGER GROUPS

Joint work with Xaro Soler-Escrivà.

Abstract:

In this paper we study flag codes of maximum distance. We characterize these codes in terms of, at most, two relevant constant dimension codes naturally associated to them. We do this first for general flag codes and then particularize to those arising as orbits under the action of arbitrary subgroups of the general linear group. We provide two different systematic orbital constructions of flag codes attaining both maximum distance and size. To this end, we use the action of Singer groups and take advantage of the good relation between these groups and Desarguesian spreads, as well as the fact that they act transitively on lines and hyperplanes.
Keywords: Network coding, flag codes, orbit codes, Desarguesian spreads, Singer group actions.

1 Introduction

Network Coding appeared in [1] as a method for maximize the information rate of a network modelled as an acyclic directed multigraph with possibly several senders and receivers. Afterwards, Koetter and Kchischang stated in [16] an algebraic approach for coding in non-coherent networks (*Random Network Coding*). In this setting, *subspace codes* stand as the most appropriate codes for error correction. Given a finite field $\mathbb{F}_q$, a subspace code is just a set of subspaces of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$, for a positive integer n. Since their definition in [16], research works on the structure, construction and decoding methods of this type of codes have proliferated considerably, e.g. in [9, 10, 15, 21, 28]. In many of these articles, all the codewords are vector subspaces having the same dimension, in which case we speak about *constant dimension codes*. Among them, *spread codes*, that is, constant dimension codes which partition the ambient space, appear as a remarkable subfamily, since they have the best distance and the largest size for that distance. Besides, a particular way of constructing constant dimension codes is based on the natural and transitive action of the general linear group, $\mathrm{GL}(n,q)$, on the Grassmannian $\mathcal{G}_q(k,n)$, which is the set of all k-dimensional vector subspaces of $\mathbb{F}_q^n$ over $\mathbb{F}_q$. In this context, an *orbit code* is a constant dimension code which is the orbit under the action of some subgroup of $\mathrm{GL}(n,q)$ acting on $\mathcal{G}_q(k,n)$. When the acting group is cyclic, we call them *cyclic orbit codes*. First studied in [27], orbit codes have a rich mathematical structure due to the group action point of view. Concerning cyclic orbit codes, *Singer groups*, that is, cyclic subgroups of $\mathrm{GL}(n,q)$ of order q^n-1, play a very important role. The structure of these groups, as well as their multiple mathematical properties, have permitted to obtain relevant information about the orbit codes they generate (see [8, 13, 22, 24, 26], for instance).

Flag codes can be seen as a generalization of constant dimension codes. In this case, the codewords are flags on $\mathbb{F}_q^n$, that is, tuples of nested vector subspaces of prescribed dimensions. The use of these codes is particularly interesting when there are limitations on the size q of the field or the length n of the information packets to be transmitted [23]. The use of flags in the Network Coding setting began with the work [20] by Liebhold *et al.* and has continued with several articles that have extended and deepened this line of research [4, 5, 17]. The action of the general linear group on the Grassmannian can be easily extended to any variety of flags. Consequently, it also makes sense to study those flag codes arising as orbits of some relevant groups (see [2, 3, 20]). The paper at hand is also involved in this research. Concretely, we deal with flag codes of maximum distance (*optimum distance flag codes*), with special emphasis on those having an orbital structure under the action of a suitable Singer group.

The paper is structured as follows. In Section 2 we give all the background we need on finite fields, constant dimension codes and Singer group actions. Section 3 is devoted to highlighting the well-known relationship between Desarguesian spreads and Singer groups, which will be very important for the subsequent derivation of our flag codes. In Section 4 we start with a summary of known results on flag codes, after which we delve into optimum distance flag codes. In Theorem 4.8 we characterize these flag codes in terms of at most two constant dimension codes, improving considerably on the, up to now, only known result in this respect (see [4]). Moreover, we give several characterizations of optimum distance flag codes when arising as orbits of the action of an arbitrary subgroup of the general linear group (Theorem 4.11). In Section 5, we use the action of appropriate Singer groups and the theoretical results previously obtained, in order to provide two systematic orbital constructions of optimum distance flag codes having the best cardinality. First, in Section 5.1, we give a construction starting from a Desarguesian spread of $\mathbb{F}_q^n$ and the results set out in Section 3. Since this kind of construction entails a restriction on the type vector of the flags (see [5]), we complete our research in Section 5.2, where we deal with the construction of orbit flag codes of full type having maximum distance and size.

2 Preliminaries

2.1 Finite fields

We begin this section by recalling some definitions and results that can be found in any textbook on finite fields (see [19], for instance).

Let $\mathbb{F}_q$ denote the finite field of q elements, for a prime power q. Given a positive integer k, we put $\mathbb{F}_q^{k\times k}$ for the set of all $k \times k$ matrices with entries in $\mathbb{F}_q$ and $\mathrm{GL}(k, q)$ for the *general linear group* of degree k over $\mathbb{F}_q$, composed by all invertible matrices in $\mathbb{F}_q^{k\times k}$. A *primitive element* ω of the field $\mathbb{F}_{q^k}$ is just a generator of the cyclic group $\mathbb{F}_{q^k}^*$, which has order $q^k - 1$. Let $p(x) = x^k + \sum_{i=0}^{k-1} p_i x^i \in \mathbb{F}_q[x]$ be the minimal polynomial of ω over $\mathbb{F}_q$. It turns out that

$p(x)$ is the characteristic polynomial of the matrix

$$M_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{k-1} \end{pmatrix} \in \mathrm{GL}(k, q),$$

which is called the *companion matrix* of $p(x)$. In particular, M_k can be seen as a root of $p(x)$ and the finite field $\mathbb{F}_{q^k}$ can be realized as $\mathbb{F}_{q^k} \cong \mathbb{F}_q[\omega] \cong \mathbb{F}_q[M_k]$, where the last field isomorphism is given by:

$$\begin{array}{cccc} \phi : & \mathbb{F}_q[\omega] & \longrightarrow & \mathbb{F}_q\left[M_k\right] \\ & \sum_{i=0}^{k-1} a_i\omega^i & \longmapsto & \sum_{i=0}^{k-1} a_i M_k^i, \end{array} \tag{6.1}$$

Consequently, the multiplicative order of M_k is $q^k - 1$, that is, M_k generates a cyclic subgroup $\langle M_k \rangle$ of order $q^k - 1$ in $\mathrm{GL}(k, q)$. Equivalently, M_k is a primitive element of the finite field $\mathbb{F}_q[M_k] \subseteq \mathbb{F}_q^{k\times k}$.

For any positive integer n and $i \in \{0, \ldots, n\}$, we denote by $\mathcal{G}_q(i, n)$ the set of all i-dimensional vector subspaces of $\mathbb{F}_q^n$ over $\mathbb{F}_q$, which is called the *Grassmann variety* (or simply the *Grassmannian*). For any positive integer $s \geqslant 2$, the field isomorphism ϕ is useful to map vector subspaces of $\mathbb{F}_{q^k}^s$ into vector subspaces of $\mathbb{F}_q^{ks}$. For $m \in \{1, \ldots, s\}$, one has the following embedding map

$$\begin{array}{cccc} \varphi : & \mathcal{G}_{q^k}(m, s) & \longrightarrow & \mathcal{G}_q(km, ks) \\ & \mathrm{rowsp}\begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{ms} \end{pmatrix} & \longmapsto & \mathrm{rowsp}\left(\begin{array}{c|c|c} \phi(a_{11}) & \cdots & \phi(a_{1s}) \\ \hline \vdots & \ddots & \vdots \\ \hline \phi(a_{m1}) & \cdots & \phi(a_{ms}) \end{array}\right), \end{array} \tag{6.2}$$

which is called a *field reduction map*. In particular, $\varphi(\mathcal{U}) \cap \varphi(\mathcal{V}) = \varphi(\mathcal{U} \cap \mathcal{V})$, for all $\mathcal{U}, \mathcal{V} \in \mathcal{G}_{q^k}(m, s)$, since φ is an injective map.

Besides, we can also use ϕ to obtain the following group monomorphism (see [29, Th. 2.4]):

$$\begin{array}{cccc} \psi : & \mathrm{GL}(s, q^k) & \longrightarrow & \mathrm{GL}(ks, q) \\ & \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix} & \longmapsto & \left(\begin{array}{c|c|c} \phi(a_{11}) & \cdots & \phi(a_{1s}) \\ \hline \vdots & \ddots & \vdots \\ \hline \phi(a_{s1}) & \cdots & \phi(a_{ss}) \end{array}\right). \end{array} \tag{6.3}$$

2.2 Subspace codes, orbit codes and Singer group actions

For any integers $1 \leqslant k < n$, the Grassmannian $\mathcal{G}_q(k, n)$ can be seen as a metric space endowed with the following *subspace distance* (see [16]):

$$d_S(\mathcal{U},\mathcal{V}) = \dim(\mathcal{U}+\mathcal{V}) - \dim(\mathcal{U}\cap\mathcal{V}) = 2(k - \dim(\mathcal{U}\cap\mathcal{V})), \tag{6.4}$$

for all $\mathcal{U},\mathcal{V} \in \mathcal{G}_q(k,n)$. Using this metric, a *constant dimension code* is just a nonempty subset $\mathcal{C}$ of $\mathcal{G}_q(k,n)$ and its *minimum distance* is defined as

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U},\mathcal{V}) \mid \mathcal{U},\mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\} \leqslant \begin{cases} 2k & \text{if } \ 2k \leqslant n, \\ 2(n-k) & \text{if } \ 2k \geqslant n. \end{cases} \tag{6.5}$$

In case that $|\mathcal{C}| = 1$, we put $d_S(\mathcal{C}) = 0$. In any other case, $d_S(\mathcal{C})$ is always a positive even integer. When the upper bound provided by (6.5) is attained we say that $\mathcal{C}$ is a *constant dimension code of maximum distance*.

Notice that the distance $d_S(\mathcal{C}) = 2k$ can only be attained if $2k \leqslant n$ and different subspaces in $\mathcal{C}$ pairwise intersect trivially. This class of codes of maximum distance were introduced in [10] as *partial spread codes* since they generalize the class of *spread codes*, previously studied in [21]. A spread code is just a spread in the geometrical sense, that is, its elements pairwise intersect trivially and cover the whole space $\mathbb{F}_q^n$ (see [12]). The size of a partial spread code of dimension k (or k-partial spread code) is always upper bounded by

$$\frac{q^n - q^r}{q^k - 1}, \tag{6.6}$$

where r is the reminder obtained dividing n by k (see [10]). In turn, k-spreads exist if, and only if, k divides n and, in this case, they have cardinality $(q^n - 1)/(q^k - 1)$, which is the largest size among k-partial spreads of $\mathbb{F}_q^n$.

On the other hand, if $2k \geqslant n$ and $\mathcal{C} \subseteq \mathcal{G}_q(k,n)$, then the *dual code* of $\mathcal{C}$ is the set $\mathcal{C}^\perp = \{\mathcal{V}^\perp \mid \mathcal{V} \in \mathcal{C}\}$. It is a constant dimension code of dimension $n-k$ with the same cardinality and distance than $\mathcal{C}$ (see [16]). In particular, if $d_S(\mathcal{C}) = 2(n-k)$, then $\mathcal{C}^\perp$ is an $(n-k)$-partial spread code and the size of $\mathcal{C}$ can be also upper bounded in terms of (6.6).

Notice that any code $\mathcal{C}$ included in the Grassmannian of lines $\mathcal{G}_q(1,n)$ or in the Grassmannian of hyperplanes $\mathcal{G}_q(n-1,n)$, with $|\mathcal{C}| \geqslant 2$, is a constant dimension code of maximum distance and its size is upper bounded by $(q^n-1)/(q-1)$.

Orbit codes were introduced in [27] as constant dimension codes arising as orbits under the action of some subgroup of the general linear group. Given a subspace $\mathcal{V} \in \mathcal{G}_q(k,n)$ and a full-rank matrix $V \in \mathbb{F}_q^{k\times n}$ generating $\mathcal{V}$, that is, $\mathcal{V} = \mathrm{rowsp}(V)$, the map

$$\begin{array}{ccc} \mathcal{G}_q(k,n) \times \mathrm{GL}(n,q) & \longrightarrow & \mathcal{G}_q(k,n) \\ (\mathcal{V}, A) & \longmapsto & \mathcal{V}\cdot A = \mathrm{rowsp}(VA), \end{array}$$

is independent from the choice of V and it defines a group action on $\mathcal{G}_q(k,n)$ (see [27]). For a subgroup $\mathbf{H}$ of $\mathrm{GL}(n,q)$, the *orbit code* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{V})$ is just:

$$\mathrm{Orb}_{\mathbf{H}}(\mathcal{V}) = \{\mathcal{V}\cdot A \mid A \in \mathbf{H}\} \subseteq \mathcal{G}_q(k,n).$$

The size of an orbit code can be computed as $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{V})| = \frac{|\mathbf{H}|}{|\mathrm{Stab}_{\mathbf{H}}(\mathcal{V})|}$, where $\mathrm{Stab}_{\mathbf{H}}(\mathcal{V}) = \{A \in \mathbf{H} \mid \mathcal{V} \cdot A = \mathcal{V}\}$ is the stabilizer subgroup of the subspace $\mathcal{V}$ under the action of $\mathbf{H}$. If $\mathbf{H} = \mathrm{Stab}_{\mathbf{H}}(\mathcal{V})$, then $\mathrm{Orb}_{\mathbf{H}}(\mathcal{V}) = \{\mathcal{V}\}$ and $d_S(\mathrm{Orb}_{\mathbf{H}}(\mathcal{V})) = 0$. In any other case, the minimum distance of the orbit code $\mathrm{Orb}_{\mathbf{H}}(\mathcal{V})$ can be computed as (see [27]) :

$$d_S(\mathrm{Orb}_{\mathbf{H}}(\mathcal{V})) = \min\{d_S(\mathcal{V}, \mathcal{V} \cdot A) \mid A \in \mathbf{H} \setminus \mathrm{Stab}_{\mathbf{H}}(\mathcal{V})\}.$$

When the acting group is cyclic, the corresponding orbit codes are called *cyclic orbit codes*. These particular type of orbit codes have been deeply studied in several papers (see [8, 22, 24, 26] for instance).

In this paper we will use the action of *Singer cyclic subgroups* of $\mathrm{GL}(n, q)$, which are generated by the so called *Singer cycles* of $\mathrm{GL}(n, q)$. These are elements of $\mathrm{GL}(n, q)$ having order $q^n - 1$, which is the largest element order in $\mathrm{GL}(n, q)$. Although Singer cycles are not necessarily conjugate, the generated subgroups are always conjugate subgroups of $\mathrm{GL}(n, q)$ (see [11, 14] for more information on Singer groups).

The following seminal result about the action of a Singer group on the Grassmannian of lines and the Grassmannian of hyperplanes is due to Singer (1938) and will be used extensively throughout this article:

Theorem 2.1. *[6, Th. 6.2] Any Singer cyclic subgroup* $\mathbf{S}$ *of* $\mathrm{GL}(n, q)$ *acts transitively on both* $\mathcal{G}_q(1, n)$ *and* $\mathcal{G}_q(n-1, n)$*. Moreover, for any* $l \in \mathcal{G}_q(1, n)$ *and any* $h \in \mathcal{G}_q(n-1, n)$*, it holds*

$$\mathrm{Stab}_{\mathbf{S}}(l) = \mathrm{Stab}_{\mathbf{S}}(h) = \{aI_s \mid a \in \mathbb{F}_q^*\},$$

which is the unique cyclic subgroup of $\mathbf{S}$ *of order* $q - 1$*.*

3 Desarguesian spread codes and Singer groups

In this section we focus on the action of Singer groups in order to obtain certain k-spreads of $\mathbb{F}_q^n$ as their orbits, for $n = ks$ and $s \geqslant 2$. Later on, in Section 5.1, we will use the results obtained here to construct orbit flag codes by considering Singer groups and their subgroups.

Consider the field reduction map φ defined in (6.2) which maps vector subspaces of $\mathbb{F}_{q^k}^s$ into vector subspaces of $\mathbb{F}_q^n$. Applying φ to a constant dimension code $\mathcal{C} \subseteq \mathcal{G}_{q^k}(m, s)$, we obtain another constant dimension code $\varphi(\mathcal{C}) \subseteq \mathcal{G}_q(km, n)$. Since φ is injective, it preserves intersections and therefore it follows that $d_S(\varphi(\mathcal{C})) = kd_S(\mathcal{C})$. In particular, if $\mathcal{C}$ attains the maximum possible distance, then $\varphi(\mathcal{C})$ do it as well. Even more, if $\mathcal{C}$ is a m-spread code of $\mathbb{F}_{q^k}^s$, then $\varphi(\mathcal{C})$ is a km-spread code of $\mathbb{F}_q^n$.

We will use two constant dimension codes of $\mathbb{F}_q^n$ constructed in this way. First, from the spread of lines of $\mathbb{F}_{q^k}^s$, we consider

$$\mathcal{S} = \varphi(\mathcal{G}_{q^k}(1, s)) \subseteq \mathcal{G}_q(k, n), \tag{6.7}$$

which is a k-spread of $\mathbb{F}_q^n$. Originally due to Segre (see [25]), in the network coding setting, this construction appears for the first time in [21]. Secondly, from the Grassmannian of hyperplanes of $\mathbb{F}_{q^k}^s$, we obtain

$$\mathcal{H} = \varphi(\mathcal{G}_{q^k}(s-1, s)) \subseteq \mathcal{G}_q(n-k, n), \tag{6.8}$$

which is a constant dimension code of $\mathbb{F}_q^n$ with maximum distance.

Notice that the field reduction map φ together with the group monomorphism ψ defined in (6.3) make it possible to establish the following relation between the group action of $\mathrm{GL}(s, q^k)$ on $\mathcal{G}_{q^k}(m, s)$ and the group action of $\mathrm{GL}(n, q)$ on $\mathcal{G}_q(km, n)$:

$$\varphi(\mathcal{V} \cdot A) = \varphi(\mathcal{V}) \cdot \psi(A), \tag{6.9}$$

for all $\mathcal{V} \in \mathcal{G}_{q^k}(m, s)$ and $A \in \mathrm{GL}(s, q^k)$. In particular, we will use this equality to relate the respective actions of two Singer cyclic subgroups in which we are very interested.

Let α be a primitive element of $\mathbb{F}_{q^n}$. Recall that, given the minimal polynomial of α over $\mathbb{F}_{q^k}$ and its companion matrix, $M_s \in \mathrm{GL}(s, q^k)$, then $\mathbb{F}_{q^n} \cong \mathbb{F}_{q^k}[\alpha] \cong \mathbb{F}_{q^k}[M_s]$. Therefore, the multiplicative order of M_s is $q^n - 1$ and $\mathbb{F}_{q^k}[M_s] = \{0_{s\times s}\} \cup \langle M_s \rangle \subseteq \mathbb{F}_{q^k}^{s\times s}$. In particular, M_s is a Singer cycle of $\mathrm{GL}(s, q^k)$, generating the Singer cyclic subgroup $\langle M_s \rangle$ of $\mathrm{GL}(s, q^k)$. Furthermore, $\psi(\langle M_s \rangle) = \langle \psi(M_s) \rangle$ is a Singer cyclic subgroup of $\mathrm{GL}(n, q)$. These two Singer groups will be crucial in Section 5.1.

Coming back to the spread $\mathcal{S} = \varphi(\mathcal{G}_{q^k}(1, s))$ defined in (6.7), let us denote $\mathcal{G}_{q^k}(1, s) = \{l_1, l_2, \dots, l_r\}$ and $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r\}$, where $r = \frac{q^n-1}{q^k-1}$ and $\mathcal{S}_i = \varphi(l_i)$ for all $i = 1, \dots, r$. In accordance with (6.9) and Theorem 2.1, for every $i \in \{1, \dots, r\}$, we can write

$$\begin{aligned} \mathcal{S} &= \varphi(\mathcal{G}_{q^k}(1, s)) = \varphi(\mathrm{Orb}_{\langle M_s \rangle}(l_i)) = \{\varphi(l_i \cdot A) \mid A \in \langle M_s \rangle\} \\ &= \{\mathcal{S}_i \cdot \psi(A) \mid \psi(A) \in \langle \psi(M_s) \rangle\} = \mathrm{Orb}_{\langle \psi(M_s) \rangle}(\mathcal{S}_i). \end{aligned} \tag{6.10}$$

In an analogous way, we denote $\mathcal{G}_{q^k}(s-1, s) = \{h_1, h_2, \dots, h_r\}$ and the code $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_r\}$, where $\mathcal{H}_i = \varphi(h_i)$, for all $i = 1, \dots, r$. Then, for every $h_i \in \mathcal{G}_{q^k}(s-1, s)$, we obtain that

$$\mathcal{H} = \varphi(\mathcal{G}_{q^k}(s-1, s)) = \varphi(\mathrm{Orb}_{\langle M_s \rangle}(h_i)) = \mathrm{Orb}_{\langle \psi(M_s) \rangle}(\mathcal{H}_i). \tag{6.11}$$

That is, the transitive action of $\langle M_s \rangle$ on the lines and hyperplanes of $\mathbb{F}_{q^k}^s$ is translated to the transitive action of $\langle \psi(M_s) \rangle$ on the constant dimension codes of maximum distance $\mathcal{S}$ and $\mathcal{H}$. Moreover, from Theorem 2.1 we also obtain that, for any $\mathcal{S}_i \in \mathcal{S}$ and $\mathcal{H}_i \in \mathcal{H}$ it holds

$$\begin{aligned}\mathrm{Stab}_{\langle\psi(M_s)\rangle}(\mathcal{S}_i) &= \mathrm{Stab}_{\langle\psi(M_s)\rangle}(\mathcal{H}_i) = \psi\left(\mathrm{Stab}_{\langle M_s\rangle}(l_i)\right) = \{\psi(aI_s) \mid a \in \mathbb{F}_{q^k}^*\} \\ &= \left\{ \left(\begin{array}{c|c|c} \phi(a) & \cdots & 0_{k\times k} \\ \hline \vdots & \ddots & \vdots \\ \hline 0_{k\times k} & \cdots & \phi(a) \end{array} \right) \;\middle|\; a \in \mathbb{F}_{q^k}^* \right\}, \end{aligned} \tag{6.12}$$

which has order $q^k - 1$ (see also [29, Lemma 3.5]).

Remark 3.1. Following [18] and [29, Cor. 3.8], we can say that a k-spread $\mathcal{D}$ of $\mathbb{F}_q^n$ is *Desarguesian* if it is $\mathrm{GL}(n,q)$-equivalent to $\mathcal{S}$, that is, if there exists $B \in \mathrm{GL}(n,q)$ such that

$$\mathcal{D} = \mathcal{S} \cdot B = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r\} \cdot B = \{\mathcal{S}_1 \cdot B, \mathcal{S}_2 \cdot B, \ldots, \mathcal{S}_r \cdot B\}.$$

From [7, 29], we know that, given a Singer cyclic subgroup of $\mathrm{GL}(n,q)$, there exists a unique k-spread of $\mathbb{F}_q^n$ which appears as its orbit. Moreover this k-spread is Desarguesian. Consequently, $\mathcal{S}$ is the unique k-spread of $\mathbb{F}_q^n$ which arises as an orbit of the Singer cyclic subgroup $\langle\psi(M_s)\rangle$. Besides, given another Desarguesian k-spread $\mathcal{D}$ of $\mathbb{F}_q^n$ and $B \in \mathrm{GL}(n,q)$ such that $\mathcal{S} \cdot B = \mathcal{D}$, it follows that

$$\begin{aligned}\mathcal{D} &= \mathcal{S} \cdot B = \mathrm{Orb}_{\langle\psi(M_s)\rangle}(\mathcal{S}_i) \cdot B \\ &= \{\mathcal{S}_i \cdot A \mid A \in \langle\psi(M_s)\rangle\} \cdot B \\ &= \{\mathcal{S}_i \cdot AB \mid A \in \langle\psi(M_s)\rangle\} \\ &= \{(\mathcal{S}_i \cdot B) \cdot B^{-1}AB \mid A \in \langle\psi(M_s)\rangle\} \\ &= \mathrm{Orb}_{\langle\psi(M_s)\rangle^B}(\mathcal{S}_i \cdot B), \end{aligned} \tag{6.13}$$

that is, the k-spread $\mathcal{D}$ appears as the orbit under the action of the Singer cyclic subgroup $\langle\psi(M_s)\rangle^B = B^{-1}\langle\psi(M_s)\rangle B$. In Section 5.1, we will construct flag codes with maximum distance and an orbital structure. To do so, we will make use of the k-spread $\mathcal{S}$ defined in (6.7) and its orbital structure under the action of the Singer group $\langle\psi(M_s)\rangle$. By virtue of (6.13), to work with any other Desarguesian k-spread $\mathcal{D}$, it would be enough to consider the group $\langle\psi(M_s)\rangle^B$ instead of $\langle\psi(M_s)\rangle$.

4 On Flag codes

The present section is devoted to a theoretical study of flag codes. We start in Section 4.1 with a revision of some known results. Next, in Section 4.2 we focus on flag codes attaining the maximum possible distance and give a characterization of them that considerably improves the one obtained in [4]. We finish the section by studying how to construct these flag codes as orbits (or union of orbits) of arbitrary subgroups of the general linear group.

4.1 Background of flag codes

Given integers $0 < t_1 < \cdots < t_r < n$, a *flag* $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a sequence of nested subspaces of $\mathbb{F}_q^n$,

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n,$$

with $\mathcal{F}_i \in \mathcal{G}_q(t_i, n)$, for all $i = 1, \ldots, r$. We say that $\mathcal{F}_i$ is the *i-th subspace* of the flag $\mathcal{F}$ and when the type vector is $(1, 2, \ldots, n-1)$ we speak about *full flags.*

The set of all flags of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is known as the *flag variety* of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ and will be denoted here by $\mathcal{F}_q((t_1, \ldots, t_r), n)$.

The subspace distance defined in (6.4) for the Grassmann variety can be naturally extended to $\mathcal{F}_q((t_1, \ldots, t_r), n)$ as follows. Given $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ two flags in $\mathcal{F}_q((t_1, \ldots, t_r), n)$, their *flag distance* is

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i).$$

The use of flags in the network coding setting appears for the first time in [20]. Since then, several papers on this subject have recently appeared (see, for instance, [2, 3, 4, 5, 17]). If $\emptyset \neq \mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$, we say that $\mathcal{C}$ is a *flag code of type* $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. The *minimum distance* of $\mathcal{C}$ is given by

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}.$$

As for subspace codes, if $|\mathcal{C}| = 1$, we put $d_f(\mathcal{C}) = 0$. Notice that $d_f(\mathcal{C})$ is upper bounded by (see [4])

$$d_f(\mathcal{C}) \leqslant 2\left(\sum_{2t_i \leqslant n} t_i + \sum_{2t_i > n} (n - t_i)\right). \tag{6.14}$$

For any $i \in \{1, \ldots, r\}$, the *i-projected code* $\mathcal{C}_i$ of $\mathcal{C}$ is defined in [4] as the constant dimension code

$$\mathcal{C}_i = \{\mathcal{F}_i \mid (\mathcal{F}_1, \ldots, \mathcal{F}_r) \in \mathcal{C}\} \subseteq \mathcal{G}_q(t_i, n).$$

Observe that $|\mathcal{C}_i| \leqslant |\mathcal{C}|$, for every $1 \leqslant i \leqslant r$. When we have the equalities $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$, the flag code $\mathcal{C}$ is said to be *disjoint* (see [4]). In the same paper, *optimum distance flag codes* (ODFCs, for short) are defined as flag codes attaining the upper bound given in (6.14). The close relationship between a flag code $\mathcal{C}$ and its projected codes can be used to characterize this class of flag codes by using the concept of disjointness.

Theorem 4.1. *[4, Th. 3.11] Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$. The following statements are equivalent:*

(i) $\mathcal{C}$ is an ODFC.

(ii) $\mathcal{C}$ *is a disjoint flag code and every projected code* $\mathcal{C}_i$ *is a constant dimension code of maximum distance.*

As a result, the size of an ODFC can be upper bounded in terms of the upper bound given by (6.6) of Section 2.2.

Theorem 4.2. *[4, Th. 3.12] Let* k *be a divisor of* n *and assume that* $\mathcal{C}$ *is an ODFC of type* $(t_1, \ldots, t_r)$ *on* $\mathbb{F}_q^n$*. If* k *is a dimension in the type vector, say* $t_i = k$*, then* $|\mathcal{C}| \leqslant \frac{q^n-1}{q^k-1}$*. Equality holds if, and only if, the projected code* $\mathcal{C}_i$ *is a* k*-spread of* $\mathbb{F}_q^n$*.*

On the other hand, requiring an ODFC to have a k-spread as one of its projected codes leads to a condition on its type vector.

Theorem 4.3. *[5, Th. 3.3] Let* $\mathcal{C}$ *be an ODFC of type* $(t_1, \ldots, t_r)$ *on* $\mathbb{F}_q^n$*. Assume that some dimension* $t_i = k$ *divides* n *and the associated projected code* $\mathcal{C}_i$ *is a* k*-spread. Then, for each* $j \in \{1, \ldots, r\}$*, either* $t_j \leqslant k$ *or* $t_j \geqslant n-k$ *holds.*

Consequently, when $n = ks$ and $s \geqslant 2$, the *full admissible type vector* for an ODFC having a k-spread as a projected code is $(1, \ldots, k, n-k, \ldots, n-1)$. We will come back to this situation in Section 5.1. Besides, notice that for $s = 2$ the full admissible type vector is just the full type vector. A construction of ODFCs with the largest possible size in this particular case, together with a decoding algorithm, can be found in [4].

The action of the general linear group on the Grassmannian seen in Section 2 can be naturally extended to flags as follows. This approach already appears in [2, 3, 20]. Given a flag $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r) \in \mathcal{F}_q((t_1, \ldots, t_r), n)$ and a subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$, the *orbit flag code* generated by $\mathcal{F}$ under the action of $\mathbf{H}$ is

$$\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) = \{\mathcal{F} \cdot A \mid A \in \mathbf{H}\} = \{(\mathcal{F}_1 \cdot A, \ldots, \mathcal{F}_r \cdot A) \mid A \in \mathbf{H}\} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n).$$

Its associated stabilizer subgroup is $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \{A \in \mathbf{H} \mid \mathcal{F} \cdot A = \mathcal{F}\}$ and it holds $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| = \frac{|\mathbf{H}|}{|\mathrm{Stab}_{\mathbf{H}}(\mathcal{F})|}$. Moreover, the minimum distance of the orbit flag code can be obtained as

$$d_f(\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})) = \min\{d_f(\mathcal{F}, \mathcal{F} \cdot A) \mid A \in \mathbf{H} \setminus \mathrm{Stab}_{\mathbf{H}}(\mathcal{F})\}$$

and it holds $d_f(\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})) = 0$ if, and only if, $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \mathbf{H}$. The projected codes of an orbit flag code are orbit (subspace) codes. More precisely, for every $1 \leqslant i \leqslant r$, we have $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})_i = \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) \subseteq \mathcal{G}_q(t_i, n)$. Besides, the stabilizer subgroup of $\mathcal{F}$ is closely related to the ones of its subspaces:

$$\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \bigcap_{i=1}^{r} \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i). \tag{6.15}$$

Remark that, fixed an acting subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$, the cardinality of the flag code $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ and their projected codes are determined by the orders of the corresponding stabilizer subgroups of $\mathbf{H}$. The equality given in (6.15) allows to obtain that disjoint flag codes, in the orbital scenario, involve an equality of subgroups and not only of cardinalities.

Proposition 4.4. *[3, Prop. 3.5] Given a flag $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ and a subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$, the following statements are equivalent:*

(i) $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ *is a disjoint flag code.*

(ii) $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}) = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_1) = \cdots = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_r)$.

(iii) $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_1) = \cdots = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_r)$.

4.2 On Optimum Distance Flag Codes

Next, we will go deeper into the theoretical study of ODFCs, obtaining some important properties that will be useful for the subsequent orbital constructions (Section 5). In Theorem 4.1, ODFCs are characterized in terms of all their projected codes. In this section, we go one step further and present a new criterion to characterize them just regarding, at most, two of its projected codes. To achieve this result, we start studying how the fact of having a projected code with maximum distance gives us some information about the cardinality and distance of some other projected codes.

Proposition 4.5. *Let $\mathcal{C}$ be a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ having a projected code $\mathcal{C}_i$ of maximum distance, for some $i \in \{1, \ldots, r\}$, and take flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_i \neq \mathcal{F}'_i$.*

(a) If $2t_i \leqslant n$, then $d_S(\mathcal{F}_j, \mathcal{F}'_j) = 2t_j$, for every $1 \leqslant j \leqslant i$. In particular, $|\mathcal{C}_i| \leqslant |\mathcal{C}_j|$, for $1 \leqslant j \leqslant i$.

(b) If $2t_i \geqslant n$, then $d_S(\mathcal{F}_j, \mathcal{F}'_j) = 2(n - t_j)$ for all $i \leqslant j \leqslant r$. As a consequence, $|\mathcal{C}_i| \leqslant |\mathcal{C}_j|$, for values $i \leqslant j \leqslant r$.

Proof. Assume that $\mathcal{C}_i$ is a constant dimension code of maximum distance. In particular, $|\mathcal{C}_i| > 1$ and we can consider flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_i \neq \mathcal{F}'_i$. We distinguish two possibilities:

(a) If $2t_i \leqslant n$, then $d_S(\mathcal{F}_i, \mathcal{F}'_i) = d_S(\mathcal{C}_i) = 2t_i$ or, equivalently, $\mathcal{F}_i \cap \mathcal{F}'_i = \{0\}$. Hence, for every $1 \leqslant j \leqslant i$, it holds that $\mathcal{F}_j \cap \mathcal{F}'_j = \{0\}$ and then $d_S(\mathcal{F}_j, \mathcal{F}'_j) = 2t_j$, which is the maximum possible distance between t_j-dimensional subspaces. In particular, $\mathcal{F}_j \neq \mathcal{F}'_j$ and we conclude that $|\mathcal{C}_i| \leqslant |\mathcal{C}_j|$, for all $1 \leqslant j \leqslant i$.

(b) If $2t_i \geqslant n$, then $d_S(\mathcal{F}_i, \mathcal{F}'_i) = d_S(\mathcal{C}_i) = 2(n-t_i)$ or, equivalently, $\mathcal{F}_i + \mathcal{F}'_i = \mathbb{F}_q^n$. As a result, when we consider higher dimensions $t_j \geqslant t_i$ in the type vector, we obtain $\mathcal{F}_j + \mathcal{F}'_j = \mathbb{F}_q^n$ as well. Consequently, $d_S(\mathcal{F}_j, \mathcal{F}'_j) = 2(n-t_j)$, which is the maximum distance between t_j-dimensional subspaces of $\mathbb{F}_q^n$. Moreover, $\mathcal{F}_j$ and $\mathcal{F}'_j$ are different and we obtain $|\mathcal{C}_i| \leqslant |\mathcal{C}_j|$, for all $i \leqslant j \leqslant r$.

∎

Remark 4.6. Notice that having a projected code attaining the maximum distance it is not enough to deduce that other projected codes satisfy the same property. The following example shows this situation.

Example 4.7. *Consider the canonical basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_6\}$ *of* $\mathbb{F}_q^6$ *and let* $\mathcal{C}$ *be the flag code of type* $(2,3)$ *on* $\mathbb{F}_q^6$ *given by the flags*

$$\begin{array}{rcl} \mathcal{F}^1 & = & (\langle \mathbf{e}_1, \mathbf{e}_2\rangle, \ \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle), \\ \mathcal{F}^2 & = & (\langle \mathbf{e}_1, \mathbf{e}_3\rangle, \ \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle), \\ \mathcal{F}^3 & = & (\langle \mathbf{e}_4, \mathbf{e}_5\rangle, \ \ \langle \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\rangle). \end{array}$$

Notice that $\mathcal{C}_2 = \{\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle, \langle \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\rangle\}$ *is a code of maximum distance,* $d_S(\mathcal{C}_2) = 6$. *However, the first projected code* $\mathcal{C}_1 = \{\langle \mathbf{e}_1, \mathbf{e}_2\rangle, \langle \mathbf{e}_1, \mathbf{e}_3\rangle, \langle \mathbf{e}_4, \mathbf{e}_5\rangle\}$ *has distance* $d_S(\mathcal{C}_1) = d_S(\mathcal{F}_1^1, \mathcal{F}_1^2) = 2$, *whereas the maximum distance for its dimension is* 4.

We will use Proposition 4.5 in order to characterize ODFCs in terms of, at most, two of their projected codes. These constant dimension codes are the ones (if they exist) of closest dimensions to $\frac{n}{2}$ in the type vector, both at left and right. Let us make this idea precise. Given an arbitrary but fixed type vector $(t_1, \dots, t_r)$ and an ambient space $\mathbb{F}_q^n$, we give special attention to two indices defined as

$$\begin{array}{rcl} a & = & \max\{i \in \{1, \dots, r\} \mid 2t_i \leqslant n\} \quad \text{and} \\ b & = & \min\{i \in \{1, \dots, r\} \mid 2t_i \geqslant n\}. \end{array} \tag{6.16}$$

Note that the sets

$$\{i \in \{1, \dots, r\} \mid 2t_i \leqslant n\} \text{ and } \{i \in \{1, \dots, r\} \mid 2t_i \geqslant n\}$$

cover the family of indices $\{1, \dots, r\}$. Hence, at least one of them must be nonempty. Even more, the first set is empty if, and only if, $b = 1$ and every dimension in the type vector is lower bounded by $\frac{n}{2}$. Similarly, the second one does not contain any element if, and only if, $a = r$, i.e., if all the dimensions in the type vector are upper bounded by $\frac{n}{2}$. In any other situation, a and b are well-defined and $a \leqslant b$. The equality holds if, and only if, n is even and the dimension $\frac{n}{2}$ appears in the type vector. In this case $t_a = t_b = \frac{n}{2}$. In any other situation, these two sets partition

$$\{1, \dots, r\} = \{1, \dots, a\} \,\dot{\cup}\, \{b, \dots, r\},$$

with $b = a + 1$.

For sake of simplicity, the following results are presented in terms of both indices a and b. Despite the fact that these two indices do not need to exist simultaneously, at least one of them is always well defined. In that way, for type vectors in which the index a (resp. b) does not exist, the next result still holds true and gives a characterization of ODFCs just in terms of the projected code of dimension $t_b = t_1$ (resp. $t_a = t_r$). In any case, it represents a significant improvement with respect to Theorem 4.1 (see [4, Th. 3.11]).

Theorem 4.8. *Let $\mathcal{C}$ be a flag code of type $(t_1, \dots, t_r)$ on $\mathbb{F}_q^n$ and consider indices a and b as in (6.16). The following statements are equivalent:*

(*i*) *The flag code $\mathcal{C}$ is an ODFC.*

(*ii*) *$\mathcal{C}_a$ and $\mathcal{C}_b$ are constant dimensions codes of maximum distance with cardinality $|\mathcal{C}_a| = |\mathcal{C}_b| = |\mathcal{C}|$.*

Proof. The implication $(i) \Longrightarrow (ii)$ follows straightforwardly from Theorem 4.1. To show $(ii) \Longrightarrow (i)$, notice that if $\mathcal{C}_a$ and $\mathcal{C}_b$ have the maximum possible distance, by means of Proposition 4.5, we have

$$\begin{array}{rcl} |\mathcal{C}| & = & |\mathcal{C}_a| \leqslant |\mathcal{C}_i| \text{ for every } i \leqslant a \text{ and} \\ |\mathcal{C}| & = & |\mathcal{C}_b| \leqslant |\mathcal{C}_j| \text{ for every } j \geqslant b. \end{array}$$

Since the cardinality of every projected code is upper bounded by $|\mathcal{C}|$, we conclude that $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$, i.e., the flag code $\mathcal{C}$ is disjoint. Now, in order to see that every projected code attains the maximum possible distance, we argue as follows. Take an index $1 \leqslant i \leqslant r$ and consider a pair of different subspaces $\mathcal{F}_i, \mathcal{F}'_i \in \mathcal{C}_i$. These subspaces come from different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ and, since $\mathcal{C}$ is disjoint, we have $\mathcal{F}_j \neq \mathcal{F}'_j$ for every $1 \leqslant j \leqslant r$. In particular, $\mathcal{F}_a \neq \mathcal{F}'_a$ and $\mathcal{F}_b \neq \mathcal{F}'_b$. Hence, by means of Proposition 4.5, in any case, the distance $d_S(\mathcal{F}_i, \mathcal{F}'_i)$ is the maximum possible one for dimension t_i and, as a result, every projected code $\mathcal{C}_i$ is a constant dimension code of maximum distance. Thus, by application of Theorem 4.1, the flag code $\mathcal{C}$ is an ODFC. ∎

Here below we translate the previous results into the orbital scenario. As before, we express them in terms of both indices a and b defined in (6.16), but always having in mind that one of them might not exist. In such a case, the result is fulfilled by the remaining index.

The following proposition was already stated in [3] for $n = 2k$ and flags of full type vector.

Proposition 4.9. *Let $\mathcal{F}$ be a flag of type $(t_1, \dots, t_r)$ on $\mathbb{F}_q^n$ and $\mathbf{H}$ a subgroup of $\mathrm{GL}(n, q)$ such that $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)$ and $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)$ are subspace codes of maximum distance. Then* (*i*) $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a)$ *for all* $i \leqslant a$ *and*

(*ii*) $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b)$ *for all* $i \geqslant b$.

Proof. (*i*) Consider a matrix $A \in \mathbf{H} \backslash \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a)$. Then $\mathcal{F}_a \neq \mathcal{F}_a \cdot A$ and, by means of Proposition 4.5, for every $i \leqslant a$, it holds $d_S(\mathcal{F}_i, \mathcal{F}_i \cdot A) = 2t_i$. In particular, it is clear that $A \notin \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$, whenever $i \leqslant a$. Equivalently, we have that $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a)$ for all $i \leqslant a$.
(*ii*) If $A \in \mathbf{H} \backslash \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b)$, then $\mathcal{F}_b \neq \mathcal{F}_b \cdot A$ and Proposition 4.5 leads to $d_S(\mathcal{F}_i, \mathcal{F}_i \cdot A) = 2(n - t_i)$ for all $i \geqslant b$. Consequently, $A \notin \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$, for every $i \geqslant b$, and the result holds. ∎

Remark 4.10. Observe that Proposition 4.5 gives us some conditions on the cardinality of the projected codes. From that result, in the orbital scenario, if we assume that $\mathcal{C}_a$ and $\mathcal{C}_b$ attain the maximum possible distance, it follows that the order of every $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$ must be upper bounded either by $|\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a)|$ or $|\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b)|$. However, in Proposition 4.9, we obtain a stronger condition: a subgroup relationship.

Next we summarize several different characterizations for ODFCs arising as orbits of the action of an arbitrary subgroup of $\mathrm{GL}(n, q)$.

Theorem 4.11. *Consider a flag* $\mathcal{F}$ *of type* $(t_1, \dots, t_r)$ *on* $\mathbb{F}_q^n$ *and a subgroup* $\mathbf{H}$ *of* $\mathrm{GL}(n, q)$. *The following statements are equivalent:*

(*i*) *The flag code* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ *is an ODFC.*

(*ii*) *The subspace codes* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)$ *and* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)$ *are of maximum distance and* $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a) = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_i)$, *for every* $1 \leqslant i \leqslant r$.

(*iii*) *The subspace codes* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)$ *and* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)$ *are of maximum distance and* $\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a) = \mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b) \subseteq \mathrm{Stab}_{\mathbf{H}}(\mathcal{F})$.

(*iv*) *The subspace codes* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)$ *and* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)$ *are of maximum distance and* $|\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_a)| = |\mathrm{Stab}_{\mathbf{H}}(\mathcal{F}_b)| \leqslant |\mathrm{Stab}_{\mathbf{H}}(\mathcal{F})|$.

The cardinality of such a flag code is $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)|$.

Proof. Observe that, by means of Theorem 4.1, together with Proposition 4.4, statement (*i*) clearly implies the other ones. On the other hand, by application of Proposition 4.9 and expression (6.15), all conditions (*ii*), (*iii*) and (*iv*) are equivalent and make the code $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ be disjoint by Proposition 4.4. In particular, it holds $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_a)| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_b)|$ with projected codes of dimensions t_a and t_b attaining the maximum possible distances. Theorem 4.8 finishes the proof. ∎

Recall that the cardinality of an orbit flag code is completely determined by the orders of the acting group and the stabilizer subgroup of the generating flag. Fixed the acting group, a natural way of obtaining codes with better cardinalities is to consider the union of different orbits. We finish this section by characterizing when the union of orbit flag codes is an ODFC. To this purpose, notice that every nonempty subset of an ODFC is either an ODFC too or a trivial code having just one element. In both cases, such a subset is a disjoint flag code. With the goal of obtaining better cardinalities in mind, we proof first the following lemma where we work with the union of two disjoint flag codes that are orbits of the same group.

Lemma 4.12. *Let $\mathcal{F}$ and $\mathcal{F}'$ be flags of type $(t_1, \dots, t_r)$ on $\mathbb{F}_q^n$ and take a subgroup $\mathbf{H}$ of $\mathrm{GL}(n, q)$ such that the orbits $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ and $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ are disjoint flag codes. The following statements hold:*

(*a*) *If $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) \neq \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)$, for some $i \in \{1, \dots, r\}$, then $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| + |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')|$.*

(*b*) *If $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) \neq \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)$, for all $i = 1, \dots, r$, then $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ is a disjoint flag code.*

(*c*) *If $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) = \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)$, for some $i \in \{1, \dots, r\}$, then $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ is a disjoint flag code if, and only if, $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) = \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$.*

Proof. (*a*) Since $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) \neq \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)$, their intersection is the empty set and, since $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})$ and $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ are disjoint flag codes, it follows that

$$\begin{aligned} |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')| &\leqslant |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| + |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i)| + |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)| \\ &= |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)| = |(\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'))_i| \\ &\leqslant |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')|. \end{aligned} \tag{6.17}$$

Therefore, $|\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')| = |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F})| + |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')|$ and the statement holds.

(*b*) Follows directly from applying (6.17) for all $i = 1, \dots, r$.

(*c*) Assume that the union code $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ is a disjoint flag code. Given that $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}_i) = \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}'_i)$, there must exist a matrix $A \in \mathbf{H}$ such that $\mathcal{F}_i = \mathcal{F}'_i \cdot A$. Hence, both flags $\mathcal{F}$ and $\mathcal{F}' \cdot A$ are in $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) \cup \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$ and they share their i-th subspace. Thus, we conclude that $\mathcal{F} = \mathcal{F}' \cdot A$ and therefore $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}) = \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}')$. The converse statement trivially holds. ∎

Now, with the benefit of the above lemma and using Theorem 4.8, the next result states when the union of a family of disjoint flag codes, arising as orbits of the same group, provides ODFCs of larger cardinality. To do so, we consider indices a and b as in (6.16).

Theorem 4.13. *Let* $\{\mathcal{F}^j = (\mathcal{F}^j_1, \ldots, \mathcal{F}^j_r)\}_{j=1}^m$ *be a family of flags of type* $(t_1, \ldots, t_r)$ *on* $\mathbb{F}_q^n$ *and consider a subgroup* $\mathbf{H}$ *of* $\mathrm{GL}(n, q)$ *such that every orbit* $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)$ *is a disjoint flag code, for* $1 \leqslant j \leqslant m$*. If the subspaces* $\mathcal{F}^1_a, \ldots, \mathcal{F}^m_a$ *and* $\mathcal{F}^1_b, \ldots, \mathcal{F}^m_b$ *lie in different orbits under the action of* $\mathbf{H}$*, then*

$$|\bigcup_{j=1}^{m} \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)| = \sum_{j=1}^{m} |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)|.$$

Moreover, the following statements are equivalent:

(*i*) *The union flag code* $\bigcup_{j=1}^m \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)$ *is an ODFC.*

(*ii*) *The projected union codes* $\bigcup_{j=1}^m \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j_a)$ *and* $\bigcup_{j=1}^m \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j_b)$ *have the maximum possible distance.*

Proof. Since every orbit $\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)$ is a disjoint flag code, we can argue as in (6.17) of the previous lemma to obtain that

$$|\bigcup_{j=1}^{m} \mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)| = \sum_{j=1}^{m} |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j)| = \sum_{j=1}^{m} |\mathrm{Orb}_{\mathbf{H}}(\mathcal{F}^j_i)|, \tag{6.18}$$

for $i = a, b$. Hence, by means of Theorem 4.8, the union flag code is an ODFC if, and only if, its projected codes of dimensions t_a and t_b attain the maximum possible distance. ∎

We will use these theoretical results in the following section in order to give specific constructions of orbit ODFCs having the maximum possible cardinalities for the corresponding type vectors.

5 ODFC From Singer Groups

This section is devoted to construct flag codes of maximum distance having an orbital structure. For this, we will use suitable Singer groups (or their subgroups) and their transitive action on lines and hyperplanes (Theorem 2.1). The goal of Section 5.1 is to obtain ODFCs on $\mathbb{F}_q^n$ having a k-spread as a projected code, for k a divisor of n. To do so, according to Theorem 4.3, we consider first flags of type $(1, \ldots, k, n-k, \ldots, n-1)$. Such a construction leads to full flag codes whenever $n = 2k$ or $k = 1$ and $n = 3$. In Section 5.2, we build ODFCs of full type vector for the remaining cases.

5.1 Orbit ODFC from Desarguesian spreads

In this section we address the orbital construction of ODFCs on $\mathbb{F}_q^n$ with the k-spread $\mathcal{S}$ defined in (6.7) as a projected code. To this end, write $n = ks$ for some $s \geqslant 2$. Recall that, by virtue of Theorem 4.2, such a code has also the largest possible size. Throughout the rest of this section, and for sake of simplicity, we will write $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k, \mathcal{F}_{n-k}, \dots, \mathcal{F}_{n-1})$ to denote an arbitrary flag of the *full admissible type vector* $(1, \dots, k, n-k, \dots, n-1)$. In these conditions, indices a and b of (6.16) are such that $t_a = k$ and $t_b = n-k$. Consider the Singer group $\langle \psi(M_s) \rangle$ of $\mathrm{GL}(n, q)$ defined in Section 3. Recall that the constant dimension codes $\mathcal{S}$ and $\mathcal{H}$ arise as their orbits (see (6.10) and (6.11)). Next, we use Theorem 4.11 in order to characterize those subgroups of $\langle \psi(M_s) \rangle$ that are appropriate to construct ODFCs.

Theorem 5.1. *Let $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k, \mathcal{F}_{n-k}, \dots, \mathcal{F}_{n-1})$ be a flag of full admissible type vector such that $\mathcal{F}_k \in \mathcal{S}$ and $\mathcal{F}_{n-k} \in \mathcal{H}$. For any positive integer t dividing $q^n - 1$, consider the unique subgroup $\mathbf{T}$ of $\langle \psi(M_s) \rangle$ of order t. Then:*

(i) $|\mathrm{Orb}_{\mathbf{T}}(\mathcal{F})| = \frac{t}{\gcd(t, q-1)}$.

(ii) $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F})$ *is an ODFC if, and only if,* $\gcd(t, q^k - 1) = \gcd(t, q-1) \neq t$.

Proof. (i) By (6.15) and Theorem 2.1, it follows that

$$\mathrm{Stab}_{\langle \psi(M_s) \rangle}(\mathcal{F}) = \mathrm{Stab}_{\langle \psi(M_s) \rangle}(\mathcal{F}_1) = \mathrm{Stab}_{\langle \psi(M_s) \rangle}(\mathcal{F}_{n-1}) = \{aI_n \mid a \in \mathbb{F}_q^*\},$$

which has order $q-1$. As a result, $\mathrm{Stab}_{\mathbf{T}}(\mathcal{F}) = \mathbf{T} \cap \{aI_n \mid a \in \mathbb{F}_q^*\}$, is a group of order $\gcd(t, q-1)$ and the statement holds. Notice that, in particular, $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}) = \{\mathcal{F}\}$ exactly when $t \mid q-1$.

(ii) By using (6.12), it follows that

$$\begin{aligned} \mathrm{Stab}_{\mathbf{T}}(\mathcal{F}_k) &= \mathbf{T} \cap \mathrm{Stab}_{\langle \psi(M_s) \rangle}(\mathcal{F}_k) = \mathbf{T} \cap \{\psi(aI_s) \mid a \in \mathbb{F}_{q^k}^*\} = \\ &= \mathbf{T} \cap \mathrm{Stab}_{\langle \psi(M_s) \rangle}(\mathcal{F}_{n-k}) = \mathrm{Stab}_{\mathbf{T}}(\mathcal{F}_{n-k}), \end{aligned}$$

which is a group of order $\gcd(t, q^k - 1)$. Since we are working with cyclic groups, we obtain that $\mathrm{Stab}_{\mathbf{T}}(\mathcal{F}_k) = \mathrm{Stab}_{\mathbf{T}}(\mathcal{F}_{n-k}) = \mathrm{Stab}_{\mathbf{T}}(\mathcal{F})$ if, and only if, $\gcd(t, q^k - 1) = \gcd(t, q-1)$. On the other hand, since $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}_k) \subseteq \mathcal{S}$ and $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}_{n-k}) \subseteq \mathcal{H}$, these projected codes of the flag code $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F})$ will be constant dimension codes of maximum distance whenever they have at least 2 elements. As a result, the statement (ii) follows from Theorem 4.11 and (i). ∎

As stated in Theorem 4.2, the size of the k-spread $\mathcal{S}$, that is, $\frac{q^n-1}{q^k-1}$, determines the maximum size of an ODFC having $\mathcal{S}$ as a projected code. In order to achieve this optimal size, we will consider unions of orbits under the action of a suitable subgroup $\mathbf{T}$ of $\langle \psi(M_s) \rangle$ and then apply Theorem 4.13 as follows.

Theorem 5.2. *Let* $\mathbf{T}$ *be a subgroup of order* t *of* $\langle\psi(M_s)\rangle$ *such that* $\gcd(t, q^k-1) = \gcd(t, q-1)$*. For* $m \geqslant 2$*, let* $\{\mathcal{F}^j = (\mathcal{F}^j_1, \ldots, \mathcal{F}^j_k, \mathcal{F}^j_{n-k}, \ldots, \mathcal{F}^j_{n-1})\}_{j=1}^m$ *be a family of flags of full admissible type such that* $\mathcal{F}^1_k, \ldots, \mathcal{F}^m_k \in \mathcal{S}$ *and* $\mathcal{F}^1_{n-k}, \ldots, \mathcal{F}^m_{n-k} \in \mathcal{H}$ *lie in different orbits under the action of* $\mathbf{T}$*. Then*

(*i*) $\cup_{j=1}^m \mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ *is an ODFC of size* $\frac{mt}{\gcd(t,q-1)}$*.*

(*ii*) *If* $m = \frac{(q^n-1)\gcd(t,q-1)}{(q^k-1)t}$*, then* $\cup_{j=1}^m \mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ *is an ODFC of the maximum size, that is,* $\frac{q^n-1}{q^k-1}$*.*

Proof. (*i*) By Theorem 5.1 we know that $|\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)| = \frac{t}{\gcd(t,q-1)}$, for all $1 \leqslant j \leqslant m$. Moreover, since $\gcd(t, q^k-1) = \gcd(t, q-1)$, the same theorem ensures that either $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j) = \{\mathcal{F}^j\}$ or $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ is an ODFC, for all $1 \leqslant j \leqslant m$. In any case, one has that $\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ is a disjoint flag code, for all $1 \leqslant j \leqslant m$. Thus, we have the hypotheses of Theorem 4.13 and following (6.18) we can write

$$|\cup_{j=1}^m \mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)| = \sum_{j=1}^m |\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)| = \sum_{j=1}^m |\mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j_i)| = \frac{mt}{\gcd(t, q-1)} \geqslant 2,$$

for $i = k, n-k$. In particular, the projected codes of dimensions k and $n-k$ of the union flag code $\cup_{j=1}^m \mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ are subsets of $\mathcal{S}$ and $\mathcal{H}$, respectively, having at least two elements. Hence they are subspace codes of maximum distance and Theorem 4.13 states that $\cup_{j=1}^m \mathrm{Orb}_{\mathbf{T}}(\mathcal{F}^j)$ is an ODFC.

Statement (*ii*) follows just by computing the number of orbits of the action of $\mathbf{T}$ on $\mathcal{S}$. ∎

Remark 5.3. Theorem 5.1 states which subgroups of $\langle\psi(M_s)\rangle$ allow the construction of ODFCs as a single orbit of them. Notice that bigger subgroups not always will provide bigger orbit flag codes. In addition, it may eventually happen that some subgroup provides an orbit ODFC of the maximum possible size, $\frac{q^n-1}{q^k-1}$. Otherwise, Theorem 5.2 leads to an optimal construction consisting of the union of several orbits. Clearly, the larger the size of each orbit, the fewer orbits we need to join to reach the maximum size and vice versa. In particular, the degenerate case where an ODFC is constructed as a union of $\frac{q^n-1}{q^k-1}$ orbits with just one element is also contemplated in Theorem 5.2.

All these considerations are reflected in the following examples, in which we apply Theorems 5.1 and 5.2 for different values of the parameters.

Example 5.4. *With the notation of Theorem 5.1, we consider all the divisors* t *of* q^n-1 *such that* $\gcd(t, q^k-1) = \gcd(t, q-1)$ *and the corresponding subgroup* $\mathbf{T}$ *of* $\langle\psi(M_k)\rangle$ *of order* t*. Consider a flag* $\mathcal{F}$ *of the full admissible type* $(1, \ldots, k, n-k, \ldots, n-1)$ *such that* $\mathcal{F}_k \in \mathcal{S}$ *and* $\mathcal{F}_{n-k} \in \mathcal{H}$*. Finally, denote by* m *the number of required orbits of* $\mathbf{T}$ *to attain the maximum size,* $\frac{q^n-1}{q^k-1}$*, for an ODFC with these parameters.*

(1) Put $q = 3$, $k = 3$ and $n = 6$. Thus, $k = n - k$, $q^n - 1 = 728$, $q^k - 1 = 26$ and $\frac{q^n-1}{q^k-1} = 28$. Then

t	1	2	4	7	8	14	28	56
$\lvert\mathrm{Orb}_{\mathbf{T}}(\mathcal{F})\rvert$	1	1	2	7	4	7	14	28
m	28	28	14	4	7	4	2	1

Table 6.1: $q = 3$, $k = 3$ and $n = 6$.

Notice that, in this case, the subgroup of order $t = 56$ allows us to obtain ODFCs of full type vector and having the best possible size, i.e., 28, by using a single orbit. In this sense, for odd characteristic, Theorem 5.1 eventually improves the construction presented in [3, Prop. 4.15], where two orbits were always needed. Moreover, remark that the subgroup of order $t = 8$ gives an orbit ODFC of smaller size than the obtained with the subgroup of order $t = 7$.

(2) Put $q = 4$, $k = 3$ and $n = 9$. Thus, $n - k = 6$, $q^n - 1 = 262143$, $q^k - 1 = 63$ and $\frac{q^n-1}{q^k-1} = 4161$. Then

t	1	3	19	57	73	219	1387	4161
$\lvert\mathrm{Orb}_{\mathbf{T}}(\mathcal{F})\rvert$	1	1	19	19	73	73	1387	1387
m	4161	4161	219	219	57	57	3	3

Table 6.2: $q = 4$, $k = 3$ and $n = 9$.

The largest orbit size is 1387 and it is obtained when the acting group has order either 1387 or 4161. On the other hand, the maximum possible size of an ODFC with these parameters is 4161. Hence, in order to achieve that cardinality, we must consider the union of, at least, 3 different orbits.

The orbital constructions of ODFC provided in this section present a restriction on the type vector, coming from the condition of having a spread as a projected code. However, there are two possible situations in which flag codes of full type can be given by using Theorems 5.1 and 5.2. First, for even values of n, taking the divisor $k = \frac{n}{2}$ leads to a construction of full type in which the values k and $n - k$ coincide. This particular case was first studied in [3], where a construction using the action of a Singer subgroup of $\mathrm{SL}(2k, q)$ is presented. On the other hand, for $n = 3$ and $k = 1$, the action of a Singer subgroup of $\mathrm{GL}(3, q)$ on the Grassmannian of lines and hyperplanes gives us a construction of type $(1, 2)$, this construction is also known and the reader can find it in [17, Prop. 2.5], where the author shows that it is the one with the biggest cardinality among ODFC of full type when $n = 3$. In the following section, we consider the remaining situations, that is, we address the construction of orbit ODFCs of full type vector on $\mathbb{F}_q^n$ for odd values of $n > 3$.

5.2 Orbit ODFC of full type vector

Throughout this section, we work with full flags on $\mathbb{F}_q^{2k+1}$, for some $k > 1$. In this case, by virtue of Theorem 4.8, the construction of ODFCs can be done by giving appropriate constant dimension codes for dimensions k and $k+1$ (see (6.16)). To this end, we present the next subgroup of $\mathrm{GL}(2k+1, q)$. Let $M_{k+1} \in \mathrm{GL}(k+1, q)$ be the companion matrix of a primitive polynomial of degree $k+1$ in $\mathbb{F}_q[x]$. Recall that, as pointed out in Section 2.2, M_{k+1} is a Singer cycle of $\mathrm{GL}(k+1, q)$ and $\mathbb{F}_q[M_{k+1}] = \langle M_{k+1} \rangle \cup \{0_{(k+1)\times(k+1)}\}$ is a matrix representation of the finite field of q^{k+1} elements. Let us write

$$g = \left(\begin{array}{c|c} I_k & 0_{k\times(k+1)} \\ \hline 0_{(k+1)\times k} & M_{k+1} \end{array}\right) \in \mathrm{GL}(2k+1, q)$$

and consider the cyclic group

$$\mathbf{G} = \langle g \rangle = \left\{ g^i \mid 0 \leqslant i \leqslant q^{k+1} - 2 \right\}. \tag{6.19}$$

Clearly, $\mathbf{G}$ is a subgroup of order $q^{k+1} - 1$ of $\mathrm{GL}(2k+1, q)$, isomorphic to the Singer subgroup $\langle M_{k+1} \rangle$ of $\mathrm{GL}(k+1, q)$. In the rest of this section, the orbit codes considered will be always generated by the action of this particular group $\mathbf{G}$.

We start by characterizing the subspaces of dimensions k and $k+1$ of $\mathbb{F}_q^{2k+1}$ whose orbits under the action of $\mathbf{G}$ are constant dimension codes of maximum distance. Given arbitrary subspaces $\mathcal{U} = \mathrm{rowsp}(U) \in \mathcal{G}_q(k, 2k+1)$ and $\mathcal{V} = \mathrm{rowsp}(V) \in \mathcal{G}_q(k+1, 2k+1)$, the respective full-rank generator matrices $U \in \mathbb{F}_q^{k\times(2k+1)}$ and $V \in \mathbb{F}_q^{(k+1)\times(2k+1)}$ can split into two blocks as

$$U = (U_1 \mid U_2) \quad \text{and} \quad V = (V_1 \mid V_2), \tag{6.20}$$

where U_1 (resp. V_1) denotes the first k columns of U (resp. V). Therefore, $U_1 \in \mathbb{F}_q^{k\times k}$, $U_2 \in \mathbb{F}_q^{k\times(k+1)}$, $V_1 \in \mathbb{F}_q^{(k+1)\times k}$ and $V_2 \in \mathbb{F}_q^{(k+1)\times(k+1)}$. Using this notation, we can write

$$\begin{aligned} \mathrm{Orb}_{\mathbf{G}}(\mathcal{U}) &= \{\mathcal{U} \cdot g^i \mid 0 \leqslant i \leqslant q^{k+1} - 2\} \\ &= \{\mathrm{rowsp}(U_1 \mid U_2 M_{k+1}^i) \mid 0 \leqslant i \leqslant q^{k+1} - 2\} \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} \mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) &= \{\mathcal{V} \cdot g^i \mid 0 \leqslant i \leqslant q^{k+1} - 2\} \\ &= \{\mathrm{rowsp}(V_1 \mid V_2 M_{k+1}^i) \mid 0 \leqslant i \leqslant q^{k+1} - 2\}. \end{aligned} \tag{6.22}$$

With this notation, the following results hold. To make this section easier to read, their proofs, and all the ones concerning subspace codes, are included in the final Appendix of the article, so that only proofs concerning results on flag codes appear here.

Proposition 5.5. *The orbit code* $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})$ *defined in (6.21) is a partial spread of dimension* k *of* $\mathbb{F}_q^{2k+1}$ *if, and only if,* $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$*. Its cardinality is* $|\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})| = |\mathbf{G}| = q^{k+1} - 1$.

Proposition 5.6. *The orbit code* $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ *defined in (6.22) attains the maximum possible distance if, and only if,* $\mathrm{rk}(V_1) = k$ *and* $\mathrm{rk}(V_2) = k+1$*. Its size is* $|\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})| = |\mathbf{G}| = q^{k+1} - 1$.

Here below, we use the previous characterizations for constant dimension codes of maximum distance in order to provide orbit ODFCs of full type on $\mathbb{F}_q^{2k+1}$. To do so, we need to consider nested subspaces $\mathcal{U} \subsetneq \mathcal{V}$ of dimensions k and $k+1$, respectively. Using the notation of (6.20), we can formulate the problem in a matrix approach: given a full-rank generator matrix $U = (U_1 \mid U_2) \in \mathbb{F}_q^{k\times(2k+1)}$ of $\mathcal{U}$, we consider a subspace $\mathcal{V}$ spanned by the rows of a matrix $V \in \mathbb{F}_q^{(k+1)\times(2k+1)}$, obtained by adding an appropriate row to U. In other words, we choose vectors $\mathbf{v}_1 \in \mathbb{F}_q^k$ and $\mathbf{v}_2 \in \mathbb{F}_q^{k+1}$ such that the matrix

$$V = (V_1 \mid V_2) = \left(\begin{array}{c|c} U_1 & U_2 \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right) \tag{6.23}$$

has rank equal to $k+1$. Using this notation, we present the next construction of ODFCs arising from the action of the group $\mathbf{G}$ defined in (6.19).

Theorem 5.7. *Let* $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_{2k})$ *be a full flag on* $\mathbb{F}_q^{2k+1}$ *such that*

$$\mathcal{F}_k = \mathcal{U} = \mathrm{rowsp}(U_1 \mid U_2) \text{ and } \mathcal{F}_{k+1} = \mathcal{V} = \mathrm{rowsp}(V_1 \mid V_2),$$

with generator matrix $(V_1 \mid V_2)$ *as in (6.23) and consider the group* $\mathbf{G}$ *defined in (6.19). The following statements are equivalent:*

(i) the flag code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})$ *is an ODFC.*

(ii) $U_1 \in \mathbb{F}_q^{k\times k}$ *and* $V_2 \in \mathbb{F}_q^{(k+1)\times(k+1)}$ *are invertible matrices.*

In this situation, $|\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})| = |\mathbf{G}| = q^{k+1} - 1$.

Proof. Assume that $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})$ is an ODFC. In particular, the projected codes $\mathrm{Orb}(\mathcal{F}_k)$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_{k+1})$ must attain the maximum distance. By means of Propositions 5.5 and 5.6, it must hold $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = \mathrm{rk}(V_1) = k$ and $\mathrm{rk}(V_2) = k+1$. Consequently, U_1 and V_2 are invertible matrices.

Conversely, assume now that $\mathrm{rk}(U_1) = k$ and $\mathrm{rk}(V_2) = k+1$. Since U_2 is composed by the first k rows of the invertible matrix V_2, we clearly obtain that

$\mathrm{rk}(U_2) = k$. On the other hand, observe that $V_1 \in \mathbb{F}_q^{(k+1)\times k}$ contains U_1 as a submatrix. Hence, its rank is k as well. Now, by using Propositions 5.5 and 5.6, we conclude that both $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_k)$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_{k+1})$ are constant dimension codes of maximum distance such that $\mathrm{Stab}_{\mathbf{G}}(\mathcal{F}_k) = \mathrm{Stab}_{\mathbf{G}}(\mathcal{F}_{k+1}) = \{I_{2k+1}\} \subseteq \mathrm{Stab}_{\mathbf{G}}(\mathcal{F}_i)$, for every $1 \leqslant i \leqslant 2k$. Hence, by application of Theorem 4.11, the orbit flag code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})$ is an ODFC of size $|\mathrm{Orb}_{\mathbf{G}}(\mathcal{F})| = |\mathbf{G}| = q^{k+1} - 1$. ∎

The ODFC constructed in Theorem 5.7 contains $q^{k+1}-1$ flags. It is the largest size for orbits under the action of the group $\mathbf{G}$. On the other hand, as proved in [17, Prop. 2.4], the maximum possible cardinality for ODFCs of full type on $\mathbb{F}_q^{2k+1}$ is exactly $q^{k+1}+1$. Consequently, our orbital construction is only two flags away from reaching the mentioned bound.

We devote the rest of the section to complete the construction in Theorem 5.7 into an ODFC on $\mathbb{F}_q^{2k+1}$ with the largest possible size, that is, $q^{k+1}+1$. Notice that this value coincides with the largest size of a partial spread of $\mathbb{F}_q^{2k+1}$ of dimension k or, equivalently, the largest size of a constant dimension code of dimension $k+1$ of $\mathbb{F}_q^{2k+1}$ having maximum distance. Therefore, by virtue of Theorem 4.8, the problem can be reduced to adding to $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_k)$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_{k+1})$ two appropriate respective subspaces such that the resulting subspace codes of dimensions k and $k+1$ are still of maximum distance. We start tackling this problem for dimension k. The following remark will help us in this research.

Remark 5.8. With the notation of (6.21), observe that, in the particular case where $U_1 = I_k$ and $U_2 = (I_{k+1})_{(k)}$, i.e., the matrix given by the last k rows of I_{k+1}, we obtain

$$\mathrm{Orb}_{\mathbf{G}}(\mathrm{rowsp}(I_k \mid (I_{k+1})_{(k)})) = \left\{\mathrm{rowsp}(I_k \mid (M_{k+1}^i)_{(k)}) \mid 0 \leqslant i \leqslant q^{k+1}-2\right\},$$

where $(M_{k+1}^i)_{(k)}$ is the matrix composed by the last k rows of M_{k+1}^i. Therefore, in this case, $\mathrm{Orb}_{\mathbf{G}}(\mathrm{rowsp}(I_k \mid (I_{k+1})_{(k)}))$ is a subset of the partial spread of $\mathbb{F}_q^{2k+1}$ of dimension k given in [10, Th. 13], which attains the maximum possible size, that is, $q^{k+1}+1$, and can be written as

$$\mathrm{Orb}_{\mathbf{G}}(\mathrm{rowsp}(I_k \mid (I_{k+1})_{(k)})) \cup \{\mathrm{rowsp}(I_k \mid 0_{k\times(k+1)}),\ \mathrm{rowsp}(0_{k\times k} \mid (I_{k+1})_{(k)})\}.$$

Inspired by this fact, for any election of full-rank matrices $U_1 \in \mathbb{F}_q^{k\times k}$ and $U_2 \in \mathbb{F}_q^{k\times(k+1)}$, we suggest the subspaces

$$\mathcal{U}' = \mathrm{rowsp}(U_1 \mid 0_{k\times(k+1)}) \ \text{ and } \ \mathcal{U}'' = \mathrm{rowsp}(0_{k\times k} \mid U_2). \tag{6.24}$$

as candidates to make $\mathrm{Orb}_{\mathbf{G}}(\mathrm{rowsp}(U_1 \mid U_2)) \cup \{\mathcal{U}', \mathcal{U}''\}$ be a partial spread of $\mathbb{F}_q^{2k+1}$ of dimension k. The proof of the following result appears in the final Appendix of the paper.

Proposition 5.9. *Let $U_1 \in \mathbb{F}_q^{k\times k}$ and $U_2 \in \mathbb{F}_q^{k\times(k+1)}$ matrices such that* $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$ *and form the k-dimensional subspaces $\mathcal{U} = \mathrm{rowsp}(U_1 \mid U_2)$ and $\mathcal{U}', \mathcal{U}''$ as in (6.24). Then the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U}) \cup \{\mathcal{U}', \mathcal{U}''\}$ is a partial spread of $\mathbb{F}_q^{2k+1}$ of dimension k with cardinality $q^{k+1} + 1$.*

Now we address the same problem for dimension $k+1$. To do so, we consider full-rank matrices $U_1 \in \mathbb{F}_q^{k\times k}, U_2 \in \mathbb{F}_q^{k\times(k+1)}$ and vectors $\mathbf{v}_1 \in \mathbb{F}_q^{k}$, $\mathbf{v}_2 \in \mathbb{F}_q^{k+1}$ such that the matrix $V = (V_1 \mid V_2)$ defined in (6.23) has $\mathrm{rk}(V) = \mathrm{rk}(V_2) = k+1$ and put $\mathcal{V} = \mathrm{rowsp}(V) \in \mathcal{G}_q(k+1, 2k+1)$. In these conditions, by Proposition 5.6, the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ attains the maximum possible distance and has size $q^{k+1} - 1$. Hence, we wonder if it is possible to find two suitable subspaces $\mathcal{V}'$ and $\mathcal{V}''$ such that the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}$ still has the maximum distance and achieve the best cardinality. Moreover, in order to use such a code, together with the one given in Proposition 5.9, to construct ODFCs of full type vector on $\mathbb{F}_q^{2k+1}$, we also require the condition $\mathcal{U}' \subset \mathcal{V}'$ and $\mathcal{U}'' \subset \mathcal{V}''$. Taking into account the form of the matrix V defined in (6.23), it seems quite natural to use subspaces $\mathcal{V}'$ and $\mathcal{V}''$, spanned by the rows of matrices

$$V' = \left(\begin{array}{c|c} U_1 & 0_{k\times(k+1)} \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right) \text{ and } V'' = \left(\begin{array}{c|c} 0_{k\times k} & U_2 \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right), \tag{6.25}$$

respectively. Observe that the vector space spanned by the first k rows of V' (resp. V'') is precisely $\mathcal{U}'$ (resp. $\mathcal{U}''$). The next result states that this pair of subspaces works if, and only if, $\mathbf{v}_1 = \mathbf{0}_k$. The corresponding proof is also postponed to the final Appendix of the paper.

Proposition 5.10. *Let V be the matrix defined in (6.23), taking* $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$ *and $\mathbf{v}_2 \notin \mathrm{rowsp}(U_2)$. Consider $\mathcal{V} = \mathrm{rowsp}(V)$ and subspaces $\mathcal{V}'$ and $\mathcal{V}''$ as in (6.25). Then the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}$ has the maximum possible distance (i.e., $2k$) if, and only if, $\mathbf{v}_1 = \mathbf{0}_k$. In such a case, the code attains the largest size, that is, $q^{k+1} + 1$.*

Now, making use of Propositions 5.9 and 5.10, we are ready to present the next construction of ODFC of full type on $\mathbb{F}_q^{2k+1}$ having the maximum size.

Take matrices $U_1 \in \mathbb{F}_q^{k\times k}$, $U_2 \in \mathbb{F}_q^{k\times(k+1)}$ such that $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$ and vectors $\mathbf{v}_1 \in \mathbb{F}_q^{k}$, $\mathbf{v}_2 \in \mathbb{F}_q^{k+1} \setminus \mathrm{rowsp}(U_2)$ and form subspaces

$$\begin{array}{ll} \mathcal{U} = \mathrm{rowsp}(U_1 \mid U_2), & \mathcal{V} = \mathrm{rowsp}\left(\begin{array}{c|c} U_1 & U_2 \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right), \\ \mathcal{U}' = \mathrm{rowsp}(U_1 \mid 0_{k\times(k+1)}), & \mathcal{V}' = \mathrm{rowsp}\left(\begin{array}{c|c} U_1 & 0_{k\times(k+1)} \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right), \\ \mathcal{U}'' = \mathrm{rowsp}(0_{k\times k} \mid U_2), & \mathcal{V}'' = \mathrm{rowsp}\left(\begin{array}{c|c} 0_{k\times k} & U_2 \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right). \end{array} \tag{6.26}$$

With this notation, the next result holds.

Theorem 5.11. *Let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be full flags on $\mathbb{F}_q^{2k+1}$ such that*

$$\begin{array}{ll} \mathcal{F}_k = \mathcal{U}, & \mathcal{F}_{k+1} = \mathcal{V}, \\ \mathcal{F}'_k = \mathcal{U}', & \mathcal{F}'_{k+1} = \mathcal{V}', \\ \mathcal{F}''_k = \mathcal{U}'', & \mathcal{F}''_{k+1} = \mathcal{V}''. \end{array}$$

defined as in (6.26). Then the flag code $\mathcal{C} = \mathrm{Orb}_{\mathbf{G}}(\mathcal{F}) \cup \{\mathcal{F}', \mathcal{F}''\}$ is an ODFC with the maximum possible cardinality, i.e., $q^{k+1}+1$, if, and only if, $\mathbf{v}_1 = \mathbf{0}_k$.

Proof. By means of Theorem 4.8, we just need to check that the projected codes of dimensions k and $k+1$ attain the maximum distance, which is $2k$ in both cases, and $|\mathcal{C}| = |\mathcal{C}_k| = |\mathcal{C}_{k+1}|$. As proved in Propositions 5.5 and 5.6, the orbits $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_k)$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{F}_{k+1})$ give us the maximum distance. Moreover, by using Proposition 5.9, we obtain that $\mathcal{C}_k$ attains the maximum distance and size for every choice of $\mathbf{v}_1$. On the other hand, for $\mathcal{C}_{k+1}$ this happens if, and only if, $\mathbf{v}_1 = \mathbf{0}_k$, by Proposition 5.10. In this case, it follows that

$$|\mathcal{C}| = |\mathcal{C}_k| = |\mathcal{C}_{k+1}| = q^{k+1} + 1,$$

as stated. ∎

6 Conclusions and open problems

In this paper we have addressed the study of flag codes having maximum distance (ODFCs). We have obtained a characterization of such codes in terms of, at most, two of their projected codes. We have done this first in a general context and then in an orbital scenario (Section 4.2). In particular, these results improve on those obtained in [4] in this respect. Next, we have focused on the construction of ODFCs with an orbital structure. To do this, we have used the action of suitable Singer groups. We have provided two different systematic constructions, both of them reaching the maximum possible cardinality. For the first one, we have exploited the good relationship between Singer groups and Desarguesian spreads to obtain ODFCs having a specific Desarguesian spread as a projected code (Section 5.1). For the second construction, we have used the transitive action of Singer groups on hyperplanes and worked with flags of full type vector, thus covering the cases that cannot be considered in the previous construction (Section 5.2).

Given that the theoretical results obtained in Section 4.2 do not in any case require working with Desarguesian spreads, a possible research to be done along these lines could include the specific construction of orbital ODFCs having a non-Desarguesian spread among their projected codes. In a wider context, it would be interesting to address the study of flag codes associated with a fixed distance, as well as to provide systematic constructions of them.

7 Appendix

Proof of Proposition 5.5: Assume that the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U}) = \{\mathcal{U} \cdot g^i \mid 0 \leqslant i \leqslant q^{k+1} - 2\}$ is a partial spread code of dimension k of $\mathbb{F}_q^{2k+1}$. In other words, $d_S(\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})) = 2k$ and $\mathrm{Stab}_{\mathbf{G}}(\mathcal{U})$ is a proper subgroup of $\mathbf{G}$. In particular, for every $g^j \in \mathbf{G} \setminus \mathrm{Stab}_{\mathbf{G}}(\mathcal{U})$, it holds that $d_S(\mathcal{U}, \mathcal{U} \cdot g^j) = 2k$ or, equivalently, $\mathcal{U} \cap \mathcal{U} \cdot g^j = \{\mathbf{0}\}$. Let us see that this necessarily implies that $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$. First, suppose that $\mathrm{rk}(U_2) < k$. Then at least one of the rows of U_2 is a linear combination of the other ones. Hence, there must exist a nonzero vector $\mathbf{a} \in \mathbb{F}_q^k$ such that $\mathbf{a}U_2 = \mathbf{0}_{k+1}$. Then, for every $0 \leqslant i \leqslant q^{k+1} - 2$, one has that $\mathbf{a}U_2 M_{k+1}^i = \mathbf{0}_{k+1}$ and

$$\mathbf{x} = (\mathbf{a}U_1 \mid \mathbf{0}_{k+1}) = \mathbf{a}(U_1 \mid U_2) = \mathbf{a}(U_1 \mid U_2 M_{k+1}^i) \in \mathcal{U} \cap \mathcal{U} \cdot g^i.$$

Moreover, since the k rows of $U = (U_1 \mid U_2)$ are linearly independent and $\mathbf{a} \neq \mathbf{0}_k$, it follows that $\mathbf{x} \neq \mathbf{0}_{2k+1}$. That is a contradiction with $\mathcal{U} \cap \mathcal{U} \cdot g^j = \{\mathbf{0}\}$, for those $g^j \in \mathbf{G} \setminus \mathrm{Stab}_{\mathbf{G}}(\mathcal{U})$. Hence, it must hold that $\mathrm{rk}(U_2) = k$. Now, let us prove that U_1 has also rank k. If not, as before, there must exist a nonzero vector $\mathbf{a} \in \mathbb{F}_q^k$ such that $\mathbf{a}U_1 = \mathbf{0}_k$. Denote $\mathbf{x}_2 = \mathbf{a}U_2$, which is a nonzero vector of $\mathcal{U}_2 = \mathrm{rowsp}(U_2)$, since the rows of U_2 are linearly independent. Besides, notice that $\mathcal{U}_2$ is a hyperplane of $\mathbb{F}_q^{k+1}$. Consider now, another hyperplane (different from $\mathcal{U}_2$) containing $\mathbf{x}_2$ too. Since the Singer subgroup $\langle M_{k+1} \rangle$ acts transitively on $\mathcal{G}_q(k, k+1)$, we can write such a hyperplane as $\mathcal{U}_2 \cdot M_{k+1}^i$, for some not scalar matrix M_{k+1}^i. In particular, we have $0 \neq M_{k+1}^i + I_{k+1} = M_{k+1}^j$, for some $j \in \{0, \ldots, q^{k+1} - 2\}$. This, in turn, implies that M_{k+1}^j neither is a scalar matrix. Thus, by Theorem 2.1, necessarily $\mathcal{U}_2 \neq \mathcal{U}_2 \cdot M_{k+1}^j$. Since $\mathbf{0}_{k+1} \neq \mathbf{x}_2 \in \mathcal{U}_2 \cdot M_{k+1}^i = \mathrm{rowsp}(U_2 M_{k+1}^i)$, there must exist a nonzero vector $\mathbf{b} \in \mathbb{F}_q^k$ such that

$$\mathbf{x}_2 = \mathbf{b}U_2 M_{k+1}^i = \mathbf{b}U_2(M_{k+1}^i + I_{k+1} - I_{k+1}) = \mathbf{b}U_2 M_{k+1}^j - \mathbf{b}U_2.$$

Now, consider the vector

$$\begin{aligned}(\mathbf{a}+\mathbf{b})(U_1 \mid U_2) &= (\mathbf{a}U_1 + \mathbf{b}U_1 \mid \mathbf{a}U_2 + \mathbf{b}U_2) \\ &= (\mathbf{b}U_1 \mid \mathbf{x}_2 + \mathbf{b}U_2 M_{k+1}^j - \mathbf{x}_2) = \mathbf{b}(U_1 \mid U_2 M_{k+1}^j).\end{aligned}$$

Finally, since $\mathbf{b} \neq \mathbf{0}_k$ and the rows of $(U_1 \mid U_2 M_{k+1}^j)$ are linearly independent, we have found a nonzero vector lying on $\mathcal{U} \cap \mathcal{U} \cdot g^j$. Moreover, since $\mathcal{U}_2 \neq \mathcal{U}_2 \cdot M_{k+1}^j$, we have that $\mathcal{U} \neq \mathcal{U} \cdot g^j$. This represents a contradiction with the fact that $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})$ is a partial spread. As a result, we conclude that $\mathrm{rk}(U_1) = k$.

Conversely, assume that $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$. Let us see that $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})$ is a partial spread with $q^{k+1} - 1$ elements. To do so, given $g^i \in \mathbf{G} \setminus \{I_{2k+1}\}$, consider a vector $\mathbf{x}$ in $\mathcal{U} \cap \mathcal{U} \cdot g^i$. Hence, we can find vectors $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^k$ such that

$$\mathbf{x} = \mathbf{a}(U_1 \mid U_2) = \mathbf{b}(U_1 \mid U_2 M_{k+1}^i).$$

In particular, it holds $\mathbf{a}U_1 = \mathbf{b}U_1$ or, equivalently, $(\mathbf{a}-\mathbf{b})U_1 = \mathbf{0}_k$. Since the matrix U_1 is invertible, we conclude that $\mathbf{a} = \mathbf{b}$. In this case, we have that $\mathbf{a}U_2 = \mathbf{a}U_2M_{k+1}^i$, i.e., $\mathbf{a}U_2(M_{k+1}^i - I_{k+1}) = \mathbf{0}_{k+1}$. Notice that, since $g^i \neq I_{2k+1}$, the matrix $M_{k+1}^i - I_{k+1}$ is invertible and then, it must hold $\mathbf{a}U_2 = \mathbf{0}_{k+1}$. Moreover, as the k rows of U_2 are linearly independent, it follows $\mathbf{a} = \mathbf{0}_k$. In other words, the intersection subspace $\mathcal{U} \cap \mathcal{U} \cdot g^i$ is trivial and then $d_S(\mathcal{U}, \mathcal{U} \cdot g^i) = 2k$, for every $g^i \neq I_{2k+1}$. As a consequence $\mathrm{Stab}_{\mathbf{G}}(\mathcal{U}) = \{I_{2k+1}\}$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})$ is a partial spread of cardinality $q^{k+1} - 1$. ∎

Proof of Proposition 5.6: Assume that the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ attains the maximum possible distance, i.e., $2k$. Hence, $\mathrm{Stab}_{\mathbf{G}}(\mathcal{V})$ is a proper subgroup of $\mathbf{G}$ and, for every $g^i \in \mathbf{G} \setminus \mathrm{Stab}_{\mathbf{G}}(\mathcal{V})$, it holds $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^i) = 1$.

Let us start proving that V_2 must be an invertible matrix. To do so, arguing by contradiction, we assume that $\mathrm{rk}(V_2) < k+1$. We proceed in two steps:

(1) If $\mathrm{rk}(V_2) \leqslant k - 1 = (k+1) - 2$, then we can find, at least, two independent vectors $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^{k+1}$ such that $\mathbf{b}V_2 = \mathbf{0}_{k+1} = \mathbf{a}V_2$. In this case, both vectors

$$\begin{array}{rclclcll} \mathbf{x} & = & \mathbf{a}(V_1 \mid V_2) & = & \mathbf{a}(V_1 \mid \mathbf{0}_{k+1}) & = & \mathbf{a}(V_1 \mid V_2M_{k+1}^i) & \text{and} \\ \mathbf{y} & = & \mathbf{b}(V_1 \mid V_2) & = & \mathbf{b}(V_1 \mid \mathbf{0}_{k+1}) & = & \mathbf{b}(V_1 \mid V_2M_{k+1}^i) & \end{array}$$

lie on every subspace of $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$. In particular, $\mathbf{x}, \mathbf{y} \in \mathcal{V} \cap \mathcal{V} \cdot g^i$, for every $g^i \in \mathbf{G} \setminus \mathrm{Stab}_{\mathbf{G}}(\mathcal{V})$. Since in this case we know that $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^i) = 1$, there must happen that $\mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in \mathbb{F}_q$. Then, it holds

$$\mathbf{0}_{2k+1} = \mathbf{x} - \lambda\mathbf{y} = (\mathbf{a} - \lambda\mathbf{b})(V_1 \mid V_2).$$

Moreover, since the rows of $(V_1 \mid V_2)$ are linearly independent, we conclude that $\mathbf{a} = \lambda\mathbf{b}$, which is a contradiction with the independence of $\mathbf{a}$ and $\mathbf{b}$. Hence, it must hold $k \leqslant \mathrm{rk}(V_2) \leqslant k+1$.

(2) If $\mathrm{rk}(V_2) = k$, then the subspace $\mathcal{V}_2 = \mathrm{rowsp}(V_2)$ is a hyperplane of $\mathbb{F}_q^{k+1}$. On the other hand, since $V_1 \in \mathbb{F}_q^{(k+1)\times k}$, there must exist a nonzero vector $\mathbf{a} \in \mathbb{F}_q^{k+1}$ such that $\mathbf{a}V_1 = \mathbf{0}_k$. Moreover, notice that the vector $\mathbf{a}(V_1 \mid V_2)$ is not zero since $\mathrm{rk}(V_1 \mid V_2) = k+1$. Hence, $\mathbf{x}_2 = \mathbf{a}V_2$ is a nonzero vector in the hyperplane $\mathcal{V}_2$ of $\mathbb{F}_q^{k+1}$. Let us consider a different hyperplane of $\mathbb{F}_q^{k+1}$ containing $\mathbf{x}_2$ as well. As the action of $\langle M_{k+1}\rangle$ is transitive on $\mathcal{G}_q(k, k+1)$, such a hyperplane is of the form $\mathcal{V}_2 \cdot M_{k+1}^i$, for some not scalar matrix M_{k+1}^i. Then we can write $\mathbf{x}_2 = \mathbf{b}V_2M_{k+1}^i$ for some vector $\mathbf{b} \in \mathbb{F}_q^{k+1}$. Observe that $M_{k+1}^i + I_{k+1}$ is again a power of M_{k+1}, say $M_{k+1}^i + I_{k+1} = M_{k+1}^j$, which is not a scalar matrix too. Thus, by Theorem 2.1, $\mathcal{V}_2 \neq \mathcal{V}_2 \cdot M_{k+1}^j$ and then $\mathcal{V} \neq \mathcal{V} \cdot g^j$. Now, notice that $\mathbf{x}_2 = \mathbf{b}V_2M_{k+1}^j - \mathbf{b}V_2 = \mathbf{a}V_2$ and

$$\mathbf{x} = (\mathbf{a} + \mathbf{b})(V_1 \mid V_2) = \mathbf{b}(V_1 \mid V_2M_{k+1}^j) \in \mathcal{V} \cap \mathcal{V} \cdot g^j$$

Moreover, since $\mathbf{x}_2 \neq \mathbf{0}_{k+1}$, it follows that $\mathbf{b}V_2 \neq \mathbf{0}_{k+1}$ and then $\mathbf{b}V_2 M_{k+1}^j \neq \mathbf{0}_{k+1}$. On the other hand, since $\mathrm{rk}(V_2) = k$, there must exist a nonzero vector $\mathbf{c} \in \mathbb{F}_q^{k+1}$ such that $\mathbf{c}V_2 = \mathbf{0}_{k+1}$. Observe that, given that $\mathrm{rk}(V_1 \mid V_2) = k+1$, then $\mathbf{c}V_1 \neq \mathbf{0}_k$. Hence, the nonzero vector

$$\mathbf{y} = (\mathbf{c}V_1 \, | \mathbf{0}_{k+1}) = \mathbf{c}(V_1 \mid V_2) = \mathbf{c}(V_1 \mid V_2 M_{k+1}^j)$$

lies as well on $\mathcal{V} \cap \mathcal{V} \cdot g^j$. We conclude that, $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^j) \geqslant 2$, which contradicts the hypothesis of $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ attaining the maximum possible distance.

Hence, assuming that $d_S(\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})) = 2k$, then it must holds $\mathrm{rk}(V_2) = k+1$. We will see now that V_1 needs to have rank equal to k. To do so, we assume that $\mathrm{rk}(V_1) \leqslant k-1$. First of all, notice that if $V_1 = 0_{(k+1)\times k}$, since V_2 is an invertible matrix, one has that $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) = \{\mathcal{V}\}$ and its distance is zero. Hence, we can assume that $1 \leqslant \mathrm{rk}(V_1) \leqslant k-1$. In this situation, there exist at least two independent vectors $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^{k+1}$ such that $\mathbf{a}V_1 = \mathbf{b}V_1 = \mathbf{0}_k$. Now, since $\mathrm{rk}(V_2) = k+1$, the rows of every matrix $V_2 M_{k+1}^i$ span the whole space $\mathbb{F}_q^{k+1}$. Take $M_{k+1}^i \neq -I_{k+1}$, then the matrix $M_{k+1}^i + I_{k+1}$ is again a power of M_{k+1}, say $M_{k+1}^i + I_{k+1} = M_{k+1}^j$, for some $0 \leqslant j \leqslant q^{k+1} - 2$. Now we express both (nonzero) vectors $\mathbf{a}V_2$ and $\mathbf{b}V_2$ as linear combinations of the rows of $V_2 M_{k+1}^i$ and obtain the following equalities:

$$\begin{array}{lllllll} \mathbf{a}V_2 & = & \mathbf{c}V_2 M_{k+1}^i & = & \mathbf{c}V_2(M_{k+1}^i + I_{k+1} - I_{k+1}) & = & \mathbf{c}V_2 M_{k+1}^j - \mathbf{c}V_2, \\ \mathbf{b}V_2 & = & \mathbf{d}V_2 M_{k+1}^i & = & \mathbf{d}V_2(M_{k+1}^i + I_{k+1} - I_{k+1}) & = & \mathbf{d}V_2 M_{k+1}^j - \mathbf{d}V_2, \end{array} \tag{6.27}$$

for some nonzero vectors $\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^{k+1}$. Hence, both vectors

$$\begin{array}{lllllll} \mathbf{x} & = & (\mathbf{a}+\mathbf{c})(V_1 \mid V_2) & = & (\mathbf{c}V_1 \mid \mathbf{a}V_2 + \mathbf{c}V_2) & = & \mathbf{c}(V_1 \mid V_2 M_{k+1}^j), \\ \mathbf{y} & = & (\mathbf{b}+\mathbf{d})(V_1 \mid V_2) & = & (\mathbf{d}V_1 \mid \mathbf{b}V_2 + \mathbf{d}V_2) & = & \mathbf{d}(V_1 \mid V_2 M_{k+1}^j), \end{array}$$

lie on $\mathcal{V} \cap \mathcal{V} \cdot g^j$. Let us see that they are independent. Otherwise, $\mathbf{x} = \lambda \mathbf{y}$, for some $\lambda \in \mathbb{F}_q$ and it must hold

$$\mathbf{0}_{k+1} = (\mathbf{c} - \lambda \mathbf{d}) V_2 M_{k+1}^j.$$

Since $V_2 M_{k+1}^j$ is invertible, we conclude that $\mathbf{c} = \lambda \mathbf{d}$. As a result, by (6.27), we obtain

$$\mathbf{a}V_2 = \mathbf{c}V_2 M_{k+1}^i = \lambda \mathbf{d} V_2 M_{k+1}^i = \lambda \mathbf{b} V_2$$

and, given that $\mathrm{rk}(V_2) = k+1$, it follows $\mathbf{a} = \lambda \mathbf{b}$, which is not possible. Hence, $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^j) \geqslant 2$. Moreover, let us see that $\mathcal{V} \neq \mathcal{V} \cdot g^j$. Note that $\dim(\mathcal{V} + \mathcal{V} \cdot g^j)$ is exactly the rank

$$\mathrm{rk}\left(\begin{array}{c|c} V_1 & V_2 \\ \hline V_1 & V_2 M_{k+1}^j \end{array}\right) = \mathrm{rk}\left(\begin{array}{c|c} V_1 & V_2 \\ \hline 0_{(k+1)\times k} & V_2 M_{k+1}^i \end{array}\right) \geqslant \mathrm{rk}(V_1) + \mathrm{rk}(V_2 M_{k+1}^i) \geqslant k+2,$$

which is greater than $\dim(\mathcal{V}) = \dim(\mathcal{V} \cdot g^j) = k+1$. Hence, $\mathcal{V} \neq \mathcal{V} \cdot g^j$ and we obtain a contradiction with $d_S(\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})) = 2k$. Then, it must happen $\mathrm{rk}(V_1) = k$ and $\mathrm{rk}(V_2) = k+1$.

Now, let us prove that the converse is also true. Assume that $\mathrm{rk}(V_1) = k$ and $\mathrm{rk}(V_2) = k+1$ and take $g^i \in \mathbf{G} \setminus \{I_{2k+1}\}$. We show that $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^i) = 1$. To do so, consider two arbitrary vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V} \cap \mathcal{V} \cdot g^i$ and write them as

$$\begin{array}{rclcl} \mathbf{x} & = & \mathbf{a}(V_1 \mid V_2) & = & \mathbf{c}(V_1 \mid V_2 M_{k+1}^i) \text{ and} \\ \mathbf{y} & = & \mathbf{b}(V_1 \mid V_2) & = & \mathbf{d}(V_1 \mid V_2 M_{k+1}^i), \end{array}$$

for some vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{F}_q^{k+1}$. Observe that $(\mathbf{a}-\mathbf{c})V_1 = (\mathbf{b}-\mathbf{d})V_1 = \mathbf{0}_k$. Since $V_1 \in \mathbb{F}_q^{(k+1)\times k}$ with $\mathrm{rk}(V_1) = k$, it follows that $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{d}$ must be proportional vectors. Let us write $(\mathbf{a}-\mathbf{c}) = \lambda(\mathbf{b}-\mathbf{d})$, for some $\lambda \in \mathbb{F}_q$. Equivalently, $\mathbf{a} - \lambda\mathbf{b} = \mathbf{c} - \lambda\mathbf{d}$. Moreover, since $\mathbf{a}V_2 = \mathbf{c}V_2 M_{k+1}^i$ and $\mathbf{b}V_2 = \mathbf{d}V_2 M_{k+1}^i$, we obtain

$$(\mathbf{a} - \lambda\mathbf{b})V_2 = (\mathbf{c} - \lambda\mathbf{d})V_2 M_{k+1}^i = (\mathbf{a} - \lambda\mathbf{b})V_2 M_{k+1}^i,$$

or, equivalently, $(\mathbf{a} - \lambda\mathbf{b})V_2(M_{k+1}^i - I_{k+1}) = \mathbf{0}_{k+1}$. Last, since $g^i \neq I_{2k+1}$, then $V_2(M_{k+1}^i - I_{k+1})$ is an invertible matrix and it follows $\mathbf{a} = \lambda\mathbf{b}$ and $\mathbf{x} = \lambda\mathbf{y}$. As a result, for every $g^i \in \mathbf{G} \setminus \{I_{2k+1}\}$, it occurs $\dim(\mathcal{V} \cap \mathcal{V} \cdot g^i) = 1$ or, equivalently, $d_S(\mathcal{V}, \mathcal{V} \cdot g^i) = 2k$. Consequently, $\mathrm{Stab}_{\mathbf{G}}(\mathcal{V}) = \{I_{2k+1}\}$ and $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ has maximum distance and cardinality $q^{k+1} - 1$. ∎

Proof of Proposition 5.9: Since $\mathrm{rk}(U_1) = \mathrm{rk}(U_2) = k$, Proposition 5.5 proves that the orbit $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U})$ is a k-partial spread of $\mathbb{F}_q^{2k+1}$. Hence, it suffices to see that adding the two subspaces $\mathcal{U}'$ and $\mathcal{U}''$ defined in (6.24) does not decrease the distance. Observe that two k-dimensional subspaces in $\mathbb{F}_q^{2k+1}$ attain the maximum possible distance if, and only if, they intersect trivially or, equivalently, if their sum subspace has dimension $2k$. It is clear that

$$\dim(\mathcal{U} \cdot g^i + \mathcal{U}') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^i \\ \hline U_1 & 0_{k\times(k+1)} \end{array}\right) = \mathrm{rk}\left(\begin{array}{c|c} 0_{k\times k} & U_2 M_{k+1}^i \\ \hline U_1 & 0_{k\times(k+1)} \end{array}\right) = 2k,$$

$$2k \geqslant \dim(\mathcal{U} \cdot g^i + \mathcal{U}'') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^i \\ \hline 0_k & U_2 \end{array}\right) \geqslant \mathrm{rk}(U_1) + \mathrm{rk}(U_2) = 2k$$

and

$$\dim(\mathcal{U}' + \mathcal{U}'') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & 0_{k\times(k+1)} \\ \hline 0_k & U_2 \end{array}\right) = \mathrm{rk}(U_1) + \mathrm{rk}(U_2) = 2k.$$

As a result, in these three situations we obtain distance equal to $2k$ and we conclude that the code $\mathrm{Orb}_{\mathbf{G}}(\mathcal{U}) \cup \{\mathcal{U}', \mathcal{U}''\}$ is a partial spread with $q^{k+1} + 1$ elements. ∎

Proof of Proposition 5.10: Notice that, by virtue of Proposition 5.6, we have that $d_S(\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})) = 2k$. Hence, in order to give the minimum distance of the code

$\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}$, we just need to compute distances between pairs of different subspaces not both in $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$. Moreover, recall that two $(k+1)$-dimensional subspaces of $\mathbb{F}_q^{2k+1}$ have the maximum possible distance if, and only if, their sum is the whole vector space $\mathbb{F}_q^{2k+1}$.

Let us start proving that taking $\mathbf{v}_1 = \mathbf{0}_k$ leads to a construction of maximum distance. To do so, just notice that

$$\begin{aligned}\dim(\mathcal{V} \cdot g^i + \mathcal{V}') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^i \\ \hline \mathbf{0}_k & \mathbf{v}_2 M_{k+1}^i \\ \hline U_1 & 0_{k\times(k+1)} \\ \hline \mathbf{0}_k & \mathbf{v}_2 \end{array}\right) &= \mathrm{rk}\left(\begin{array}{c|c} 0_{k\times k} & U_2 M_{k+1}^i \\ \hline \mathbf{0}_k & \mathbf{v}_2 M_{k+1}^i \\ \hline U_1 & 0_{k\times(k+1)} \end{array}\right) \\ &= \mathrm{rk}(U_1) + \mathrm{rk}(V_2 M_{k+1}^i) = 2k+1,\end{aligned}$$

$$\dim(\mathcal{V} \cdot g^i + \mathcal{V}'') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^i \\ \hline \mathbf{0}_k & \mathbf{v}_2 M_{k+1}^i \\ \hline 0_{(k+1)\times k} & U_2 \\ \hline \mathbf{0}_k & \mathbf{v}_2 \end{array}\right) \geqslant \mathrm{rk}(U_1) + \mathrm{rk}(V_2) = 2k+1$$

and

$$\dim(\mathcal{V}' + \mathcal{V}'') = \mathrm{rk}\left(\begin{array}{c|c} U_1 & 0_{k\times(k+1)} \\ \hline \mathbf{0}_k & \mathbf{v}_2 \\ \hline 0_{(k+1)\times k} & U_2 \\ \hline \mathbf{0}_k & \mathbf{v}_2 \end{array}\right) = \mathrm{rk}(U_1) + \mathrm{rk}(V_2) = 2k+1.$$

Hence, we obtain that $\mathcal{V} \cdot g^i + \mathcal{V}' = \mathcal{V} \cdot g^i + \mathcal{V}'' = \mathcal{V}' + \mathcal{V}'' = \mathbb{F}_q^{2k+1}$, for every $0 \leqslant i \leqslant q^{k+1} - 2$. Consequently, the distance of the code is the maximum possible one. In particular, $\mathcal{V}'$ and $\mathcal{V}''$ are different and they do not lie in the orbit of $\mathcal{V}$. Thus, it follows $|\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}| = q^{k+1} + 1$, i.e., the largest size for its distance.

Conversely, we show that taking $\mathbf{v}_1 = \mathbf{0}_k$ is the only possibility for $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}$ to attain the maximum distance. To do so, assume that $\mathbf{v}_1$ is a nonzero vector of $\mathbb{F}_q^k$. Hence, since $\mathrm{rk}(U_1) = k$, there exists a nonzero vector $\mathbf{a} \in \mathbb{F}_q^k$ such that $\mathbf{v}_1 = \mathbf{a} U_1$. We will exhibit a explicit subspace $\mathcal{V} \cdot g^j \in \mathrm{Orb}_{\mathbf{G}}(\mathcal{V})$ such that $d_S(\mathcal{V} \cdot g^j, \mathcal{V}'') < 2k$.

Since $U_2 \in \mathbb{F}_q^{k\times(k+1)}$ has $\mathrm{rk}(U_2) = k$, the subspace $\mathcal{U}_2 = \mathrm{rowsp}(U_2)$ is a hyperplane of $\mathbb{F}_q^{k+1}$ and, as $\mathrm{rk}(V_2) = k+1$, it follows that $\mathbf{v}_2 \notin \mathcal{U}_2$. In particular, $\mathbf{v}_2$ and $\mathbf{a} U_2$ are linearly independent vectors and so they are $\mathbf{v}_2$ and $\mathbf{v}_2 - \mathbf{a} U_2$. Consider a hyperplane of $\mathbb{F}_q^{k+1}$ containing the vector $\mathbf{v}_2 - \mathbf{a} U_2$ but not $\mathbf{v}_2$. Recall that, since the Singer subgroup generated by M_{k+1} acts transitively on $\mathcal{G}_q(k, k+1)$ (see Theorem 2.1), such a hyperplane is of the form $\mathcal{U}_2 \cdot M_{k+1}^i$, for some $i \in \{0, \dots, q^{k+1} - 2\}$. Observe that, for this choice of i, we have $\mathbf{v}_2 - \mathbf{a} U_2 \in \mathcal{U}_2 \cdot M_{k+1}^i$ and $\mathbf{v}_2 \notin \mathcal{U}_2 \cdot M_{k+1}^i$. Thus, at this point, we have that

$$(\mathbf{v}_2 - \mathbf{a} U_2) M_{k+1}^{-i} \in \mathcal{U}_2 \quad \text{and} \quad \mathbf{v}_2,\ \mathbf{v}_2 M_{k+1}^{-i} \notin \mathcal{U}_2.$$

On the other hand, as $V_2 \in \mathrm{GL}(k+1, q)$, every vector in $\mathbb{F}_q^{k+1}$ can be written as a linear combination of its rows, which are the ones of U_2 together with $\mathbf{v}_2$. In particular, there must exist $\lambda \in \mathbb{F}_q$ and $\mathbf{b} \in \mathbb{F}_q^k$ such that $\mathbf{v}_2 M_{k+1}^{-i} = \lambda \mathbf{v}_2 + \mathbf{b} U_2$. Moreover, since $\mathbf{v}_2 M_{k+1}^{-i} \notin \mathcal{U}_2$, it follows $\lambda \neq 0$. Then the matrix $\lambda^{-1} M_{k+1}^{-i} \in \mathbb{F}_q[M_{k+1}]$ is a power of M_{k+1} and we can write $\lambda^{-1} M_{k+1}^{-i} = M_{k+1}^{j}$, for certain exponent $j \in \{0, \ldots, q^{k+1}-2\}$. Observe that, for this matrix M_{k+1}^j, we have that $\mathbf{v}_2 M_{k+1}^j - \mathbf{v}_2 \in \mathcal{U}_2$ and also $(\mathbf{v}_2 - \mathbf{a} U_2) M_{k+1}^j \in \mathcal{U}_2$. As a result, their difference, i.e., the vector $\mathbf{v}_2 - \mathbf{a} U_2 M_{k+1}^j \in \mathcal{U}_2$ as well. Now, consider the subspace $\mathcal{V} \cdot g^j$. Let us see that $d_S(\mathcal{V} \cdot g^j, \mathcal{V}'')$ is not the maximum one or, equivalently, that $\mathcal{V} \cdot g^j + \mathcal{V}'' \neq \mathbb{F}_q^{2k+1}$. Observe that

$$
\begin{aligned}
\dim(\mathcal{V} \cdot g^j + \mathcal{V}'') &= \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^j \\ \hline \mathbf{v}_1 & \mathbf{v}_2 M_{k+1}^j \\ \hline 0_{(k+1)\times k} & U_2 \\ \hline \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right) \\
&= \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^j \\ \hline \mathbf{0}_k & \mathbf{v}_2 M_{k+1}^j - \mathbf{a} U_2 M_{k+1}^j \\ \hline 0_{(k+1)\times k} & U_2 \\ \hline \mathbf{0}_k & \mathbf{v}_2 - \mathbf{a} U_2 M_{k+1}^j \end{array}\right) \\
&= \mathrm{rk}\left(\begin{array}{c|c} U_1 & U_2 M_{k+1}^j \\ \hline 0_{(k+1)\times k} & U_2 \end{array}\right) \\
&= \dim(\mathcal{U} \cdot g^j + \mathcal{U}'') = 2k.
\end{aligned}
$$

Since $k > 1$, it follows that $2k > k+1$. Consequently, for this precise value of j, it holds that $\mathcal{V} \cdot g^j \neq \mathcal{V}''$ and $d_S(\mathcal{V} \cdot g^j, \mathcal{V}'') < 2k$. We conclude that $\mathrm{Orb}_{\mathbf{G}}(\mathcal{V}) \cup \{\mathcal{V}', \mathcal{V}''\}$ does not attain the maximum distance. ∎

BIBLIOGRAPHY

[1] R. Ahlswede, N. Cai, R. Li and R. W. Yeung, *Network Information Flow*, IEEE Transactions on Information Theory, Vol. 46 (2000), 1204-1216.

[2] C. Alonso-González and M. A. Navarro-Pérez, *Cyclic Orbit Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2331–2356.

[3] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An Orbital Construction of Optimum Distance Flag Codes*, Finite Fields and Their Applications, Vol. 73 (2021), 101861.

[4] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes from Planar Spreads in Network Coding*, Finite Fields and Their Applications, Vol. 68 (2020), 101745.

[5] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.

[6] Th. Beth, D. Jungnickel and H. Lenz, *Desing Theory*, Cambridge University Press, Cambridge, 1999.

[7] K. Drudge, *On the Orbits of Singer Groups and Their Subgroups*, The Electronic Journal of Combinatorics, Vol. 9 (2002), research paper 15.

[8] H. Gluesing-Luerssen, K. Morrison and C. Troha, *Cyclic Orbit Codes and Stabilizer Subfields*, Advances in Mathematics of Communications, Vol. 9 (2) (2015), 177-197.

[9] E. Gorla, F. Manganiello and J. Rosenthal, *An Algebraic Approach for Decoding Spread Codes*, Advances in Mathematics of Communications, Vol. 6 (4) (2012), 443-466.

[10] E. Gorla and A. Ravagnani, *Partial Spreads in Random Network Coding*, Finite Fields and Their Applications, Vol. 26 (2014), 104-115.

[11] M. D. Hestenes, *Singer Groups*, Canadian Journal of Mathematics, Vol. XXII (3) (1970), 492-513.

[12] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[13] A.-L. Horlemann-Trautmann, *Message Encoding and Retrieval for Spread and Cyclic Orbit Codes*, Designs, Codes and Cryptography, Vol. 86 (2018), 365-386.

[14] B. Huppert, *Endliche Gruppen*, Vol. I, Springer-Verlag, New York, 1967.

[15] A. Kohnert and S. Kurz, *Construction of Large Constant Dimension Codes with a Prescribed Minimum Distance*, in: J. Calmet *et al.* (Eds.), Mathematical Methods in Computer Science. Lecture Notes in Computer Science, Vol. 5393. Springer, Berlin, Heidelberg, 2008, pp. 31-42.

[16] R. Koetter and F. Kschischang, *Coding for Errors and Erasures in Random Network Coding*, in: IEEE Transactions on Information Theory, Vol. 54 (2008), 3579-3591.

[17] S. Kurz, *Bounds for Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2759–2785.

[18] M. Lavrauw and G. Van de Voorde, *Field Reduction and Linear Sets in Finite Geometry*, Contemporary Mathematics, Vol. 632 (2015), 271-293.

[19] R. Lidl and H. Niederreiter, *Introduction to Finite Fields and their Applications*, Cambridge University Press, Cambridge, London, 1994. Revised edition.

[20] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Network Coding with Flags*, Designs, Codes and Cryptography, Vol. 86(2) (2018), 269-284.

[21] F. Manganiello, E. Gorla and J. Rosenthal, *Spread Codes and Spread Decoding in Network Coding*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), Toronto, Canada, 2008, pp. 851-855.

[22] F. Manganiello, A.-L. Trautmann and J. Rosenthal, *On Conjugacy Classes of Subgroups of the General Linear Group and Cyclic Orbit Codes*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), St. Petersburg, Russia, 2011, pp. 1916-1920.

[23] R. W. Nóbrega and B. F. Uchôa-Filho, *Multishot Codes for Network Coding: Bounds and a Multilevel Construction*, in: 2009 IEEE International Symposium on Information Theory, Proceedings (ISIT), Seoul, South Korea, 2009, pp. 428-432.

[24] J. Rosenthal and A.-L. Trautmann, *A Complete Characterization of Irreducible Cyclic Orbit codes and their Plücker Embedding*, Designs, Codes and Cryptography, Vol. 66 (2013), 275-289.

[25] B. Segre, *Teoria di Galois, Fibrazioni Proiettive e Geometrie non Desarguesiane*, Annali di Matematica Pura ed Applicata, Vol. 64 (1964), 1-76.

[26] A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal, *Cyclic Orbit Codes*, in: IEEE Transactions on Information Theory, Vol. 59 (11) (2013), 7386-7404.

[27] A.-L. Trautmann, F. Manganiello and J. Rosenthal, *Orbit Codes - A New Concept in the Area of Network Coding*, in: 2010 IEEE Information Theory Workshop, ITW 2010 - Proceedings, Dublin, Ireland, 2010, pp. 1-4.

[28] A.-L. Trautmann and J. Rosenthal, *Constructions of Constant Dimension Codes*, in: M. Greferath *et al.* (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 25-42.

[29] G. Van de Voorde, *Desarguesian Spreads and Field Reduction for Elements of the Semilinear Group*, Linear Algebra and its Applications, Vol. 507 (2016), 96-120.

CHAPTER 7

ON GENERALIZED GALOIS CYCLIC ORBIT FLAG CODES

Joint work with Clementa Alonso-González.

Abstract:

Flag codes that are orbits of a cyclic subgroup of the general linear group acting on flags of a vector space over a finite field, are called *cyclic orbit flag codes*. In this paper we present a new contribution to the study of such codes started in [3], by focusing this time on the generating flag. More precisely, we examine those ones whose generating flag has at least one subfield among its subspaces. In this situation, two important families arise: the already known *Galois flag codes*, in case we have just fields, or the *generalized Galois flag codes* in other case. We investigate the parameters and properties of the latter ones and explore the relationship with their underlying Galois flag code.
Keywords: Network coding, flag codes, cyclic orbit flag codes.

1 Introduction

Network coding represents a procedure to data transfer within a network that is a directed multigraph without cycles, where the information travels from one or several senders to several receivers. In [1], it was proved that one can improve the efficiency if the intermediate nodes can linearly combine the information vectors. We speak about *random network coding* whenever the underlying network topology is unknown. Due to the fact that vector subspaces are invariant under linear combinations, they are proposed as suitable codewords in [13], giving raise to the concept of *subspace codes*. When all the subspaces have a fixed dimension, we get *constant dimension codes*. Research in this area has been very intense in latter years (consult [26] and references inside).

One method devised in [25] to build subspace codes consists of making subgroups of the general linear group $\mathrm{GL}(n,q)$ act on the set of subspaces of $\mathbb{F}_q^n$ and thus, consider the corresponding orbits. This idea leads to the concept of *orbit codes*. In particular, when the acting group is cyclic, we obtain the so-called *cyclic orbit codes*, widely studied in the last times (see [8, 9, 10, 11, 20, 22, 24, 25, 27]CHAP7, for instance). Of special relevance for our purposes is the paper [11], where the authors treat β-*cyclic orbit codes* as collections of $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_{q^n}$ that are orbits under the natural action of a subgroup $\langle\beta\rangle$ of $\mathbb{F}_{q^n}^*$ on $\mathbb{F}_q$-vector spaces (if β is primitive, the corresponding orbit is called just *cyclic orbit code*). In that paper, it is introduced an interesting tool to analyze β-cyclic orbit codes: the *best friend* of the code, that is, the largest subfield of $\mathbb{F}_{q^n}$ over which the *generating subspace* is a vector space.

Flag codes can be seen as a generalization of constant dimension codes. The codewords of a flag code are flags, that is, sequences of chained subspaces of prescribed dimensions. In the network coding context, they appeared for the first time in the paper [15] where the multiplicative action of $\mathrm{GL}(n,q)$ is translated from subspaces to flags to provide different constructions of *orbit flag codes*. This seminal work has sparked an incipient interest in flag codes reflected in the recent works [2, 3, 4, 5, 6, 7, 14, 19].

In [3], the authors undertake the study of *β-cyclic orbit flag codes* inspired by the ideas in [11]. More precisely, they consider flags on $\mathbb{F}_{q^n}$ given by nested $\mathbb{F}_q$-subspaces of the field $\mathbb{F}_{q^n}$ constructed as orbits of subgroups $\langle\beta\rangle \subseteq \mathbb{F}_{q^n}^*$, and coin the concept of *best friend* of a cyclic flag code as the largest subfield of $\mathbb{F}_{q^n}$ over which every subspace in the *generating flag* is a vector space. The knowledge of the best friend turns out to be extremely useful to determine the parameters of the code as well as other features such as the necessary conditions on the type vector to reach the maximum distance. It is also presented the particular family of *Galois flag codes*, that consists of β-cyclic orbit flag codes generated by flags given by nested fields. For that class of codes it is possible to precisely establish a nice correspondence between the set of attainable distances and the subgroups of $\mathbb{F}_{q^n}^*$.

In the current paper we extend the study performed in [3] by focusing on the generating flag. More precisely, we examine β-cyclic orbit flag codes whose generating flag has at least one subfield among its subspaces. We distinguish two situations: either all the subspaces are fields, then we have the already known *Galois flag codes*, or there is also at least one subspace not being a field. The last case entails the definition of a new kind of β-cyclic orbit flag codes called *generalized Galois flag codes*. We discuss the properties of this new class of codes by taking into account that a generalized Galois flag code has always an underlying Galois flag code that influences on it to a greater or lesser extent.

The text is structured as follows. In section 2 we remember the basics on subspace codes as well as some notions and results related to cyclic orbit (subspace) codes developed in [11]. In Section 3 we recall, on the one hand, some background on flag codes and the most important facts on cyclic orbit flag codes that appear in [3]. On the other hand, we present some new results on the interplay between type vectors, best friend and the flag distance parameter. The family of generalized β-Galois flag codes is introduced here as an extension of the β-Galois flag codes. We discuss in which way the properties of a generalized Galois flag code are driven by its underlying Galois flag code and launch a related question. Section 4 is devoted to provide a systematic construction of generalized β-Galois flag codes with a prescribed underlying β-Galois flag code by using generating flags written in a precise regular form. In Subsections 4.1, 4.2 and 4.3 we analyze the particular properties of the previous construction in case β is primitive, and propose a decoding algorithm over the erasure channel taking advantage of such properties. In Subsection 4.4 we address the case when β is not primitive and present some specific results. To finish, we show how our construction allows us to give an answer to the question previously formulated.

2 Preliminaries

Consider q a prime power and $\mathbb{F}_q$ the finite field with q elements. We denote by $\mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$ for any natural number $n \geqslant 1$ and by $\mathcal{P}_q(n)$ the set of all the subspaces of $\mathbb{F}_q^n$. For every $0 \leqslant k \leqslant n$, the set of k-dimensional subspaces of $\mathbb{F}_q^n$, that is, the *Grassmannian*, will be denoted by $\mathcal{G}_q(k,n)$. The set $\mathcal{P}_q(n)$ can be equipped with a metric called the *subspace distance*: for any pair $\mathcal{U}, \mathcal{V} \in \mathcal{P}_q(n)$, we set

$$d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}). \tag{7.1}$$

In particular, the subspace distance between two subspaces $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(k,n)$ becomes

$$d_S(\mathcal{U}, \mathcal{V}) = 2(k - \dim(\mathcal{U} \cap \mathcal{V})). \tag{7.2}$$

A *constant dimension code* $\mathcal{C}$ of dimension k and length n is a nonempty subset of $\mathcal{G}_q(k,n)$ whose *minimum subspace distance* is given by

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U}, \mathcal{V}) \mid \mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\}.$$

If $|\mathcal{C}| = 1$, we put $d_S(\mathcal{C}) = 0$. For further details on this family of codes, consult [26] and the references inside.

It is clear that the minimum distance of a constant dimension code $\mathcal{C} \subseteq \mathcal{G}_q(k,n)$ is attained when the intersection of every pair of codewords has the minimum possible dimension. In this case, we have that

$$d_S(\mathcal{C}) \leqslant \begin{cases} 2k & \text{if } \ 2k \leqslant n, \\ 2(n-k) & \text{if } \ 2k > n. \end{cases} \tag{7.3}$$

A constant dimension code with dimension $k \leqslant \lfloor \frac{n}{2} \rfloor$ attaining the previous bound is called *partial spread code*. A partial spread code being also a partition of $\mathbb{F}_q^n$ into k-dimensional subspaces is known as a *spread code* or just a *k-spread*. In [23] it is proved that k-spread exist if, and only if, k divides n. As a result, the size of any k-spread is exactly $\frac{q^n-1}{q^k-1}$. See [12, 16, 17, 26] for more information concerning spread codes in the network coding setting.

Among all the special families of constant dimension codes, here we are interested in *orbit codes*, that is, those that arise as orbits of the action of subgroups of the general linear group $\mathrm{GL}(n,q)$ on the Grassmannian. This family of codes was introduced in [25]. More precisely, fixed a k-dimensional subspace $\mathcal{U} \subset \mathbb{F}_q^n$ and a subgroup $G \subseteq \mathrm{GL}(n,q)$, the orbit of $\mathcal{U}$ under the action of G is the constant dimension code given by $\mathrm{Orb}_G(\mathcal{U}) = \{\mathcal{U} \cdot A \mid A \in G\}$, where $\mathcal{U} \cdot A = \mathrm{rowsp}(UA)$, for any full-rank generator matrix U of $\mathcal{U}$. The *stabilizer* of $\mathcal{U}$ under the action of G is the subgroup $\mathrm{Stab}_G(\mathcal{U}) = \{A \in G \mid \mathcal{U} \cdot A = \mathcal{U}\}$. As a consequence,

$$|\mathrm{Orb}_G(\mathcal{U})| = \frac{|G|}{|\mathrm{Stab}_G(\mathcal{U})|} \tag{7.4}$$

and the minimum distance can be computed as

$$d_S(\mathrm{Orb}_G(\mathcal{U})) = \min\{d_s(\mathcal{U},\mathcal{U}\cdot A) \mid A \in G \setminus \mathrm{Stab}_G(\mathcal{U})\}.$$

Whenever the acting group G is cyclic, the orbit $\mathrm{Orb}_G(\mathcal{U})$ is called *cyclic orbit code*. The works [11, 18, 21, 24] are devoted to the study of this family of codes. In the current paper we are especially interested in the viewpoint developed in [24] and [11] where, taking advantage of the natural $\mathbb{F}_q$-linear isomorphism between $\mathbb{F}_q^n$ and $\mathbb{F}_{q^n}$, cyclic orbit codes are seen as collections of subspaces in $\mathbb{F}_{q^n}$. More precisely, in [11] the authors consider a nonzero element β and define β*-cyclic orbit codes* as orbits of the group $\langle\beta\rangle$ on $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_{q^n}$. In particular, if $1 \leqslant k < n$ and $\mathcal{U} \subset \mathbb{F}_{q^n}$ is a k-dimensional subspace over $\mathbb{F}_q$, the β*-cyclic orbit code generated by* $\mathcal{U}$ is the following set of $\mathbb{F}_q$-subspaces of dimension k

$$\mathrm{Orb}_\beta(\mathcal{U}) = \{\mathcal{U}\beta^i \mid 0 \leqslant i \leqslant |\beta| - 1\},$$

where $|\beta|$ denotes the *multiplicative order* of β. The *stabilizer* of the subspace $\mathcal{U}$ under the action of $\langle\beta\rangle$ is the cyclic subgroup $\mathrm{Stab}_\beta(\mathcal{U}) = \{\beta^i \mid \mathcal{U}\beta^i = \mathcal{U}\}$. An important example of such kind of codes, already developed in [24], is the following k-spread code, where k is a divisor of n and α is a primitive element of $\mathbb{F}_{q^n}$:

$$\mathrm{Orb}_{\langle\alpha\rangle}(\mathbb{F}_{q^k}) = \{\mathbb{F}_{q^k}\alpha^i \mid i = 0, \ldots, q^n - 2\}. \tag{7.5}$$

Remark 2.1. Following the notation used in [11], when the acting group is $\mathbb{F}_{q^n}^*$, we simply denote the corresponding orbit by $\mathrm{Orb}(\mathcal{U})$ and call it just the *cyclic orbit code generated by* $\mathcal{U}$. In this situation, we also remove the subscript β and write $\mathrm{Stab}(\mathcal{U})$ to denote the stabilizer of $\mathcal{U}$.

Concerning the cardinality and distance of a β-cyclic orbit code, in [11] the authors study these parameters with the aid of the *best friend* of the generating subspace. This concept is closely linked to the stabilizer of the subspace. Let us recall the definition.

Definition 2.2. A subfield $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^n}$ is said to be a *friend* of a subspace $\mathcal{U} \subset \mathbb{F}_{q^n}$ if $\mathcal{U}$ is an $\mathbb{F}_{q^m}$-vector space. The largest friend of $\mathcal{U}$ is called its *best friend*.

The knowledge of the best friend of a subspace $\mathcal{U}$ provides straightforwardly the cardinality of the cyclic orbit code as well as a lower bound for its distance.

Proposition 2.3. *([11, Prop. 3.3, 3.12, 3.13 and 4.1]) Let $\mathcal{U}$ be a subspace of $\mathbb{F}_{q^n}$ with the subfield $\mathbb{F}_{q^m}$ as its best friend. Then*

$$|\mathrm{Orb}(\mathcal{U})| = \frac{q^n - 1}{q^m - 1}.$$

Moreover, the value $2m$ divides the distance between every pair of subspaces in $\mathrm{Orb}(\mathcal{U})$ and, hence, we have that $d_S(\mathrm{Orb}(\mathcal{U})) \geqslant 2m$.

We finish this section recalling a construction of cyclic orbit codes with prescribed distance and cardinality from the choice of a subspace $\mathcal{U}$ written in a specific regular form.

Proposition 2.4. *([11, Prop. 4.3]) Consider the subspace $\mathcal{U} = \bigoplus_{i=0}^{t-1} \mathbb{F}_{q^m}\alpha^{li}$ for some $1 \leqslant l < \frac{q^n-1}{q^m-1}$ such that $\mathbb{F}_{q^m}$ is the best friend of $\mathcal{U}$. Then $d_S(\mathrm{Orb}(\mathcal{U})) = 2m$.*

It is clear that the subfield $\mathbb{F}_{q^m}$ is a friend of a subspace $\mathcal{U}$ written as in previous proposition, although it is not necessarily its best friend. In fact, there are just two possibilities for the best friend of $\mathcal{U}$.

Proposition 2.5. *([11, Proposition 4.4]) Given the subspace $\mathcal{U} = \bigoplus_{i=0}^{t-1} \mathbb{F}_{q^m}\alpha^{li}$ for some $1 \leqslant l < \frac{q^n-1}{q^m-1}$. If $f(x)$ is the minimal polynomial of α^l over $\mathbb{F}_{q^m}$, then its degree is at least t and*

$$\mathcal{U} = \mathbb{F}_{q^{mt}} \;\Leftrightarrow\; \deg(f) = t \;\Leftrightarrow\; \alpha^l \in \mathrm{Stab}(\mathcal{U}) \;\Leftrightarrow\; \mathbb{F}_{q^m} \textit{ is not the best friend of } \mathcal{U}.$$

We will come back to this family of subspaces and to the β-cyclic orbit codes generated by them in Section 4, where we provide a specific construction of generalized Galois cyclic orbit flag codes by using subspaces written as in Proposition 2.4.

3 Cyclic orbit flag codes

Part of this section is dedicated to gather the basic background on flag codes that already appears in [6, 7, 15], and to recall the main definitions and results that pertain to the particular class of cyclic orbit flag codes introduced in [3]. In Subsection 3.3, we present new results concerning the interdependence between the minimum distance of a β-cyclic orbit flag code, its best friend and the set of dimensions appearing in the type vector. In addition, the class of generalized Galois flag codes is introduced in Subsection 3.4.

3.1 Flags and flag codes

A *flag* $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ on the extension field $\mathbb{F}_{q^n}$ is a sequence of nested $\mathbb{F}_q$-vector subspaces

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_{q^n}.$$

The subspace $\mathcal{F}_i$ is called the *i-th subspace* of $\mathcal{F}$ and the *type* of $\mathcal{F}$ is the vector $(\dim(\mathcal{F}_1), \ldots, \dim(\mathcal{F}_r))$. When the type vector is $(1, 2, \ldots, n-1)$, we say that $\mathcal{F}$ is a *full flag*. Given two different flags $\mathcal{F}$, $\mathcal{F}'$ on $\mathbb{F}_{q^n}$, we say that $\mathcal{F}'$ is a *subflag* of $\mathcal{F}$ if each subspace of $\mathcal{F}'$ is a also subspace of $\mathcal{F}$.

The *flag variety* of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_{q^n}$ is the set of flags of this type and will be denoted by $\mathcal{F}_q((t_1, \ldots, t_r), n)$. Note that $\mathcal{F}_q((t_1, \ldots, t_r), n)$ embeds in the product of Grassmannians $\mathcal{G}_q(t_1, n) \times \cdots \times \mathcal{G}_q(t_r, n)$ and, hence, this variety can be endowed with a metric that extends the subspace distance defined in (7.1). Given two flags $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \dots, \mathcal{F}'_r)$ in $\mathcal{F}_q((t_1, \dots, t_r), n)$, their *flag distance* is

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i).$$

Definition 3.1. A *flag code* $\mathcal{C}$ of type $(t_1, \dots, t_r)$ on $\mathbb{F}_{q^n}$ is a nonempty subset of $\mathcal{F}_q((t_1, \dots, t_r), n)$. The *minimum distance* of $\mathcal{C}$ is given by

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}.$$

whenever $\mathcal{C}$ has more that two elements. In case $|\mathcal{C}| = 1$, we put $d_f(\mathcal{C}) = 0$. For type $(t_1, \dots, t_r)$, it always holds

$$d_f(\mathcal{C}) \leqslant 2 \left(\sum_{t_i \leqslant \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right). \tag{7.6}$$

There are constant dimension codes intrinsically correlated with a flag code $\mathcal{C}$ that play an important role in the study of parameters and properties of $\mathcal{C}$.

Definition 3.2. Given a flag code $\mathcal{C}$ of type $(t_1, \dots, t_r)$, the *i-projected code* of $\mathcal{C}$ is the set

$$\mathcal{C}_i = \{\mathcal{F}_i \mid (\mathcal{F}_1, \dots, \mathcal{F}_i, \dots, \mathcal{F}_r) \in \mathcal{C}\} \subseteq \mathcal{G}(t_i, n).$$

Remark 3.3. Concerning the relationship between the size of a flag code and the ones of its projected codes, it is clear that $|\mathcal{C}_i| \leqslant |\mathcal{C}|$ for every $i = 1, \dots, r$. In case $|\mathcal{C}_1| = \cdots = |\mathcal{C}_r| = |\mathcal{C}|$, we say that $\mathcal{C}$ is *disjoint*. Under this condition, it is possible to establish also a clear connection between the minimum distance of a given flag code and the ones of its projected codes. More precisely, if $\mathcal{C}$ is a disjoint flag code, then

$$d_f(\mathcal{C}) \geqslant \sum_{i=1}^{r} d_S(\mathcal{C}_i).$$

In [2] the authors introduced a family of flag codes such that the distance and size of the projected codes completely determine the ones of the corresponding flag code.

Definition 3.4. A flag code $\mathcal{C}$ is *consistent* if the following conditions hold:

(1) $\mathcal{C}$ is disjoint.

(2) $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i)$.

In the same paper, the authors develop a decoding algorithm for consistent flag codes over the erasure channel and provide important families of such a class of codes. Among them, we can find the one of *optimum distance flag codes*. This class of flag codes has been already studied in [4, 6, 7, 19]. In these works, the reader can find specific constructions of them as well as the following characterization.

Theorem 3.5. *[6, Th. 3.11] A flag code is an optimum distance flag code if, and only if, it is disjoint and every projected code attains the maximum possible distance for its dimension.*

We will come back to this ideas in Subsection 4.3 in order to adapt the consistent flag codes decoding algorithm designed in [2] to the constructions proposed in the present paper.

3.2 Cyclic orbit flag codes

Let us recall the concept of *cyclic orbit flag code* as the orbit of the multiplicative action of (cyclic) subgroups of $\mathbb{F}_{q^n}^*$ on flags on $\mathbb{F}_{q^n}$. This concept of cyclic orbit flag code was first introduced in [3] following the approach of [11] for cyclic orbit subspace codes.

The cyclic group $\mathbb{F}_{q^n}^*$ acts on flags on $\mathbb{F}_{q^n}$ as follows: given $\beta \in \mathbb{F}_{q^n}^*$ and a flag $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ of type $(t_1, \dots, t_r)$, the flag $\mathcal{F}\beta$ is

$$\mathcal{F}\beta = (\mathcal{F}_1\beta, \dots, \mathcal{F}_r\beta) \tag{7.7}$$

and the orbit

$$\mathrm{Orb}_\beta(\mathcal{F}) = \{\mathcal{F}\beta^j \mid 0 \leqslant j \leqslant |\beta| - 1\} \tag{7.8}$$

is called the *β-cyclic orbit flag code* generated by $\mathcal{F}$. The *stabilizer* of $\mathcal{F}$ (w.r.t. β) is the subgroup of $\langle\beta\rangle$ given by

$$\mathrm{Stab}_\beta(\mathcal{F}) = \{\beta^j \mid \mathcal{F}\beta^j = \mathcal{F}\}. \tag{7.9}$$

If β is primitive, that is, if the acting group is $\mathbb{F}_{q^n}^*$, we simply write $\mathrm{Orb}(\mathcal{F})$ to denote the *cyclic orbit flag code generated by $\mathcal{F}$*. We also drop the subscript in $\mathrm{Stab}(\mathcal{F})$. Observe that, for every $\beta \in \mathbb{F}_{q^n}^*$, it holds $\mathrm{Stab}_\beta(\mathcal{F}) = \langle\beta\rangle \cap \mathrm{Stab}(\mathcal{F})$.

We can take advantage of the orbital structure to compute the code parameters: the cardinality of $\mathrm{Orb}_\beta(\mathcal{F})$ is given by

$$|\mathrm{Orb}_\beta(\mathcal{F})| = \frac{|\beta|}{|\mathrm{Stab}_\beta(\mathcal{F})|} = \frac{|\beta|}{|\langle\beta\rangle \cap \mathrm{Stab}(\mathcal{F})|} \tag{7.10}$$

and its minimum distance can be calculated as

$$d_f(\mathrm{Orb}_\beta(\mathcal{F})) = \min\{d_f(\mathcal{F}, \mathcal{F}\beta^j) \mid \beta^j \notin \mathrm{Stab}_\beta(\mathcal{F})\}. \tag{7.11}$$

Remark 3.6. Concerning the projected codes associated to $\mathrm{Orb}_\beta(\mathcal{F})$, there are important facts to point out. First of all, note that the projected codes of a β-cyclic orbit flag codes are also β-cyclic orbit (subspace) codes. More precisely, for every $1 \leqslant i \leqslant r$, we have

$$(\mathrm{Orb}_\beta(\mathcal{F}))_i = \mathrm{Orb}_\beta(\mathcal{F}_i). \tag{7.12}$$

Moreover, the straightforward stabilizers relationship

$$\mathrm{Stab}_\beta(\mathcal{F}) = \bigcap_{i=1}^{r} \mathrm{Stab}_\beta(\mathcal{F}_i) \tag{7.13}$$

leads to a nice rapport between cardinalities: for every $1 \leqslant i \leqslant r$, we have that $|\mathrm{Orb}_\beta(\mathcal{F}_i)|$ divides $|\mathrm{Orb}_\beta(\mathcal{F})|$ ([3, Prop. 3.6]).

Coming back to the computation of the values $|\mathrm{Orb}_\beta(\mathcal{F})|$ and $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$, in [3], it is showed that the knowledge of a specific subfield associated to $\mathcal{F}$ allows us to obtain them directly. Let us recall the concept of best friend of a flag introduced in [3] by generalization of the concept of a subspace best friend given in [11].

Definition 3.7. A subfield $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^n}$ is said to be a *friend* of a flag $\mathcal{F}$ on $\mathbb{F}_{q^n}$ if all the subspaces of $\mathcal{F}$ are $\mathbb{F}_{q^m}$-vector spaces, that is, if it is a friend of all of them. The *best friend* of the flag $\mathcal{F}$ is its biggest friend.

From this definition it clearly holds that the type vector of a flag has to satisfy a necessary condition whenever the best friend is fixed. Furthermore, as it occurs when we work with the stabilizer subgroup, there are important connections between the best friend of a flag and the ones of its subspaces.

Proposition 3.8. *[3, Lemma 3.14, Prop. 3.16, Cor. 3.18] Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ be a flag of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_{q^n}$. If $\mathbb{F}_{q^m}$ is a friend of $\mathcal{F}$, then m divides $\gcd(t_1, \ldots, t_r, n)$. Moreover, if $\mathbb{F}_{q^m}$ is the best friend of $\mathcal{F}$, then it is the intersection of the ones of $\mathcal{F}_i$, for every $i = 1, \ldots, r$, and we also have that $\mathbb{F}_{q^m} = \mathrm{Stab}(\mathcal{F}) \cup \{0\}$.*

Remark 3.9. Note that, if $1 \in \mathcal{F}_1$, then every friend of the flag $\mathcal{F}$ is contained in $\mathcal{F}_1$. Moreover, all the flags in $\mathrm{Orb}_\beta(\mathcal{F})$ have the same best friend, allowing us to speak about the *best friend of a β-cyclic orbit flag code*. Now, if $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ is a flag of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_{q^n}$ with $\mathbb{F}_{q^m}$ as its best friend, $\mathbb{F}_{q^m}$ must be a friend of all its subspaces and we can write $t_i = ms_i$ for $i = 1, \ldots, r$, where $1 \leqslant s_1 < \cdots < s_r < s = \frac{n}{m}$. Finally, we can find linearly independent elements $a_1, \ldots, a_{s_r} \in \mathbb{F}_{q^n}$ (over $\mathbb{F}_{q^m}$) such that, for every $1 \leqslant i \leqslant r$, we have

$$\mathcal{F}_i = \bigoplus_{j=1}^{s_i} \mathbb{F}_{q^m} a_j. \tag{7.14}$$

In case m is a dimension in the type vector, then $s_1 = 1$ and the cyclic orbit code $\mathrm{Orb}(\mathcal{F}_1)$ is the m-spread of $\mathbb{F}_{q^n}$ described in (7.5). Moreover, if $1 \in \mathcal{F}_1$, this subspace is exactly the subfield $\mathbb{F}_{q^m}$.

Let us recall how the knowledge of the best friend of a β-cyclic orbit flag code provides relevant information about the code parameters.

Proposition 3.10. *[3] Let $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ be a flag on $\mathbb{F}_{q^n}$ and assume that $\mathbb{F}_{q^m}$ is its best friend. Then*

$$|\mathrm{Orb}_\beta(\mathcal{F})| = \frac{|\beta|}{|\langle\beta\rangle \cap \mathbb{F}_{q^m}^*|}. \tag{7.15}$$

Moreover, the value $2m$ divides $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$ and, if the the type vector of $\mathcal{F}$ is $(ms_1, \ldots, ms_r)$, then it holds

$$2m \leqslant d_f(\mathrm{Orb}(\mathcal{F})) \leqslant 2m \left(\sum_{s_i \leqslant \lfloor \frac{s}{2} \rfloor} s_i + \sum_{s_i > \lfloor \frac{s}{2} \rfloor} (s - s_i) \right), \tag{7.16}$$

whenever $\beta \notin \mathbb{F}_{q^m}^*$. On the other hand, if $\beta \in \mathbb{F}_{q^m}^*$, then $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 0$.

Remark 3.11. From (7.15) and (7.16), it is clear that both size and cardinality depend on the generating flag (hence on its best friend), the acting subgroup and the type vector. In particular, once we have fixed the best friend $\mathbb{F}_{q^m}$, we obtain the maximum possible orbit size if β is a primitive element of $\mathbb{F}_{q^n}$. In this case, it holds $|\mathrm{Orb}(\mathcal{F})| = \frac{q^n-1}{q^m-1}$. However, if we take $\beta \in \mathbb{F}_{q^m}^*$, we obtain the minimum possible cardinality since $\mathrm{Orb}_\beta(\mathcal{F}) = \{\mathcal{F}\}$.

3.3 Flag distances, best friend and type vectors interplay

From the bounds provided in (7.16) we know that, fixed the subfield $\mathbb{F}_{q^m}$ as best friend of a flag code $\mathcal{C}$ of type $(ms_1, \ldots, ms_r)$, the possible values for the distance between flags in $\mathcal{C}$ belong to the interval

$$[2m, 2m \sum_{s_i \leqslant \lfloor \frac{s}{2} \rfloor} s_i + \sum_{s_i > \lfloor \frac{s}{2} \rfloor} (s - s_i)]. \tag{7.17}$$

Nevertheless, in the orbital flag codes setting it is very important to point out that not every possible flag distance value is compatible with every type vector. In general, the greater the flag distance, the more conditions over the corresponding type vector we will have to impose. The simplest case comes from considering cyclic flag codes of length one. In [11, Lemma 4.1], it was already shown that a cyclic (subspace) code with best friend $\mathbb{F}_{q^m}$ has, at least, distance $2m$ and constructions attaining this extreme value of the distance were also provided in

[11, Prop. 4.3]. However, when we work with flags of length $r \geqslant 2$, not even the minimum value of the distance, which is $2m$ as well, can be obtained for every type vector. This is a consequence of the link between flag distance values and the number of subspaces of a flag $\mathcal{F}$ that share the best friend of $\mathcal{F}$. Let us explain this relationship in the following result.

Theorem 3.12. *Let $\mathcal{F}$ be a flag on $\mathbb{F}_{q^n}$ with the subfield $\mathbb{F}_{q^m}$ as its best friend and take $\beta \in \mathbb{F}_{q^n}^* \setminus \mathbb{F}_{q^m}^*$.*

(1) If there are $1 \leqslant j \leqslant r$ subspaces of $\mathcal{F}$ with $\mathbb{F}_{q^m}$ as their best friend, then $d_f(\mathrm{Orb}_\beta(\mathcal{F})) \geqslant 2mj$.

(2) If $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 2m$, then $\mathbb{F}_{q^m}$ is the best friend of exactly one subspace of $\mathcal{F}$.

Proof. Let us prove (1). Assume that there exist j subspaces, say $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_j}$, of $\mathcal{F}$ having $\mathbb{F}_{q^m}$ as their best friend. Then it suffices to see that, if $\beta^l \notin \mathrm{Stab}_\beta(\mathcal{F}) = \langle\beta\rangle \cap \mathbb{F}_{q^m}^*$, then β^l does not stabilize the subspaces $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_j}$. Consequently, we have

$$d_f(\mathcal{F}, \mathcal{F}\beta^l) \geqslant \sum_{k=1}^{j} d_S(\mathcal{F}_{i_k}, \mathcal{F}_{i_k}\beta^l) \geqslant 2mj.$$

To prove (2), let us start assuming that there are at least two different subspaces $\mathcal{F}_i$ and $\mathcal{F}_j$ in $\mathcal{F}$ with $\mathbb{F}_{q^m}$ as their best friend. By (1) we have that $d_f(\mathrm{Orb}_\beta(\mathcal{F})) \geqslant 4m > 2m$. On the other hand, suppose that no subspace in $\mathcal{F}$ has $\mathbb{F}_{q^m}$ as its best friend. In this case, for every $1 \leqslant i \leqslant r$, we put $\mathbb{F}_{q^{m_i}}$ the best friend of $\mathcal{F}_i$ and, since $\mathbb{F}_{q^m} = \bigcap_{i=1}^r \mathbb{F}_{q^{m_i}}$, we have that m is a proper divisor of every m_i. In particular, $m < m_i$, for every $1 \leqslant i \leqslant r$. Now, for every $\beta^l \notin \mathrm{Stab}_\beta(\mathcal{F})$, we have at least one index $1 \leqslant i \leqslant r$ such that $\beta^l \notin \mathrm{Stab}_\beta(\mathcal{F}_i)$. Hence,

$$d_f(\mathcal{F}, \mathcal{F}\beta^l) \geqslant d_S(\mathcal{F}_i, \mathcal{F}_i\beta^l) \geqslant 2m_i > 2m.$$

Thus, $d_f(\mathcal{C}) > 2m$. ∎

Remark 3.13. Note that the converses of statements (2) and (1) in the previous result are not necessarily true. Take, for instance, $\mathcal{F} = (\mathbb{F}_{q^2}, \mathbb{F}_{q^4}, \mathbb{F}_{q^8})$ on $\mathbb{F}_{q^{16}}$, which has best friend $\mathbb{F}_{q^2}$. Let us consider $\beta = \alpha^5$ where $\langle\alpha\rangle = \mathbb{F}_{q^{16}}^*$. Then we have just one subspace of $\mathcal{F}$ with best friend $\mathbb{F}_{q^2}$ whereas $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 12 = 2 \cdot 3 \cdot 2 \geqslant 4$. At the same time there are not three subspaces in $\mathcal{F}$ sharing its best friend.

The previous theorem allows us to discard some type vectors if we work with the minimum value of the distance when the best friend is $\mathbb{F}_{q^m}$.

Corollary 3.14. *Let $\mathcal{F}$ be a flag of type $(ms_1, \ldots, ms_r)$ on $\mathbb{F}_{q^n}$ with best friend $\mathbb{F}_{q^m}$ and take $\beta \in \mathbb{F}_{q^n}^* \setminus \mathbb{F}_{q^m}^*$. If $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 2m$, then $\gcd(s_j, \frac{n}{m}) \neq 1$ for, at least $r-1$ indices $1 \leqslant j \leqslant r$.*

Proof. By means of Theorem 3.12, we know that there is exactly one subspace of $\mathcal{F}$ having best friend $\mathbb{F}_{q^m}$, say $\mathcal{F}_i$. Now, for each $j \neq i$, we put $\mathbb{F}_{q^{m_j}}$ the best friend of $\mathcal{F}_j$. In particular, we know that m is a proper divisor of every m_j. Let us write $m_j = ma_j$, with $a_j > 1$ for every $j \neq i$. In addition, $m_j = ma_j$ divides both $\dim(\mathcal{F}_j) = ms_j$ and n. Hence, $1 < a_j$ divides both s_j and $\frac{n}{m}$. We conclude that $\gcd(s_j, \frac{n}{m}) > 1$ for all $1 \leqslant j \leqslant r$, $j \neq i$. ∎

Example 3.15. *If $n = 16$ and we fix $\mathbb{F}_{q^2}$ as the best friend of our flags, the minimum distance value 4 cannot be obtained for type $(4, 6, 10)$ since $\gcd(3, 8) = \gcd(5, 8) = 1$. In contrast, this value would be attainable for type $(4, 6, 8)$, for instance. Using the same argument, if we take $n = 14$, and consider a flag $\mathcal{F}$ on $\mathbb{F}_{q^{14}}$ having the subfield $\mathbb{F}_{q^2}$ as its best friend, we can conclude that β-cyclic orbit flag codes generated by $\mathcal{F}$ will never give distance 4, unless $\mathcal{F}$ is the flag of length one $\mathcal{F} = (\mathbb{F}_{q^2})$.*

We have seen that, fixed the best friend $\mathbb{F}_{q^m}$, the minimum value of the distance $2m$ can only be obtained by codes $\mathrm{Orb}_\beta(\mathcal{F})$ in which $\mathcal{F}$ has exactly one subspace with $\mathbb{F}_{q^m}$ as its best friend as well. On the other end, as said in Theorem 3.5, a flag code $\mathcal{C}$ attains the maximum possible distance for its type if, and only if, it is disjoint and all its projected codes attain the respective maximum (subspace) distance. Recall that a flag code $\mathcal{C}$ of length r on $\mathbb{F}_{q^n}$ is disjoint if it holds

$$|\mathcal{C}_1| = \cdots = |\mathcal{C}_r| = |\mathcal{C}|. \tag{7.18}$$

In [3, Prop. 4.19] the authors prove that for cyclic orbit flag codes (β primitive) this condition is equivalent to say that each subspace of $\mathcal{F}$ has the same best friend (then the best friend of $\mathcal{F}$). Summing up, we can also draw conditions on the type vector in the case of cyclic orbit flag codes having $\mathbb{F}_{q^m}$ as their best friend and the largest possible distance, that is, the upper value of the range in (7.17).

Proposition 3.16. *[3, Cor. 4.23] Assume that the cyclic orbit code $\mathrm{Orb}(\mathcal{F})$ is an optimum distance flag code on $\mathbb{F}_{q^n}$ with the subfield $\mathbb{F}_{q^m}$ as its best friend. Then one of the following statements holds:*

(1) $\mathrm{Orb}(\mathcal{F})$ is a constant dimension code of dimension either m or $n-m$.

(2) $\mathrm{Orb}(\mathcal{F})$ has type vector $(m, n-m)$.

In any of the three cases above, the code $\mathrm{Orb}(\mathcal{F})$ has the largest possible size, that is, $\frac{q^n-1}{q^m-1}$.

Using Theorem 3.12, we obtain the next construction of cyclic orbit flag codes with the best possible distance for the above mentioned cases.

Proposition 3.17. *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ a flag of type $(n, n-m)$ on $\mathbb{F}_{q^n}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ have the subfield $\mathbb{F}_{q^m}$ as their best friend, then the cyclic orbit codes* $\mathrm{Orb}(\mathcal{F}_1)$, $\mathrm{Orb}(\mathcal{F}_2)$ *and* $\mathrm{Orb}(\mathcal{F})$ *have the maximum possible distance.*

Proof. The result holds for $\mathrm{Orb}(\mathcal{F}_1) = \mathrm{Orb}(\mathbb{F}_{q^m})$ by means of (7.5). For $\mathrm{Orb}(\mathcal{F}_2)$, it suffices to see that, if $\mathbb{F}_q^m$ is the best friend of $\mathcal{F}_2$, then $d_S(\mathrm{Orb}(\mathcal{F}_2)) = 2m$, which is the maximum possible distance for dimension $n-m$. Last, by means of Theorem 3.12, we conclude that $d_f(\mathrm{Orb}(\mathcal{F})) = 4m$, i.e., the maximum possible distance for type $(m, n-m)$. ∎

Remark 3.18. In the case of β-cyclic orbit flag codes with β non primitive, in [3, Prop. 4.19] it is proved that the code $\mathrm{Orb}_\beta(\mathcal{F})$ is disjoint if, and only if,

$$\langle\beta\rangle \cap \mathbb{F}_{q^m}^* = \langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* = \cdots = \langle\beta\rangle \cap \mathbb{F}_{q^{m_r}}^*. \tag{7.19}$$

Note that to have (7.19) it is not necessary that all the subspaces of $\mathcal{F}$ share the same best friend, contrary to what happens if β primitive (see part (2) on Example 3.19). Moreover, in [3, Thm. 4.21] the authors give also conditions on the type vector of $\mathcal{F}$ of an optimum distance β-cyclic flag code with fixed best friend if β is not primitive. Here we present some examples extracted from [3, Table 3] where they determine the set of allowed dimensions in the type vector, depending on the size of the acting subgroup $\langle\beta\rangle$ of $\mathbb{F}_{2^{12}}^* = \langle\alpha\rangle$, when the best friend is $\mathbb{F}_{2^2}$.

β	$\lvert\beta\rvert$	$\langle\beta\rangle \cap \mathbb{F}_{q^m}^*$	$\lvert\mathrm{Orb}_\beta(\mathcal{F})\rvert$	Allowed dimensions	Max. distance
α	4095	$\mathbb{F}_{2^2}^*$	1365	2, 10	8
α^5	819	$\mathbb{F}_{2^2}^*$	273	2, 4, 8, 10	24
α^9	455	$\{1\}$	455	2, 10	8
α^{63}	65	$\{1\}$	65	2, 4, 6, 8, 10	36

Table 7.1: Admissible dimensions for $q = 2$, $n = 12$, $m = 2$.

Concerning the explicit construction of such codes, in [4, 19], the authors follow the approach of [24] to build optimum distance flag codes under the action of (subgroups of) Singer groups of the special linear group and the general linear group, respectively, by placing a suitable spread among the projected codes. In our framework, this idea corresponds to the choice a generating flag that has certain subfield among its subspaces. Let us exhibit some concrete examples.

Example 3.19. *Let us work in $\mathbb{F}_{2^{12}}$ and fix $\mathbb{F}_{q^2}$ as best friend of all our flags.*

(1) *Take $\mathcal{F} = (\mathbb{F}_{2^2}, \mathcal{F}_2, \mathbb{F}_{2^6}, \mathcal{F}_4, \mathcal{F}_5)$ of type $(2, 4, 6, 8, 10)$ and consider $\beta = \alpha^{63}$, then the orbit $\mathrm{Orb}_\beta(\mathcal{F})$ is an optimum distance flag code of cardinality* 65

(see [4]), which is the maximum possible size for an optimum distance flag code of this type. With the same notation, the orbit $\mathrm{Orb}_\beta((\mathbb{F}_{2^2}, \mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_5))$ *is an optimum distance flag code, in this case of type* $(2, 4, 8, 10)$*, of the same size.*

(2) *On the other hand, following the ideas in [19], if we consider the flag* $\mathcal{F}' = (\mathbb{F}_{2^2}, \mathbb{F}_{2^4}, \mathcal{F}'_3, \mathcal{F}'_4)$ *of type* $(2, 4, 8, 10)$ *such that* $\mathbb{F}_{2^4}$ *is the best friend of* $\mathcal{F}'_3$*, and take* $\beta = \alpha^5$*, then the orbit* $\mathrm{Orb}_\beta(\mathcal{F}')$ *is an optimum distance flag code with cardinality* 273*. Note that in this example the subspaces* $\mathcal{F}'_1$ *and* $\mathcal{F}'_3$ *do not share their best friend even thought* $\mathrm{Orb}_\beta(\mathcal{F}')$ *is disjoint.*

These examples lead us to study β-cyclic orbit codes when we place one or more subfields in the generating flag.

3.4 Generating flags based on subfields

In this subsection we focus on β-cyclic orbit flag codes generated by flags having at least one subfield among their subspaces. We distinguish two situations: either every subspace in the generating flag is a subfield or there is also one subspace that is not a subfield.

Galois flag codes

Let us start with β-cyclic orbit flag codes generated by flags having just subfields of $\mathbb{F}_{q^n}$ as subspaces, that is, generated by the so-called *Galois flags*. This particular class of β-cyclic orbit flag codes was introduced in [3]. Let us recall the definition. Consider a sequence of integers $1 \leqslant t_1 < \cdots < t_r < n$ such that all of them are divisors of n and t_i divides t_{i+1}, for $1 \leqslant i \leqslant r-1$.

Definition 3.20. The *Galois flag* of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_{q^n}$ is the flag given by the sequence of nested subfields $(\mathbb{F}_{q^{t_1}}, \ldots, \mathbb{F}_{q^{t_r}})$. Given $\beta \in \mathbb{F}_{q^n}^*$, the β-cyclic orbit flag code generated by this Galois flag is called the *β-Galois cyclic orbit flag code*, or just *β-Galois flag code*, for short, of type $(t_1, \ldots, t_r)$.

In the Galois flag $\mathcal{F}$ of type vector $(t_1, \ldots, t_r)$, clearly the i-th subspace has the subfield $\mathbb{F}_{q^{t_i}}$ as best friend. Hence, the first subfield $\mathbb{F}_{q^{t_1}}$ is the best friend of any β-Galois flag code of type $(t_1, \ldots, t_r)$. For β primitive we have the following straightforward result.

Proposition 3.21 [3]**.** *Let* $\mathcal{C}$ *be the Galois flag code of type* $(t_1, \ldots, t_r)$*, then the cardinality of this flag code is* $|\mathcal{C}| = (q^n - 1)/(q^{t_1} - 1)$ *and its distance is* $d_f(\mathcal{C}) = 2t_1$*. Its i-th projected code* $\mathcal{C}_i$ *has size* $|\mathcal{C}_i| = (q^n - 1)/(q^{t_i} - 1)$ *and distance* $2t_i$*.*

Remark 3.22. Note that, if we take the Galois flag $\mathcal{F}$ of type $(t_1, \dots, t_r)$, the distance $d_f(\mathrm{Orb}(\mathcal{F})) = 2t_1$ is the lowest possible one for cyclic orbit flag codes with $\mathbb{F}_{q^{t_1}}$ as best friend, according to (7.6) (in case $|\mathcal{C}| > 1$). In fact, there is a precise set of attainable distances for the different orbits $\mathrm{Orb}_\beta(\mathcal{F})$ when we consider the action of subgroups $\langle\beta\rangle \subseteq \mathbb{F}_{q^n}^*$. Furthermore, as proved in [3], we can always select β in a controlled manner such that the code $\mathrm{Orb}_\beta(\mathcal{F})$ reaches any distance value in the set of possible distances. This choice is made by checking the relationship between the subgroup $\langle\beta\rangle$ and the subfields $\mathbb{F}_{q^{t_i}}$. Let us recall the precise result.

Theorem 3.23. *[3, Thm. 4.14] Let $\mathcal{F}$ be the Galois flag of type $(t_1, \dots, t_r)$ and consider an element $\beta \in \mathbb{F}_{q^n}^*$. Then*

$$d_f(\mathrm{Orb}_\beta(\mathcal{F})) \in \{0, 2t_1, 2(t_1+t_2), \dots, 2(t_1+t_2+\cdots+t_r)\}. \tag{7.20}$$

Moreover,

(1) $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 0$ if, and only if, $\mathrm{Stab}_\beta(\mathcal{F}_1) = \mathrm{Stab}_\beta(\mathcal{F}_r) = \langle\beta\rangle$.

(2) $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 2\sum_{i=1}^{r} t_i$ if, and only if, $\mathrm{Stab}_\beta(\mathcal{F}_1) = \mathrm{Stab}_\beta(\mathcal{F}_r) \neq \langle\beta\rangle$.

(3) $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = 2\sum_{i=1}^{j-1} t_i$ if, and only if, $\mathrm{Stab}_\beta(\mathcal{F}_1) \neq \mathrm{Stab}_\beta(\mathcal{F}_r)$ and $j \in \{2, \dots, r\}$ is the minimum index such that $\mathrm{Stab}_\beta(\mathcal{F}_1) \subsetneq \mathrm{Stab}_\beta(\mathcal{F}_j)$.

In view of the previous result, it is worth highlighting that, given a Galois flag $\mathcal{F}$, the range of attainable distances by the codes $\mathrm{Orb}_\beta(\mathcal{F})$ follows a concrete pattern in terms of the dimensions in the type vector (see (7.20)). On the other hand, we have the possibility to gradually improve the distance of $\mathrm{Orb}(\mathcal{F})$ by selecting the orbit $\mathrm{Orb}_\beta(\mathcal{F})$ for an appropriate β, even if this choice could involve a loss of size. This nice behaviour gives rise to think that Galois codes could constitute an appropriate "skeleton" to support a more general family of β-cyclic orbit flag codes whose properties, in turn, might be driven by them. To explore this idea, in the following section we introduce a new family of codes.

Generalized Galois flag codes

Let us take now generating flags having at least one subspace that is a subfield and at least another one that is not. This condition gives length at least two. Note also that, all the fields in a flag $\mathcal{F}$ constitute a Galois subflag.

Definition 3.24. We say that a flag $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k)$ of type $(s_1, \dots, s_k)$ *generalizes the Galois flag of type* $(t_1, \dots, t_r)$ if $\{t_1, \dots, t_r\} \subsetneq \{s_1, \dots, s_k\}$ and the following conditions are satisfied:

(1) The subflag of $\mathcal{F}$ of type $(t_1, \dots, t_r)$ is the Galois flag of this type,

(2) there is at least one subspace of $\mathcal{F}$ that is not a field.

Remark 3.25. Observe that the second condition in Definition 3.24 excludes Galois flags from our study of generalized Galois flags. Even more, according to the previous definition, a generalized Galois flag is just a flag having at least one field and one subspace that is not a field among its subspaces. Besides, in the conditions of the previous definition, $\mathcal{F}$ clearly generalizes every subflag of the Galois flag of type $(t_1, \ldots, t_r)$ as well. We pay special attention to the longest Galois flag being a subflag of $\mathcal{F}$.

Definition 3.26. Let $\mathcal{F}$ be a generalized Galois flag. Its longest Galois subflag is called its *underlying Galois subflag.*

Observe that the underlying Galois subflag of a generalized Galois flag always exists and, due to the nested structure of flags, it is unique.

Definition 3.27. Given $\mathcal{F}$ a generalized Galois flag and $\beta \in \mathbb{F}_{q^n}^*$, the β-cyclic orbit flag code generated by $\mathcal{F}$ is called a *generalized β-Galois (cyclic orbit) flag code.* If $\mathcal{F}'$ is the underlying Galois subflag of $\mathcal{F}$, then we say that $\mathrm{Orb}_\beta(\mathcal{F}')$ is the *underlying β-Galois flag code* of $\mathrm{Orb}_\beta(\mathcal{F})$.

Let us see some examples reflecting different situations related to this new class of flag codes.

Example 3.28. *Take $n = 8$ and primitive elements $\alpha \in \mathbb{F}_{q^8}$, $\gamma \in \mathbb{F}_{q^4}$. The sequences*

$$\mathcal{F} = (\mathbb{F}_{q^2}, \mathbb{F}_{q^4}, \mathbb{F}_{q^4} \oplus \mathbb{F}_q^2\alpha) \quad and \quad \mathcal{F}' = (\mathbb{F}_{q^2}, \mathbb{F}_{q^2} \oplus \mathbb{F}_q\gamma, \mathbb{F}_{q^4})$$

are generalized flags of type $(2,4,6)$ and $(2,3,4)$ on $\mathbb{F}_{q^8}$, respectively, with common underlying Galois subflag $(\mathbb{F}_{q^2}, \mathbb{F}_{q^4})$. Now, for any $\beta \in \mathbb{F}_{q^n}^*$, the best friend of the β-Galois flag code $\mathrm{Orb}_\beta((\mathbb{F}_{q^2}, \mathbb{F}_{q^4}))$ is the field $\mathbb{F}_{q^2}$ and this property also holds for the generalized β-Galois flag code $\mathrm{Orb}(\mathcal{F})$. However, the best friend of $\mathrm{Orb}(\mathcal{F}')$ is $\mathbb{F}_q$, which is a field not appearing in $\mathcal{F}'$.

Remark 3.29. Given a generalized β-Galois flag code $\mathcal{C}$ of type $(s_1, \ldots, s_k)$ with underlying Galois subflag $(\mathbb{F}_{q^{t_1}}, \ldots, \mathbb{F}_{q^{t_r}})$, if the subfield $\mathbb{F}_{q^m}$ is the best friend of $\mathcal{C}$, then it holds $\mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^{t_1}}$.

Concerning the attainable distance values for this class of codes, contrary what happens with Galois flag codes whose set of reachable distances is completely determined by the type, different situations can arise.

Example 3.30. *Take $n = 4$ and consider a generalized Galois flag $\mathcal{F}$ of type $(1,2,3)$ with underlying Galois subflag $(\mathbb{F}_q)$. In this case, the set of attainable values for $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$ is exactly the same as for general flag codes of this type on $\mathbb{F}_{q^4}$, that is, any even integer between 0 and 8. However, for the same choice* *of the parameters, flags generalizing the Galois flag* $(\mathbb{F}_q, \mathbb{F}_{q^2})$, *i.e., those of the form*

$$\mathcal{F}' = (\mathbb{F}_q, \mathbb{F}_{q^2}, \mathcal{F}'_3),$$

for some subspace $\mathcal{F}'_3$ *of dimension* 3 *of* $\mathbb{F}_{q^4}$, *present a restriction on the set of possible distances. Let us prove that the value* $d_f(\mathrm{Orb}_\beta(\mathcal{F}')) = 6$ *cannot be obtained for any* $\beta \in \mathbb{F}_{q^n}^*$. *Observe that the projected code* $\mathrm{Orb}_\beta(\mathbb{F}_{q^2})$ *is a partial spread of dimension* 2 *of* $\mathbb{F}_{q^4}$. *Hence, when computing the distance*

$$d_f(\mathcal{F}', \mathcal{F}'\beta^i) = d_S(\mathbb{F}_q, \mathbb{F}_q\beta^i) + d_S(\mathbb{F}_{q^2}, \mathbb{F}_{q^2}\beta^i) + d_S(\mathcal{F}'_3, \mathcal{F}'_3\beta^i),$$

we have that

$$d_S(\mathbb{F}_{q^2}, \mathbb{F}_{q^2}\beta^i) = \begin{cases} 0 & \text{if } \beta \in \mathbb{F}_{q^2}^*, \\ 4 & \text{otherwise.} \end{cases}$$

Moreover, if $d_S(\mathbb{F}_{q^2}, \mathbb{F}_{q^2}\beta^i) = 0$, *then it holds* $d_f(\mathcal{F}', \mathcal{F}'\beta^i) \leqslant 4$. *On the other hand, if* $d_S(\mathbb{F}_{q^2}, \mathbb{F}_{q^2}\beta^i) = 4$, *we have* $\dim(\mathbb{F}_{q^2} \cap \mathbb{F}_{q^2}\beta^i) = 0$ *or, equivalently,* $\dim(\mathbb{F}_{q^2} \oplus \mathbb{F}_{q^2}\beta^i) = 4$. *Hence, for this precise* β^i, *we get*

$$\dim(\mathbb{F}_q \cap \mathbb{F}_q\beta^i) = 0 \quad \text{and} \quad \dim(\mathcal{F}'_3 + \mathcal{F}'_3\beta^i) = 4.$$

In both cases, we can conclude $d_S(\mathbb{F}_q, \mathbb{F}_q\beta^i) = d_S(\mathcal{F}'_3, \mathcal{F}'_3\beta^i) = 2$. *As a consequence, it holds* $d_f(\mathcal{F}', \mathcal{F}'\beta^i) = 8$. *Thus, the value* $d = 6$ *cannot be obtained if we consider flags of type* $(1, 2, 3)$ *on* $\mathbb{F}_{q^4}$ *generalizing* $(\mathbb{F}_q, \mathbb{F}_{q^2})$.

The situation exhibited in the last example is a direct consequence of the presence of certain subfields of $\mathbb{F}_{q^n}$ as subspaces of a generalized Galois flag $\mathcal{F}$. In other words, its underlying Galois subflag affects, in some sense, the value $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$. The next result establish some conditions on the minimum distance of generalized β-Galois flag codes that allow us to discard some values of the distance.

Theorem 3.31. *Let* $\mathcal{F}$ *be a generalized Galois flag of type* $(s_1, \ldots, s_k)$ *on* $\mathbb{F}_{q^n}$ *with underlying Galois subflag* $(\mathbb{F}_{q^{t_1}}, \ldots, \mathbb{F}_{q^{t_r}})$ *and take* $\beta \in \mathbb{F}_{q^n}^*$. *Consider* $i \in \{1, \ldots, r\}$. *Then the following statements hold:*

(1) If $\beta \in \mathbb{F}_{q^{t_i}}^*$, *then* $d_S(\mathcal{F}_l, \mathcal{F}_l\beta) = 0$, *for all* $s_l \in \{t_i, \ldots, t_r\}$.

(2) If $\beta \notin \mathbb{F}_{q^{t_i}}^*$, *then* $d_S(\mathcal{F}_l, \mathcal{F}_l\beta) = 2s_l$ *for all* $s_1 \leqslant s_l \leqslant t_i$.

Proof. (1) Assume that $\beta \in \mathbb{F}_{q^{t_i}}^*$. Hence, $\beta \in \mathbb{F}_{q^{t_j}}^*$, for every $i \leqslant j \leqslant r$. In other words, we have $\mathbb{F}_{q^{t_j}}\beta = \mathbb{F}_{q^{t_j}}$, i.e., $d_S(\mathbb{F}_{q^{t_j}}, \mathbb{F}_{q^{t_j}}\beta) = 0$ for all $i \leqslant j \leqslant r$.

(2) Take now $\beta \notin \mathbb{F}_{q^{t_i}}^* = \mathrm{Stab}(\mathbb{F}_{q^{t_i}})$. In this case, $\mathbb{F}_{q^{t_i}}$ and $\mathbb{F}_{q^{t_i}}\beta$ are different subspaces in the t_i-spread $\mathrm{Orb}(\mathbb{F}_{q^{t_i}})$ and we have that $\dim(\mathbb{F}_{q^{t_i}} \cap \mathbb{F}_{q^{t_i}}\beta) = 0$. Thus, for every dimension $s_l \leqslant t_i$ in the type vector, we have $\dim(\mathcal{F}_l \cap \mathcal{F}_l\beta) = 0$ and then $d_S(\mathcal{F}_l, \mathcal{F}_l\beta) = 2s_l$.

∎

According to this result, it is clear that some combinations of subspace distances are automatically discarded when we compute the minimum distance of a generalized β-Galois flag code. Even thought we do not have a pattern to compute the distance values as it occurs for Galois flag codes (see (7.20), by means of Theorem 3.31, we can state some required conditions for potential distance values between flags on a generalized β-Galois flag code.

Definition 3.32. Take a flag $\mathcal{F}$ of type $(s_1, \dots, s_k)$ on $\mathbb{F}_{q^n}$ generalizing the Galois flag $(\mathbb{F}_{q^{t_1}}, \dots, \mathbb{F}_{q^{t_r}})$ and an element $\beta \in \mathbb{F}_{q^n}^*$. We say that an even integer d is a *potential value for* $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$ if it can be obtained as a sum of subspace distances of dimensions $s_1, \dots, s_k$ satisfying:

(1) For dimensions t_i, only distances 0 or $2t_i$ are considered.

(2) If we sum $2t_i$ for dimension t_i, then all the distances for lower dimensions in the type vector are maximum as well.

(3) If for some dimension t_i we have distance 0, then the same happens for dimensions t_j, with $i \leqslant j \leqslant r$.

Remark 3.33. Notice that, according to Definition 3.32, and as suggested in Example 3.30, having the field $\mathbb{F}_q$ as the first subspace of a flag does not affect to the set of potential distance values since every β-cyclic code of dimension 1 of $\mathbb{F}_{q^n}$ (generated or not by $\mathbb{F}_q$) has distance either 0 or 2. On the other hand, as also mentioned in Example 3.30, some distances cannot be attained when we have other fields among the subspaces of the generating flag. For instance, the single value $d = 6$ is discarded for $n = 4$, type $(1, 2, 3)$ and underlying Galois subflag of type $(1, 2)$.

The next example shows that, in general, many values of the flag distance are not compatible with the underlying structure of nested fields.

Example 3.34. *Fix $n = 16$ and the type vector $(2, 4, 5, 6, 8)$. In general, every even integer $0 \leqslant d \leqslant 50$ is a possible value for the flag distance for this choice of the parameters. Nevertheless, if $\mathcal{F}$ is a generalized Galois flag of type $(2, 4, 5, 6, 8)$ with underlying Galois subflag of type $(2, 4, 8)$, then for every $\beta \in \mathbb{F}_{q^n}^*$, the set of potential values for $\mathrm{Orb}_\beta(\mathcal{F})$ is*

$$\{0, 2, 4, 6, 8, 10, 22, 50\}.$$

In other words, no intermediate distances $12 \leqslant d \leqslant 20$ or $24 \leqslant d \leqslant 48$ can be obtained when the starting flag contains fields as its subspaces of dimensions $2, 4$ and 8 (for more details on the computation of the set of distance values, we refer the reader to [5], where a deep study on the flag distance parameter and its behaviour is presented).

At this point, we can assert that the distance of a generalized Galois flag code is strongly influenced by its underlying Galois code. Hence, it is quite natural to wonder if generalized β-Galois flag codes behave as well as Galois flag codes in the following sense:

($*$) *Given a generalized Galois flag* $\mathcal{F}$ *and a potential value* d *for the distance defined in 3.32, can we always find a suitable subgroup* $\langle\beta\rangle \subseteq \mathbb{F}_{q^n}^*$ *such that* $d_f(\mathrm{Orb}_\beta(\mathcal{F})) = d$*?*

In the following section we prove that the answer to the previous question is negative by exhibiting a specific family of generalized Galois flags.

4 A construction of generalized Galois flag codes

This section is devoted to build generalized Galois flags written in a regular form that allows us to provide constructions of β-cyclic orbit flag codes with a prescribed best friend. In Subsections 4.1, 4.2 and 4.3, we focus on the particular case of β primitive due to the fact that the obtained cyclic orbit codes present important properties that deserve to be underlined. Finally, Subsection 4.4 is devoted to deal with the β-cyclic case and, in particular, to give an answer to the question ($*$) formulated in the previous section.

Recall that, according to the definition of best friend, we can express all the subspaces of a given flag as vector spaces over its best friend (see (7.14)). In our case, we will consider a specific family of flags, whose subspaces are written in the regular form used in Proposition 2.4. Let us describe the form of such flags and obtain the parameters of cyclic orbit flag codes generated by them.

Fix $\mathbb{F}_{q^m}$ a subfield of $\mathbb{F}_{q^n}$ and consider a primitive element α of $\mathbb{F}_{q^n}$. For each positive integer l such that $1 \leqslant l < \frac{q^n-1}{q^m-1}$, let L be the degree of the minimal polynomial of α^l over $\mathbb{F}_{q^m}$. Observe that L is also the degree of the field extension $\mathbb{F}_{q^{mL}}/\mathbb{F}_{q^m}$, that is, $L = [\mathbb{F}_{q^{mL}} : \mathbb{F}_{q^m}]$. Hence, L divides $[\mathbb{F}_{q^n} : \mathbb{F}_{q^m}] = n/m = s$ and, we have that $L \leqslant s$. In addition, the set $\{1, \alpha^l, \alpha^{2l}, \ldots, \alpha^{(L-1)l}\}$ is a basis of the field extension $\mathbb{F}_{q^{mL}}/\mathbb{F}_{q^m}$. Thus, we can write

$$\bigoplus_{j=0}^{L-1} \mathbb{F}_{q^m}\alpha^{jl} = \mathbb{F}_{q^m}[\alpha^l] \cong \mathbb{F}_{q^{mL}}. \tag{7.21}$$

Now, for every $i = 1, \ldots, L$, the vector space

$$\mathcal{U}_i = \bigoplus_{j=0}^{i-1} \mathbb{F}_{q^m}\alpha^{jl} \tag{7.22}$$

has dimension mi (over $\mathbb{F}_q$) and, as stated in Section 2, it is a field if, and only if, either $i = 1$ or $i = L$. Hence, the sequence $(\mathcal{U}_1, \ldots, \mathcal{U}_L)$ forms a generalized Galois flag of type $(m, 2m, \ldots, mL)$ with underlying Galois subflag $(\mathbb{F}_{q^m}, \mathbb{F}_{q^{mL}})$.

Remark 4.1. Observe that we cannot define a direct sum of this shape with more than L terms, since every power of α^l is always an element in $\mathcal{U}_L = \mathbb{F}_{q^m}[\alpha^l] \cong \mathbb{F}_{q^{mL}}$. Hence, this regular form allows us to construct flag codes on $\mathbb{F}_{q^n}$ of length $r \leqslant L$. Moreover, in case $L = s$, as $\mathcal{U}_L = \mathbb{F}_{q^n}$, we just get $r \leqslant L - 1 = s - 1$.

The following two subsections (4.1 and 4.2) are devoted to describe a construction of generalized Galois flag codes having $\mathbb{F}_{q^m}$ as their best friend. We perform it in two steps. First, we use flags in the regular form just described above in order to obtain a "basic" construction of generalized Galois flag codes where the underlying Galois flag code has, at most, length 2. Then, we propose a procedure to overcome this restriction and present another construction of generalized Galois flag code having a prescribed underlying Galois flag code by suitably "weaving" several basic generalized Galois flag codes.

4.1 Basic constructions

By means of Proposition 2.5, we can easily determine the best friend of the subspaces $\mathcal{U}_i$ defined in (7.22): it is the subfield $\mathbb{F}_{q^m}$, for $1 \leqslant i \leqslant L-1$ whereas the subspace $\mathcal{U}_L = \mathbb{F}_{q^{mL}}$ is its own best friend. This fact implies that we will find one or two fields among the subspaces $\mathcal{U}_i$, according to Remark 4.1. Since the case of length $r = 1$ corresponds to constant dimension codes (already studied in [11]), from now on, we will assume $r \geqslant 2$. Now, we know that for every type vector given by multiples of m, say $(ms_1, \dots, ms_r)$, where $1 \leqslant s_1 < \cdots < s_r \leqslant L$, we select the subspaces defined in (7.22) corresponding respectively to these dimensions, that is,

$$\mathcal{F}_i = \mathcal{U}_{s_i} = \bigoplus_{j=0}^{s_i-1} \mathbb{F}_{q^m} \alpha^{lj}, \ 1 \leqslant i \leqslant r. \tag{7.23}$$

With this notation, the next result holds.

Theorem 4.2. *Let α be a primitive element of $\mathbb{F}_{q^n}$, l a positive integer with $1 \leqslant l < \frac{q^n-1}{q^m-1}$ and L the degree of the minimal polynomial of α^l over $\mathbb{F}_{q^m}$. Consider the flag $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ of type $(ms_1, \dots, ms_r)$ on $\mathbb{F}_{q^n}$ with subspaces defined in (7.23). Hence, the code $\mathrm{Orb}(\mathcal{F})$ has best friend $\mathbb{F}_{q^m}$ and, in particular, its cardinality is $(q^n - 1)/(q^m - 1)$. Moreover,*

(1) If $s_r < L$, the code $\mathrm{Orb}(\mathcal{F})$ is consistent with distance $d_f(\mathrm{Orb}(\mathcal{F})) = 2mr$.

(2) If $s_r = L$, we have that $d_f(\mathrm{Orb}(\mathcal{F})) = 2m(r-1)$ and we can write

$$\mathrm{Orb}(\mathcal{F}) = \dot{\bigcup_{i=0}^{c-1}} \mathrm{Orb}_{\alpha^c}(\mathcal{F}\alpha^i),$$

with $c = \frac{q^n-1}{q^{mL}-1}$.

Proof. Recall that, by means of Proposition 2.5, the best friend of every subspace in $\mathcal{F}$ is either $\mathbb{F}_{q^m}$ or $\mathbb{F}_{q^{mL}}$. Since $r \geqslant 2$, there is at least one subspace with $\mathbb{F}_{q^m}$ as best friend and, automatically, this subfield is the best friend of the flag $\mathcal{F}$. As a consequence, by Proposition 3.10, the cardinality of $\mathrm{Orb}(\mathcal{F})$ is $(q^n-1)/(q^m-1)$.

Now, suppose that $s_r < L$. In this case, every dimension in the type vector is $ms_i \leqslant ms_r < mL$ and every subspace in the flag $\mathcal{F}$ has the subfield $\mathbb{F}_{q^m}$ as its best friend. Hence, the code is disjoint and, by means of Theorem 3.12, we have $d_f(\mathrm{Orb}(\mathcal{F})) \geqslant 2mr$. Moreover, notice that, for every $1 \leqslant i \leqslant r$, the subspace $\mathcal{F}_i \cap \mathcal{F}_i\alpha^l = \bigoplus_{j=1}^{s_i-1} \mathbb{F}_{q^m}\alpha^{lj}$ has dimension $m(s_i-1)$ over $\mathbb{F}_q$ and then $d_S(\mathrm{Orb}(\mathcal{F}_i)) = d_S(\mathcal{F}_i, \mathcal{F}_i\alpha^l) = 2m$. Hence, it holds

$$d_f(\mathcal{C}) \leqslant d_f(\mathcal{F}, \mathcal{F}\alpha^l) = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}_i\alpha^l) = 2mr$$

and we conclude $d_f(\mathcal{C}) = 2mr$. Moreover, the value coincides with the sum of the ones of its r projected codes (each one of them with distance $2m$). Hence, our code is consistent (see Definition 3.4).

Assume now that $s_r = L$. Then $\mathrm{Orb}(\mathcal{F})$ is not disjoint since the subspace $\mathcal{F}_r = \mathbb{F}_{q^{mL}}$ is its own best friend. However, since $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$ have the same best friend $\mathbb{F}_{q^m}$, Theorem 3.12 ensures that $d_f(\mathrm{Orb}(\mathcal{F})) \geqslant 2m(r-1)$. On the other hand, observe that $\alpha^l \in \mathbb{F}_q^{mL} \setminus \mathbb{F}_{q^m}^*$ stabilizes $\mathcal{F}_r$ but $d_S(\mathcal{F}_i, \mathcal{F}_i\alpha^l) = 2m$ for every $1 \leqslant i < r$. Hence, we have $d_f(\mathrm{Orb}(\mathcal{F})) = d_f(\mathcal{F}, \mathcal{F}\alpha^l) = 2m(r-1)$.

Concerning the structure of this cyclic orbit flag code, it is clear that

$$\bigcup_{i=0}^{c-1} \mathrm{Orb}_{\alpha^c}(\mathcal{F}\alpha^i) \subseteq \mathrm{Orb}(\mathcal{F}).$$

Let us see that both sets have the same cardinality. To do so, observe that α^c is a primitive element of $\mathbb{F}_{q^{mL}}$. Hence, for every $1 \leqslant i \leqslant r$, it holds

$$\mathrm{Stab}_{\alpha^c}(\mathcal{F}\alpha^i) = \mathbb{F}_{q^{mL}}^* \cap \mathrm{Stab}(\mathcal{F}\alpha^i) = \mathbb{F}_{q^{mL}}^* \cap \mathbb{F}_{q^m}^* = \mathbb{F}_{q^m}^*.$$

As a consequence, we have $|\mathrm{Orb}_{\alpha^c}(\mathcal{F}\alpha^i)| = \frac{q^{mL}-1}{q^m-1}$. Now, we prove that all these orbits are different. To do so, for every choice $0 \leqslant i \leqslant c-1$, observe that flags in $\mathrm{Orb}_{\alpha^c}(\mathcal{F}\alpha^i)$ have the same last subspace $\mathcal{F}_r\alpha^i = \mathbb{F}_{q^{mL}}\alpha^i$ since α^c stabilizes $\mathcal{F}_r = \mathbb{F}_{q^{mL}} = \{0\} \cup \langle \alpha^c \rangle$. Moreover, the last projected code

$$\mathrm{Orb}(\mathcal{F}_r) = \{\mathbb{F}_{q^{mL}}\alpha^i \mid 0 \leqslant i \leqslant c-1\}$$

of $\mathrm{Orb}(\mathcal{F})$ is precisely the mL-spread $\mathrm{Orb}(\mathbb{F}_{q^{mL}})$ of $\mathbb{F}_{q^n}$. Hence, for every choice $0 \leqslant i < j \leqslant c-1$, subspaces $\mathbb{F}_{q^{mL}}\alpha^i$ and $\mathbb{F}_{q^{mL}}\alpha^j$ are different. Therefore, all the orbits $\mathrm{Orb}_{\alpha^c}(\mathcal{F}\alpha^i)$ are different and the cardinality of their union is exactly

$$c \cdot \frac{q^{mL}-1}{q^m-1} = \frac{q^n-1}{q^{mL}-1} \cdot \frac{q^{mL}-1}{q^m-1} = \frac{q^n-1}{q^m-1} = |\mathcal{C}|.$$

∎

From Theorem 4.2, and making a suitable choice of the type vector, we derive some constructions of our interest.

Corollary 4.3. *Consider the flag $\mathcal{F}$ of type $(ms_1, \ldots, ms_r)$ with $r \geqslant 2$ defined in (7.23). If $L < s$, then:*

(1) the code $\mathrm{Orb}(\mathcal{F})$ *is a Galois flag code if, and only if, the type vector is* (m, mL).

(2) $\mathrm{Orb}(\mathcal{F})$ *is a generalized Galois flag code if, and only if,*

$$\emptyset \neq \{1, L\} \cap \{s_1, \ldots, s_r\} \neq \{1, L\}.$$

Proof. In the first place, if $L < s$, it is clear that $\mathcal{F}$ is a Galois flag if it just have subfields of $\mathbb{F}_{q^n}$ as its subspaces. According to expression (7.23), just subspaces of dimensions m and mL are fields. Moreover, since $r \geqslant 2$, the result follows. ∎

Corollary 4.4. *Take the flag $\mathcal{F}$ of type $(ms_1, \ldots, ms_r)$ defined in (7.23) and assume $r \geqslant 2$ and $L = s$, then:*

(1) $\mathrm{Orb}(\mathcal{F})$ *is a generalized Galois flag code if, and only if, $s_1 = 1$.*

(2) In particular, if $s_1 = 1$ and $s_2 = L-1$, then $\mathrm{Orb}(\mathcal{F})$ *is an optimum distance generalized Galois flag code of type $(m, n-m)$ with the largest possible size.*

Proof. In this case, the only subfield of $\mathbb{F}_{q^n}$ writen in the regular form (7.22) is $\mathbb{F}_{q^m}$ and the first statement follows straightforwardly. For the second one, it suffices to notice that the subspaces $\mathcal{F}_1 = \mathcal{U}_1 = \mathbb{F}_{q^m}$ and $\mathcal{F}_2 = \mathcal{U}_{L-1}$ are of dimensions m and $n-m$ and have $\mathbb{F}_{q^m}$ as their best friend. Hence, the result holds by means of Proposition 3.17. ∎

At this point, we have all the ingredients to perfectly describe the structure of any cyclic orbit flag code $\mathrm{Orb}(\mathcal{F})$ given in Theorem 4.2, in case its distance is either the minimum or the maximum possible one. This result is closely related with to discussion in Subsection 3.3 about the interdependence of distance values and type vectors, for this particular family of codes.

Theorem 4.5. *Let α be a primitive element of $\mathbb{F}_{q^n}$, l a positive integer such that $1 \leqslant l < \frac{q^n-1}{q^m-1}$, and L the degree of the minimal polynomial of α^l over $\mathbb{F}_{q^m}$. Consider $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ the flag of type $(ms_1, ms_2, \ldots, ms_r)$ on $\mathbb{F}_{q^n}$ of length $r \geqslant 2$ with subspaces defined as in (7.23). If $s = n/m$, then:*

(1) The code $\mathrm{Orb}(\mathcal{F})$ *has distance equal to $2m$ if, and only if, its type vector is (ms_1, mL) for some $1 \leqslant s_1 < L < s$. Moreover, if $s_1 = 1$, then* $\mathrm{Orb}(\mathcal{F})$ *is the Galois flag code of type (m, mL).*

(2) The code $\mathrm{Orb}(\mathcal{F})$ *is an optimum distance flag code if, and only if,* $L = s$ *and its type vector is* $(m, m(L-1))$*. In this case,* $\mathrm{Orb}(\mathcal{F})$ *is a generalized Galois flag code that attains the largest possible size.*

Proof. We divide the proof into two parts.

(1) By means of Theorem 3.12, if the orbit $\mathrm{Orb}(\mathcal{F})$ has distance $2m$, then there is exactly one subspace of $\mathcal{F}$ with $\mathbb{F}_{q^m}$ as its best friend. Since subspaces defined in (7.23) have the subfield $\mathbb{F}_{q^m}$ as their best friend except if they are fields, we conclude that the last subspace of the generating flag must be the field $\mathbb{F}_{q^{mL}} = \mathbb{F}_{q^m}[\alpha^l]$. Thus L cannot be s and it holds $L < s$. Its first subspace can be any other subspace $\mathcal{F}_1$ of dimension $ms_1 < mL$.

To prove the converse, just note that the distance between $(\mathcal{F}_1, \mathbb{F}_{q^{mL}})$ and $(\mathcal{F}_1, \mathbb{F}_{q^{mL}})\alpha^l$ is $2m$, which is the minimum distance for cyclic orbit flag codes with $\mathbb{F}_{q^m}$ as their best friend.

To finish, if $s_1 = 1$, the only possibility for $\mathcal{F}$ is to be the Galois flag of type (m, mL) and the result holds.

(2) On the other hand, if $\mathrm{Orb}(\mathcal{F})$ is an optimum distance cyclic orbit flag code with $\mathbb{F}_{q^m}$ as its best friend, by means of Proposition 3.16, and assuming $r \geqslant 2$, its type vector must be $(m, n-m)$. Hence, we need mL to be at least $n - m = m(s-1)$. In other words, L must be greater or equal than $s - 1$. However, L has to divide s. If $L = s - 1$, the only possibility is $s = 2$ and the only type vector consisting of multiples of m is (m), which has length one. Thus, it must hold $L = s$. The converse is also true by application of Corollary 4.3.

∎

Remark 4.6. Concerning the first statement of the previous result, it is important to point out that if we consider a generating flag not necessarily written in the regular form described in (7.23), it is possible to attain the distance $2m$ with no other field among the subspaces of the generating flag than the best friend of the flag. Recall that, for general flags, even if we fix the field $\mathbb{F}_{q^m}$ as the first subspace, there are three possibilities for the best friend of each one of its subspaces: the field $\mathbb{F}_{q^m}$, the subspace itself (in case it is a field) or an intermediate extension field over $\mathbb{F}_{q^m}$. The next example contemplates this situation.

Example 4.7. *Fix* $n = 16$ *and consider a generalized Galois flag* $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ *with type vector* $(2, 8)$ *such that*

$$\begin{array}{rcl} \mathcal{F}_1 & = & \mathbb{F}_{q^2}, \\ \mathcal{F}_2 & = & \mathbb{F}_{q^2} \oplus \mathbb{F}_{q^2}\beta \oplus \mathbb{F}_{q^2}\alpha \oplus \mathbb{F}_{q^2}\beta\alpha \;\; = \;\; \mathbb{F}_{q^4} \oplus \mathbb{F}_{q^4}\alpha, \end{array}$$

where α *denotes a primitive element of* $\mathbb{F}_{q^{16}}$ *and* $\beta = \alpha^{(q^{16}-1)/(q^4-1)}$ *is a primitive element of the subfield* $\mathbb{F}_{q^4}$*. Observe that* $\mathbb{F}_{q^4}$ *is a friend of* $\mathcal{F}_2$*. Even more, it is its* *best friend by means of Proposition 2.5, since the degree of the minimal polynomial of α over $\mathbb{F}_{q^4}$ is $16/4 = 4$. According to this, $\mathbb{F}_{q^2}$ is the best friend of the cyclic orbit flag code* $\mathrm{Orb}(\mathcal{F})$*. Moreover, we have that $d_f(\mathcal{F}, \mathcal{F}\beta) = d_S(\mathcal{F}_1, \mathcal{F}_1\beta) = 4$ is the minimum possible distance for cyclic orbit flag codes with $\mathbb{F}_{q^2}$ as its best friend. However,* $\mathrm{Orb}(\mathcal{F})$ *is not the Galois flag code of type $(2, 8)$, since $\mathcal{F}$ is the only flag in the code with its first subspace containing the element $1 \in \mathbb{F}_{q^{16}}$ but $\mathcal{F}_2$ is not a field.*

Observe now that, taking again the flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ as in the previous example, we have the subfield $\mathbb{F}_{q^4}$ as an intermediate subspace. In other words, the sequence

$$\mathcal{F}' = (\mathbb{F}_{q^2}, \mathbb{F}_{q^4}, \mathbb{F}_{q^4} \oplus \mathbb{F}_{q^4}\alpha)$$

forms a generalized Galois flag longer than $\mathcal{F}$. Notice that the field $\mathbb{F}_{q^4}$ can be written in regular form as a vector space over $\mathbb{F}_{q^2}$ as $\mathbb{F}_{q^2} \oplus \mathbb{F}_{q^2}\beta$. Moreover, the subspace $\mathbb{F}_{q^4} \oplus \mathbb{F}_{q^4}\alpha$ is, at the same time, written in regular form as a vector space over $\mathbb{F}_{q^4}$. Inspired by this idea, we will describe a general procedure that allows us to obtain generalized Galois flags codes, written in regular form over suitably chained subfields.

4.2 Weaving basic generalized Galois flag codes

The previous basic construction (Theorem 4.2) presents a limitation on the number of subfields that can appear as subspaces of the generating flag. In this subsection, we focus on a systematic construction of generalized Galois flag codes with a prescribed underlying Galois subflag. More precisely, if $m_1, m_2, \ldots, m_k$ are divisors of n such that m_i divides m_{i+1}, for every $1 \leqslant i \leqslant k$, we work on the construction of generalized Galois flag codes with $(\mathbb{F}_{q^{m_1}}, \ldots, \mathbb{F}_{q^{m_k}})$ as underlying Galois subflag.

As a matter of notation, through this section we will write $m_{k+1} = n$. Let α be a primitive element of $\mathbb{F}_{q^n}$ and put $c_i = \frac{q^n-1}{q^{m_i}-1}$, for $1 \leqslant i \leqslant k+1$. It turns out that each power $\alpha_i = \alpha^{c_i}$ is a primitive element of the corresponding subfield $\mathbb{F}_{q^{m_i}}$. For every $2 \leqslant i \leqslant k+1$, the degree of the minimal polynomial of α_i over $\mathbb{F}_{q^{m_{i-1}}}$ is $L_i = \frac{m_i}{m_{i-1}}$. With this notation, we consider k flags $\mathcal{F}^1, \ldots, \mathcal{F}^k$ on $\mathbb{F}_{q^n}$, whose subspaces are given by

$$\mathcal{F}_j^i = \bigoplus_{l=0}^{j-1} \mathbb{F}_{q^{m_i}} \alpha_{i+1}^l, \tag{7.24}$$

for $1 \leqslant j \leqslant L_{i+1} - 1$ and $1 \leqslant i \leqslant k$. Observe that, for every $1 \leqslant i \leqslant k$, we have that $\mathcal{F}_1^i = \mathbb{F}_{q^{m_i}}$ and the dimension of $\mathcal{F}_j^i$ (as an $\mathbb{F}_q$-vector space) is jm_i. Hence, the type vector of $\mathcal{F}^i$ is given by all the multiples of m_i smaller than m_{i+1}, that is, $(m_i, 2m_i, \ldots, m_{i+1} - m_i)$. As a consequence of Theorem 4.2, the next result holds.

Corollary 4.8. *Let $\mathcal{F}^i$ be the flag defined in (7.24) for every $1 \leqslant i \leqslant k$. Then the generalized Galois flag code* $\mathrm{Orb}(\mathcal{F}^i)$ *is consistent with distance* $2(m_{i+1} - m_i)$ *and has the field* $\mathbb{F}_{q^{m_i}}$ *as its best friend.*

Proof. For every $1 \leqslant i \leqslant k$, we apply Theorem 4.2 (part (1)) to the generalized Galois flag code $\mathrm{Orb}(\mathcal{F}^i)$ and conclude that it is a consistent flag code with distance equal to $2m_i(L_{i+1} - 1) = 2(m_{i+1} - m_i)$. ∎

Observe that we have constructed a collection of orbit flag codes with respective generating flags in regular form over their best friend. The first subspace of each flag $\mathcal{F}^i$ is precisely its best friend and contains the last subspace of the previous flag, since

$$\mathcal{F}^i_{L_{i+1}-1} \subseteq \mathbb{F}_{q^{m_i}}[\alpha_{i+1}] = \mathbb{F}_{q^{m_{i+1}}} = \mathcal{F}^{i+1}_1,$$

for every value $1 \leqslant i \leqslant k-1$. By means of this property, we can consider a generating generalized Galois flag having all the subfields $\{\mathbb{F}_{q^{m_i}}\}_{i=1}^k$ among its subspaces just by taking

$$\mathcal{F} = (\mathcal{F}^1_1, \dots, \mathcal{F}^1_{L_1-1}, \mathcal{F}^2_1, \dots, \mathcal{F}^2_{L_2-1}, \dots, \mathcal{F}^k_1, \dots, \mathcal{F}^k_{L_k-1}), \tag{7.25}$$

whose type vector is $(m_1, \dots, m_2 - m_1, m_2, \dots, m_3 - m_2, m_3, \dots, m_k, \dots, n - m_k)$. In this way, by weaving the independent basic constructions described in Corollary 4.8, we get a generalized Galois flag code with the prescribed tower of subfields $(\mathbb{F}_{q^{m_1}}, \dots, \mathbb{F}_{q^{m_k}})$ as its underlying Galois subflag.

Proposition 4.9. *Let $\mathcal{F}$ be the generalized Galois flag on $\mathbb{F}_{q^n}$ given in (7.25). Then the generalized Galois flag code* $\mathrm{Orb}(\mathcal{F})$ *generalizes the Galois flag of type $(m_1, \dots, m_r)$. Its cardinality is $\frac{q^n-1}{q^{m_1}-1}$ and its minimum distance,* $2(m_2 - m_1)$.

Proof. By construction, it is clear that the subspace of dimension m_i of $\mathcal{F}$ is the field $\mathbb{F}_{q^{m_i}}$, for every $1 \leqslant i \leqslant k$. Moreover, subspaces of dimensions $m_i, \dots, m_{i+1} - m_i$ have the subfield $\mathbb{F}_{q^{m_i}}$ as its best friend. As a result, the best friend of the flag $\mathcal{F}$ coincides with its first subspace, that is, $\mathbb{F}_{q^{m_1}}$. This fact leads to the statement about the cardinality. Let us compute now the minimum distance of the code. First, by means of Theorem 3.12, since there are exactly $L_2 - 1$ subspaces of $\mathcal{F}$ with $\mathbb{F}_{q^{m_1}}$ as their best friend, we conclude that $d_s(\mathrm{Orb}(\mathcal{F})) \geqslant 2m_1(L_2 - 1) = 2(m_2 - m_1)$. Moreover, observe that α_2 stabilizes every subspace of the flag $\mathcal{F}$ containing $\mathbb{F}_{q^{m_2}} = \{0\} \cup \langle \alpha_2 \rangle$. Hence, $d_f(\mathrm{Orb}(\mathcal{F})) \leqslant d_f(\mathcal{F}, \mathcal{F}\alpha_2) = d_f(\mathcal{F}^1, \mathcal{F}^1\alpha_2) = 2m(L_2 - 1) = 2(m_2 - m_1)$ since, for every $1 \leqslant j \leqslant L_2 - 1$, the subspace $\mathcal{F}^1_j \cap \mathcal{F}^1_j\alpha = \bigoplus_{l=1}^{j-1} \mathbb{F}_{q^m}\alpha_2^l$ has dimension $m(j-1)$. We conclude that $d_f(\mathrm{Orb}(\mathcal{F})) = 2(m_2 - m_1)$. ∎

Remark 4.10. Note that weaving our basic constructions allows us to give generalized Galois flag codes with any given underlying Galois subflag in a systematic way. Another interesting fact to point out is that the best friends of the subspaces in the generalized Galois flag $\mathcal{F}$ defined in (7.25) form a nested sequence of subfields. This does not happen in general for arbitrary generalized Galois flag codes and it helps us to easily determine the cardinality and distance of the code $\mathrm{Orb}(\mathcal{F})$ as well as to give bounds for the distance when we consider its β-cyclic subcodes, as we will see in Subsection 4.4. On the negative side, contrary to what happens with the basic construction, the waved one is not consistent since it is not even disjoint.

4.3 Decoding our constructions over the erasure channel

The use of flags in network coding was originally introduced by Liebhold *et al.* in [15]. In that paper, a channel model for flags was presented and some constructions, together with their decoding algorithms (over the erasure channel) were provided. In [2], a decoding algorithm over the erasure channel for consistent flag codes is presented. In particular, such an algorithm can be applied to the basic construction given in Theorem 4.2, part (1). Although the rest of constructions in this paper are not consistent, we can adapt the decoding process in [2] to them. To do so, let us briefly recall some concepts related to the notion of *correctability*.

Assume that we have sent a flag $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ and hence, the receiver gets a sequence of nested subspaces $\mathcal{X} = (\mathcal{X}_1, \dots, \mathcal{X}_r)$ that, when working over an erasure channel, must satisfy $\mathcal{X}_i \subseteq \mathcal{F}_i$, for all $1 \leqslant i \leqslant r$. In this context, each value $e_i = d_S(\mathcal{F}_i, \mathcal{X}_i) = \dim(\mathcal{F}_i) - \dim(\mathcal{X}_i)$ is called *number of erasures at the i-th shot* whereas $e = d_f(\mathcal{F}, \mathcal{X}) = \sum_{i=1}^r e_i$ is the *total number of erasures*. We say that the total number of erasures e is *correctable* (by minimum distance) by a flag code $\mathcal{C}$ whenever $e \leqslant \lfloor \frac{d_f(\mathcal{C})-1}{2} \rfloor$. Analogusly, we also say that the value e_i is correctable by the projected code $\mathcal{C}_i$ if $e_i \leqslant \lfloor \frac{d_S(\mathcal{C}_i)-1}{2} \rfloor$.

Let us fix the flag code $\mathcal{C}$ as the one presented in Theorem 4.2, part (2). Recall that such a code has distance $2m(r-1)$ and r projected codes of distance $2m$. Following the ideas of [2, Proposition 8], we state the next result.

Proposition 4.11. *If the total number of erasures e is correctable by the generalized Galois flag code $\mathcal{C}$, then there exists some $1 \leqslant i \leqslant r-1$ such that the value e_i is also correctable by the corresponding projected code $\mathcal{C}_i$.*

Proof. Assume that no value e_i is correctable for every $1 \leqslant i \leqslant r-1$. Equivalently, we have that $e_i \geqslant m$ for every $1 \leqslant i \leqslant r-1$. As a consequence, we have that

$$e = \sum_{i=1}^{r} e_i \geqslant m(r-1) + e_r \geqslant m(r-1),$$

which is a contradiction, since $\mathcal{C}$ can correct up to $m(r-1)-1$ erasures. ∎

Now, if $\mathcal{C}'$ denotes the generalized Galois flag code obtained by the weaved construction given in Proposition 4.9 and $m_1 = m$, then we have $d_f(\mathcal{C}') = 2(m_2 - m) = 2m(L_2 - 1)$. Moreover, $d_S(\mathcal{C}'_i) = 2m$ holds for the first $L_2 - 1$ projected codes. Hence, the same argument in the proof of Proposition 4.11 can be used to show that a correctable total number of erasures can be detected and corrected by one of the first $L_2 - 1$ projected codes of $\mathcal{C}$.

Proposition 4.12. *If the total number of erasures e is correctable by $\mathcal{C}'$, then there exists some $1 \leqslant i \leqslant L_2 - 1$ such that e_i is correctable by the projected code $\mathcal{C}'_i$.*

Moreover, in both situations, every projected code has the same distance, which is $2m$. Hence, the number of erasures at any shot is correctable whenever it holds $e_i \leqslant m - 1$. We can easily identify if an erasure is correctable just by checking the dimension of every received subspace $\mathcal{X}_i$. The next proposition is valid for both a generalized Galois flag codes $\mathcal{C}$ and $\mathcal{C}'$.

Proposition 4.13. *The number of erasures e_i is correctable by the constant dimension code $\mathcal{C}_i$ (resp. $\mathcal{C}'_i$) if, and only if,* $\dim(\mathcal{X}_i) \geqslant \dim(\mathcal{F}_i) - m + 1$.

Proof. Assume that we send a flag $\mathcal{F} \in \mathcal{C}$ (resp. in $\mathcal{C}'$) and a stuttering flag $\mathcal{X}$ is received. Then e_i is correctable by $\mathcal{C}_i$ (resp. $\mathcal{C}'_i$) if, and only if, it holds

$$\dim(\mathcal{F}_i) - \dim(\mathcal{X}_i) = d_S(\mathcal{F}_i, \mathcal{X}_i) = e_i \leqslant m - 1$$

or equivalently, if $\dim(\mathcal{X}_i) \geqslant \dim(\mathcal{F}_i) - m + 1$. ∎

Remark 4.14. Observe that neither $\mathcal{C}$ nor $\mathcal{C}'$ are disjoint flag codes. However, Proposition 4.11 (resp. 4.12) allows us to decode at least one of the received subspaces $\mathcal{X}_i$ into the sent one $\mathcal{F}_i$ for an index i satisfying $|\mathcal{C}_i| = |\mathcal{C}|$ (resp. $|\mathcal{C}'_i| = |\mathcal{C}'|$). Hence, after having recovered $\mathcal{F}_i$, one can easily obtain the sent flag $\mathcal{F}$ as the unique flag in $\mathcal{C}$ (resp. $\mathcal{C}'$) having $\mathcal{F}_i$ as its i-th subspace.

4.4 The β-cyclic case

In this part of the paper, we consider orbits under the action of proper subgroups of $\mathbb{F}_{q^n}^*$ generated by the flag $\mathcal{F}$ given in (7.25), which has underlying Galois subflag $(\mathbb{F}_{q^{m_1}}, \mathbb{F}_{q^{m_2}}, \ldots, \mathbb{F}_{q^{m_k}})$. In other words, for every $\beta \in \mathbb{F}_{q^n}^*$, we study the generalized β-Galois flag code $\mathrm{Orb}_\beta(\mathcal{F})$. Recall that this code has type $(m_1, \ldots, m_2 - m_1, m_2, \ldots, m_3 - m_2, m_3, \ldots, m_k, \ldots, n - m_k)$ and it has the following particularity: the best friends of the subspaces of $\mathcal{F}$ are nested. More precisely, the subfield $\mathbb{F}_{q^{m_i}}$ is the best friend of the subspaces of dimensions $m_i, \ldots, m_{i+1} - m_i$ in the flag, for every $1 \leqslant i \leqslant k$, where $m_{k+1} = n$. This property makes our flag $\mathcal{F}$ be closer to the Galois flag of type $(m_1, \ldots, m_k)$ than other flags that also generalize it. As a result, we can give lower and upper bounds for the distance of $\mathrm{Orb}_\beta(\mathcal{F})$ by studying the sequence of subgroups

$$\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* \subseteq \langle\beta\rangle \cap \mathbb{F}_{q^{m_2}}^* \subseteq \cdots \subseteq \langle\beta\rangle \cap \mathbb{F}_{q^{m_k}}^*.$$

In particular, we consider two possibilities: either all these subgroups coincide or some inclusion is strict. In the latest case, we are especially interested in the first index $1 < i \leqslant k$ such that $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* \neq \langle\beta\rangle \cap \mathbb{F}_{q^{m_i}}^*$. Moreover, we exclude those elements $\beta \in \mathbb{F}_{q^{m_1}}^*$ since they provide trivial orbit flag codes with distance equal to zero.

Theorem 4.15. *Let $\mathcal{F}$ be the generalized Galois flag given in (7.25) and $\beta \in \mathbb{F}_{q^n}^* \setminus \mathbb{F}_{q^{m_1}}^*$. For every $1 \leqslant i \leqslant k$, we write $M_i = \sum_{j=1}^{i-1} m_{j+1}(L_{j+1}-1)$.*

(1) If $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* = \langle\beta\rangle \cap \mathbb{F}_{q^{m_k}}^*$, then

$$2m_k(L_{k+1}-1) + M_k \leqslant d_f(\mathrm{Orb}_\beta(\mathcal{F})) \leqslant m_k \left\lfloor \frac{L_{k+1}^2}{2} \right\rfloor + M_k.$$

(2) Otherwise, consider the minimum $1 < i \leqslant k$ such that $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* \subsetneq \langle\beta\rangle \cap \mathbb{F}_{q^{m_i}}^*$. Then it holds:

$$2m_{i-1}(L_i-1) + M_{i-1} \leqslant d_f(\mathrm{Orb}_\beta(\mathcal{F})) \leqslant m_{i-1} \left\lfloor \frac{L_i^2}{2} \right\rfloor + M_{i-1}.$$

Proof. Assume that $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* = \langle\beta\rangle \cap \mathbb{F}_{q^{m_k}}^*$. Let us compute the distance $d_f(\mathcal{F}, \mathcal{F}\beta^l)$, for every element $\beta^l \notin \mathrm{Stab}_\beta(\mathcal{F}) = \langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^*$. Since $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}}^* = \langle\beta\rangle \cap \mathbb{F}_{q^{m_k}}^*$, this power β^l does not stabilize any subspace in the flag $\mathcal{F}$. In particular, observe that $\mathbb{F}_{q^{m_k}}$ and $\mathbb{F}_{q^{m_k}}\beta^l$ are different subspaces in the spread $\mathrm{Orb}(\mathbb{F}_{q^{m_k}})$. In other words, it holds $d_S(\mathbb{F}_{q^{m_k}}, \mathbb{F}_{q^{m_k}}\beta^l) = 2m_k$ and, by means of Theorem 3.31, every subspace distance between subspaces of $\mathcal{F}$ and $\mathcal{F}\beta^l$ of dimensions lower than m_k is maximum as well, i.e., twice the corresponding dimension. Hence, for dimensions up to $m_k - m_{k-1} = m_{k-1}(L_k - 1)$, we obtain the sum of subspace distances:

$$\begin{array}{rcl} \sum_{j=1}^{k-1} \big(2m_j + \cdots + 2m_j(L_{j+1}-1)\big) & = & \sum_{j=1}^{k-1} 2m_j\big(1 + \cdots + (L_{j+1}-1)\big) \\ & = & \sum_{j=1}^{k-1} m_j L_{j+1}(L_{j+1}-1) \\ & = & \sum_{j=1}^{k-1} m_{j+1}(L_{j+1}-1) = M_k. \end{array}$$

Moreover, since subspaces of dimensions $m_k, \ldots, n - m_k = m_k(L_{k+1}-1)$ have $\mathbb{F}_{q^{m_k}}$ as their best friend, if d represents the flag distance between the subflags of type $m_k(1, \ldots, L_{k+1}-1)$ of $\mathcal{F}$ and $\mathcal{F}\beta^l$ and, by means of (7.16), we have:

$$2m_k(L_{k+1}-1) \leqslant d \leqslant m_k \left\lfloor \frac{L_{k+1}^2}{2} \right\rfloor.$$

Combining these two facts, we get the desired lower and upper bounds for $d_f(\mathcal{F}, \mathcal{F}\beta^l)$, if $\beta^l \notin \mathrm{Stab}_\beta(\mathcal{F})$. In particular, these bounds are also valid for $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$.

To prove (2), suppose that $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}} \neq \langle\beta\rangle \cap \mathbb{F}_{q^{m_k}}$ and then take the minimum $1 < i \leqslant k$ such that $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}} \subsetneq \langle\beta\rangle \cap \mathbb{F}_{q^{m_i}}$. In this case, we can always find an element $\beta^l \in \mathbb{F}^*_{q^{m_i}} \setminus \mathbb{F}^*_{q^{m_1}}$. This power β^l stabilizes every subspace in $\mathcal{F}$ having the subfield $\mathbb{F}_{q^{m_i}}$ as a friend, i.e., all those of dimensions at least m_i. This means that these dimensions do not contribute to the computation of $d_f(\mathcal{F}, \mathcal{F}\beta^l)$. On the other hand, since $\langle\beta\rangle \cap \mathbb{F}_{q^{m_1}} = \cdots = \langle\beta\rangle \cap \mathbb{F}_{q^{m_{i-1}}}$, then $\mathbb{F}_{q^{m_{i-1}}} \neq \mathbb{F}_{q^{m_{i-1}}}\beta^l$ are different spread elements and the distance between them is $2m_{i-1}$. As before, by means of Theorem 3.31, all the subspace distances are maximum for dimensions up to m_i. In particular, the distance between the subflags of type $(m_1, \dots, m_i - m_{i-1})$ of $\mathcal{F}$ and $\mathcal{F}\beta^l$ is exactly

$$\sum_{j=1}^{i-2} m_{j+1}(L_{j+1} - 1) = M_{i-1}.$$

Besides, observe that the subspaces of dimensions $m_{i-1}, \dots, m_i - m_{i-1}$ of $\mathcal{F}$ and $\mathcal{F}\beta^l$ are $\mathbb{F}_{q^{m_{i-1}}}$-subspaces of $\mathbb{F}_{q^{m_i}}$. Hence, if d denotes the distance between the corresponding subflags of $\mathcal{F}$ and $\mathcal{F}\beta^l$, by (7.16), we have

$$2m_{i-1}(L_i - 1) \leqslant d \leqslant m_{i-1} \left\lfloor \frac{L_i^2}{2} \right\rfloor.$$

As a result, we conclude

$$2m_{i-1}(L_i - 1) + M_{i-1} \leqslant d_f(\mathcal{F}, \mathcal{F}\beta^l) \leqslant m_{i-1} \left\lfloor \frac{L_i^2}{2} \right\rfloor + M_{i-1} \tag{7.26}$$

for every $\beta^l \in \mathbb{F}^*_{q^{m_i}} \setminus \mathbb{F}^*_{q^{m_1}}$. Arguing as above, if we take another power of β not in $\mathbb{F}^*_{q^{m_i}}$, say β^h, we obtain maximum subspace distances up to, at least, dimensions m_i and then

$$d_f(\mathcal{F}, \mathcal{F}\beta^h) \geqslant d_f(\mathcal{F}, \mathcal{F}\beta^l).$$

As a consequence, the minimum distance of the code is attained when we consider powers $\beta^l \in \mathbb{F}^*_{q^{m_i}} \setminus \mathbb{F}^*_{q^{m_1}}$ and we have the result. ∎

Observe that the upper bound for the distance in the first part of Theorem 4.15 is exactly the maximum possible distance for general flags of the corresponding type. Here below, we go a step further and give a sufficient condition for our construction to provide optimum distance flag codes, i.e., flag codes with the maximum possible distance for their type on $\mathbb{F}_{q^n}$.

Corollary 4.16. *Consider the generalized Galois flag $\mathcal{F}$ given in (7.25) and take $\beta \in \mathbb{F}^*_{q^n}$ such that $\langle\beta\rangle \cap \mathbb{F}^*_{q^{m_1}} = \langle\beta\rangle \cap \mathbb{F}^*_{q^{m_k}}$. If $L_{k+1} \leqslant 3$, then $\mathrm{Orb}_\beta(\mathcal{F})$ is an optimum distance flag code.*

Proof. Notice that, under these assumptions, by means of Theorem 4.15 (part (1)), we have

$$2m_k(L_{k+1}-1)+M_k \leqslant d_f(\mathrm{Orb}_\beta(\mathcal{F})) \leqslant m_k \left\lfloor \frac{L_{k+1}^2}{2} \right\rfloor + M_k.$$

Moreover, if the degree of the extension $\mathbb{F}_{q^n}/\mathbb{F}_{q^{m_k}}$, that is, the positive integer $L_{k+1} = [\mathbb{F}_{q^n} : \mathbb{F}_{q^{m_k}}] = n/m_{k+1}$, satisfies $1 < L_{k+1} \leqslant 3$, then we have

$$2(L_{k+1}-1) = \left\lfloor \frac{L_{k+1}^2}{2} \right\rfloor.$$

Hence, both lower and upper bounds for $d_f(\mathrm{Orb}_\beta(\mathcal{F}))$ coincide and the code $\mathrm{Orb}_\beta(\mathcal{F})$ attains the maximum possible distance for its type vector. ∎

To finish this subsection we address the question $(*)$ launched in Subsection 3.4. Recall that the potential distance values of generalized Galois flag code follow the rules stated in Theorem 3.31 and Definition 3.32. These conditions arise naturally from the presence of certain subfields among the subspaces of a generalized Galois flag. Concerning question $(*)$, we wonder if, given a generalized Galois flag $\mathcal{F}$, every potential value of the distance can be truly obtained by a cyclic (or β-cyclic) orbit flag code generated by $\mathcal{F}$. We answer this question by using the β-cyclic construction presented in Subsection 4.4

Example 4.17. *Consider the following parameters choice: $q=2$, $n=10$. Moreover, we take nested subfields $\mathbb{F}_2 \subset \mathbb{F}_{2^5}$ of the field $\mathbb{F}_{2^{10}}$, which correspond to the election of divisors $m_1 = 1$ and $m_2 = 5$ of $n = 10$. In this case, we have $L_2 = 5$ and $L_3 = 2$. Let us use the generalized Galois flag $\mathcal{F} = (\mathbb{F}_2, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathbb{F}_{2^5})$ of type $(1,2,3,4,5)$ with subspaces:*

$$\mathcal{F}_2 = \mathbb{F}_2 \oplus \mathbb{F}_2\gamma, \qquad \mathcal{F}_3 = \mathbb{F}_2 \oplus \mathbb{F}_2\gamma \oplus \mathbb{F}_2\gamma^2, \quad \textit{and} \quad \mathcal{F}_4 = \mathbb{F}_2 \oplus \mathbb{F}_2\gamma \oplus \mathbb{F}_2\gamma^2 \oplus \mathbb{F}_2\gamma^3,$$

where γ is a primitive element of $\mathbb{F}_{2^5}$.

The set of potential values of the distance in this case is given by:

$$\{0, 8, 10, 12, 30\}.$$

In this case, we know how to choose $\beta \in \mathbb{F}_{2^{10}}^*$ so that the orbit $\mathrm{Orb}_\beta(\mathcal{F})$ attains some of these distances. More precisely:

- *Distance $d=0$ is obtained if, and only if, $\beta \in \mathbb{F}_2^* = \{1\} = \mathrm{Stab}(\mathcal{F})$.*
- *For distance $d=8$, it suffices to take the cyclic orbit code $\mathrm{Orb}(\mathcal{F})$ that, by means of Theorem 4.2, has distance $d_f(\mathrm{Orb}(\mathcal{F})) = 8$.*

- *Last, since $L_3 = 2 \leqslant 3$, by application of Corollary 4.16, we know that every $\beta \in \mathbb{F}_{2^{10}}^*$ such that $\langle\beta\rangle \cap \mathbb{F}_{2^5} = \{1\}$ makes $\mathrm{Orb}_\beta(\mathcal{F})$ be an optimum distance flag code. For instance, it suffices to consider subgroups $\langle\beta\rangle$ of $\mathbb{F}_{2^{10}}^*$ of orders $\{3, 11, 33\}$ to attain the maximum distance, i.e., the value $d = 30$.*

Moreover, for this specific example, we have obtained the parameters of the code $\mathrm{Orb}_\beta(\mathcal{F})$, for every subgroup $\langle\beta\rangle$ of $\mathbb{F}_{2^{10}}^*$ by using GAP. First of all, since $\mathrm{Stab}(\mathcal{F}) = \mathbb{F}_2^* = \{1\}$, we have $|\mathrm{Orb}_\beta(\mathcal{F})| = |\beta|$. The next table collects the set of distances for the generating flag $\mathcal{F}$:

$\|\beta\|$	$d_f(\mathrm{Orb}_\beta(\mathcal{F}))$
1	0
3	30
11	30
31	8
33	30
93	8
341	8
1023	8

Table 7.2: Distance of all the β-cyclic orbit flag code generated by $\mathcal{F}$.

Using this example, one can see that not all the potential values of the distance can be obtained by taking a suitable subgroup of $\mathbb{F}_{2^{10}}^*$. It suffices to observe that neither distances $d = 10$ nor $d = 12$ appear in Table 7.2. Even more, despite the fact that $d_f(\mathrm{Orb}_\beta(\mathcal{F})) \neq 12$ for any $\beta \in \mathbb{F}_{2^{10}}^*$, this value still can be the distance between a couple of flags; for instance, we have

$$d_f(\mathcal{F}, \mathcal{F}\gamma^2) = 2 + 4 + 4 + 2 + 0 = 12.$$

However, this is not even true for distance $d = 10$. In other words, for every $\beta \in \mathbb{F}_{2^{10}}^*$ and any power $1 \leqslant l \leqslant |\beta|$, the distance $d_f(\mathcal{F}, \mathcal{F}\beta^l) \neq 10$.

5 Conclusions and future work

In this work we present new contributions to the study of β-cyclic orbit flag codes started in [3], also following the viewpoint of [11]. The best friend of a flag code still has a crucial role throughout the paper. In particular, we discuss the rich interplay among flag distances, best friend and type vector for this family of codes.

Nevertheless, whereas in [3] the accent was put precisely on the best friend of the flag code, this time we turn our attention to the generating flag of the orbit.

We focus especially on those ones having at least one field among their subspaces, by distinguishing the case of having just fields on the generating flag from the case where also at least one subspace not being a field appears. This dichotomy leads, on one side, to the known β-Galois flag codes and, on the other one, to the generalized β-Galois flag codes, which properties we describe.

Every generalized β-Galois flag code has an underlying β-Galois flag code. Thus, we have addressed the question of determine if the parameters and the behaviour of Galois flag codes drives, in some sense, the ones of the generalized ones. To do this, we provide a systematic construction of generalized Galois flag codes with a prescribed underlying Galois flag code that presents remarkable properties and helps to us to shed some light on the raised questions.

To future work, we want to deepen the study of β-cyclic orbit codes by determining suitable generating flags that allow us to obtain a prefixed distance value and code sizes as large as possible, even when it is necessary to take unions or orbits.

BIBLIOGRAPHY

[1] R. Ahlswede, N. Cai, R. Li and R. W. Yeung, *Network Information Flow*, IEEE Transactions on Information Theory, Vol. 46 (2000), 1204-1216.

[2] C. Alonso-González, M. A. Navarro-Pérez, *Consistent Flag Codes*, Mathematics, Vol. 8(12) (2020), 2243.

[3] C. Alonso-González and M. A. Navarro-Pérez, *Cyclic Orbit Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2331-2356.

[4] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An Orbital Construction of Optimum Distance Flag Codes*, Finite Fields and Their Applications, Vol. 73 (2021), 101861.

[5] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes: Distance Vectors and Cardinality Bounds*, https://arxiv.org/abs/2111.00910 (preprint).

[6] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes from Planar Spreads in Network Coding*, Finite Fields and Their Applications, Vol. 68 (2020), 101745.

[7] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.

[8] E. Ben-Sasson, T. Etzion, A. Gabizon, and N. Raviv, *Subspace Polynomials and Cyclic Subspace Codes*, IEEE Transactions on Information Theory, Vol. 62 (2016), 1157–1165.

[9] B. Chen and H. Liu, *Constructions of Cyclic Constant Dimension Codes*, Designs, Codes and Cryptography, Vol. 86(6) (2018), 1267–1279.

[10] T. Etzion and A. Vardy, *Error-Correcting Codes in Projective Space*, IEEE Transactions on Information Theory, Vol. 57 (2011), 1165–1173.

[11] H. Gluesing-Luerssen, K. Morrison and C. Troha, *Cyclic Orbit Codes and Stabilizer Subfields*, Advances in Mathematics of Communications, Vol. 9(2) (2015), 177-197.

[12] E. Gorla, F. Manganiello and J. Rosenthal, *An Algebraic Approach for Decoding Spread Codes*, Advances in Mathematics of Communications, Vol. 6(4) (2012), 443-466.

[13] R. Koetter and F. Kschischang, *Coding for Errors and Erasures in Random Network Coding*, IEEE Transactions on Information Theory, Vol. 54 (2008) 3579-3591.

[14] S. Kurz, *Bounds for Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2759–2785.

[15] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Network Coding with Flags*, Designs, Codes and Cryptography, Vol. 86(2) (2018), 269-284.

[16] F. Manganiello, E. Gorla and J. Rosenthal, *Spread Codes and Spread Decoding in Network Coding*, in: Proceedings of the 2008 IEEE International Symposium on Information Theory (ISIT), Toronto, Canada, 2008, pp. 851-855.

[17] F. Manganiello and A.-L. Trautmann, *Spread Decoding in Extension Fields*, Finite Fields and Their Applications, Vol. 25 (2014), 94-105.

[18] F. Manganiello, A.-L. Trautmann and J. Rosenthal, *On Conjugacy Classes of Subgroups of the General Linear Group and Cyclic Orbit Codes*, in: Proceedings of the 2011 IEEE International Symposium on Information Theory (ISIT), Saint Pettersburg, 2011, pp. 1916–1920.

[19] M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes of Maximum Distance and Constructions using Singer Groups*, https://arxiv.org/abs/2109.00270 (preprint).

[20] K. Otal and F. Ozbudak, *Cyclic Subspace Codes via Subspace Polynomials*, Designs, Codes and Cryptography, Vol. 85(2) (2017), 191-204.

[21] J. Rosenthal and A.-L. Trautmann, *A Complete Characterization of Irreducible Cyclic Orbit Codes and their Plücker Embedding*, Designs, Codes and Cryptography, Vol. 66 (2013), 275–289.

[22] R. M. Roth, N. Raviv and I. Tamo, *Construction of Sidon Spaces With Applications to Coding*, IEEE Transactions on Information Theory, Vol. 64(6) (2018), 4412-4422.

[23] B. Segre, *Teoria di Galois, Fibrazioni Proiettive e Geometrie non Desarguesiane*, Annali di Matematica Pura ed Applicata, Vol. 64 (1964), 1-76.

[24] A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal, *Cyclic Orbit Codes*, IEEE Transactions on Information Theory, Vol. 59(11) (2013), 7386-7404.

[25] A.-L. Trautmann, F. Manganiello, and J. Rosenthal, *Orbit Codes: A New Concept in the Area of Network Coding*, in: Proceedings of IEEE Information Theory Workshop, Dublin, Ireland, 2010, pp. 1–4.

[26] A.-L. Trautmann and J. Rosenthal, *Constructions of Constant Dimension Codes*, in: M. Greferath et al. (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 25-42.

[27] W. Zhao and X. Tang. *A Characterization of Cyclic Subspace Codes via Subspace Polynomials*, Finite Fields and Their Applications, Vol. 57 (2019), 1–12.

CHAPTER 8

FLAG CODES: DISTANCE VECTORS AND CARDINALITY BOUNDS

Joint work with Clementa Alonso-González and Xaro Soler-Escrivà.

Abstract:

Given $\mathbb{F}_q$ the finite field with q elements and an integer $n \geqslant 2$, a *flag* is a sequence of nested subspaces of $\mathbb{F}_q^n$ and a *flag code* is a nonempty set of flags. In this context, the distance between flags is the sum of the corresponding subspace distances. Hence, a given flag distance value might be obtained by many different combinations. To capture such a variability, in the paper at hand, we introduce the notion of *distance vector* as an algebraic object intrinsically associated to a flag code that encloses much more information than the distance parameter itself. Our study of the flag distance by using this new tool allows us to provide a fine description of the structure of flag codes as well as to derive bounds for their maximum possible size once the minimum distance and dimensions are fixed.
Keywords: Network coding, flag codes, flag distance, bounds.

1 Introduction

Network coding was introduced in [1] as a new method for sending information within networks modelled as acyclic multigraphs with possibly several senders and receivers, where intermediate nodes are allowed to send linear combinations of the received vectors, instead of simply routing them. In [14], the reader can find the first algebraic approach to network coding through non-coherent networks, i.e., those which their topology does not need to be known. In the same paper, Kötter and Kschischang present *subspace codes* as the most appropriate codes to this situation. To be precise, if $\mathbb{F}_q$ is the finite field of q elements (with q a prime power) and we consider a positive integer $n \geqslant 2$, a *subspace code* is a nonempty collection of $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_q^n$. When every codeword has the same dimension, say $1 \leqslant k < n$, we speak about *constant dimension codes*. In this context, we use the *subspace distance*, denoted by d_S (see [14]). Constant dimension codes have been widely studied in the last decade. See, for instance, [23] and the references therein.

Size and minimum distance are the most important parameters associated to an error-correcting code. The first one gives us the number of different messages that can be encoded. The second one is related with the error-correction capability of the code. According to this, there are two central problems when working with constant dimension codes. On the one hand, the study and construction of codes having the maximum possible distance for their dimension (see, for instance, [7, 8, 12, 18]). On the other hand, determining (or giving bounds for) the value $A_q(n, d, k)$, that is, the maximum possible size of a constant dimension code in $\mathcal{G}_q(k, n)$ with minimum distance equal to d, is an interesting question that has led to many research works (see [10, 13, 16, 24], for instance).

In [17], the authors propose the use of *flag codes* in network coding for the first time. This class of codes generalizes constant dimension codes and represents a possible alternative to obtain codes with good parameters in case that neither n nor q could be increased. Flags are objects coming from classic linear algebra defined as follows. Given integers $1 \leqslant t_1 < \cdots < t_r < n$, a *flag* of type $\mathbf{t} = (t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a sequence $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ of nested subspaces $\mathcal{F}_i$ of $\mathbb{F}_q^n$ such that $\dim(\mathcal{F}_i) = t_i$, for every $1 \leqslant i \leqslant r$. In this setting, codewords are flags of the prescribed type vector $\mathbf{t}$. As for constant dimension codes, describing the family of flag codes attaining the maximum possible distance for their type (*optimum distance flag codes*) is a central question that has been addressed in [3, 4, 5, 19]. On the other hand, obtaining bounds for the value $A_q^f(n, d, \mathbf{t})$, i.e., the maximum possible size of flag codes of type $\mathbf{t}$ on $\mathbb{F}_q^n$ and minimum distance equal to d, is also an important problem that, up to now, has only been addressed in the work [15], where the author focuses on the *full type vector* $(1, \ldots, n-1)$. The current paper represents a contribution in this direction.

Given two flags $\mathcal{F}, \mathcal{F}'$ of type $\mathbf{t}$ on $\mathbb{F}_q^n$, their *flag distance* is defined as $d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i)$. This definition implies that a flag distance value might be obtained as different combinations of subspace distances and it suggests that the way we obtain the flag distance is relevant information to take into account beyond the proper numerical value. We deal with this question by introducing the concept of *distance vector* $\mathbf{d}(\mathcal{F}, \mathcal{F}) = (d_S(\mathcal{F}_1, \mathcal{F}'_1), \ldots, d_S(\mathcal{F}_r, \mathcal{F}'_r))$ associated to the pair of flags $\mathcal{F}$ and $\mathcal{F}'$. This is an algebraic object strongly related not only to the value of the distance $d_f(\mathcal{F}, \mathcal{F}')$ but also to the nested linear structure of these flags. With its help, we develop techniques that allow us to study both the cardinality and the minimum distance of flag codes. First, the study of distance vectors will allow us to determine how the flag distance fluctuates when we consider flags sharing a given number of subspaces. Hence, we investigate the structure of flag codes in which different flags do not share simultaneously their subspaces of a prescribed set of dimensions. This approach leads us both to derive some structural properties of the flag code and to obtain upper bounds for the value $A_q^f(n, d, \mathbf{t})$, strongly based on the maximum number of subspaces that different flags in a flag code of type $\mathbf{t}$ and distance d can share.

The paper is organized as follows. In Section 2, we recall some definitions and known facts related with constant dimension codes and flag codes. In Section 3, the notion of distance vector and a characterization of them are presented. Section 4 is devoted to study the flag distance between flags of the same type that share certain subspaces. In Section 5, we generalize the notion of *disjointness* introduced in [4] and use the results obtained in the previous section in order to deduce structural properties of a code by simply looking at its minimum distance. In Section 6, we apply the concepts and results in Sections 4 and 5 to extract bounds for the values $A_q^f(n, d, \mathbf{t})$. Last, Section 7 is dedicated to developing a very complete example that illustrates in detail the techniques previously discussed.

2 Preliminaries

In this section we recall some known facts on subspace and flag codes that we need in this paper. We start fixing some notation. Let q be a prime power and consider the finite field $\mathbb{F}_q$ with q elements. For every positive integer $n \geqslant 2$, we write $\mathbb{F}_q^n$ to denote the n-dimensional vector space over the field $\mathbb{F}_q$. Given a positive integer $k \leqslant n$, the *Grassmann variety*, or simply the Grassmannian, of dimension k is the set $\mathcal{G}_q(k,n)$ of k-dimensional vector subspaces of $\mathbb{F}_q^n$. It is well known (see [14]) that

$$|\mathcal{G}_q(k,n)| = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n-1)\dots(q^{n-k+1}-1)}{(q^k-1)\dots(q-1)}. \tag{8.1}$$

The Grassmannian can be seen as a metric space endowed with the *subspace distance* defined as

$$d_S(\mathcal{U},\mathcal{V}) = \dim(\mathcal{U}+\mathcal{V}) - \dim(\mathcal{U}\cap\mathcal{V}) = 2(k - \dim(\mathcal{U}\cap\mathcal{V})). \tag{8.2}$$

for all $\mathcal{U},\mathcal{V} \in \mathcal{G}_q(k,n)$. A *constant dimension code* $\mathcal{C}$ in $\mathcal{G}_q(k,n)$ is a nonempty collection of k-dimensional vector subspaces of $\mathbb{F}_q^n$. These codes were introduced in [14] and studied many papers (see [23] and references therein for further information). The *minimum distance* of $\mathcal{C}$ is the value

$$d_S(\mathcal{C}) = \min\{d_S(\mathcal{U},\mathcal{V}) \mid \mathcal{U},\mathcal{V}\in\mathcal{C},\ \mathcal{U}\neq\mathcal{V}\},$$

whenever $|\mathcal{C}| \geqslant 2$. If $|\mathcal{C}| = 1$, we put $d_S(\mathcal{C}) = 0$. In any case, the subspace distance is an even integer such that

$$0 \leqslant d_S(\mathcal{C}) \leqslant \begin{cases} 2k & \text{if } \ 2k \leqslant n, \\ 2(n-k) & \text{if } \ 2k \geqslant n. \end{cases} \tag{8.3}$$

The study and construction of constant dimension codes attaining this upper bound for the distance has been addressed in several papers (see [8, 18], for instance). Another important problem is the one of determining (or giving bounds for) the value $A_q(n,d,k)$, which denotes the maximum possible size for constant dimension codes in $\mathcal{G}_q(k,n)$ having prescribed minimum distance d. The reader can find constructions of constant dimension codes as well as lower and upper bounds for $A_q(n,d,k)$ in [6, 10, 12, 13, 16, 20, 21, 22, 23, 24]. As a generalization of constant dimension codes, in [17], the authors introduced the use of flag codes in network Coding. Let us recall some basic definitions in this matter.

Given integers $1 \leqslant t_1 < \cdots < t_r < n$, a *flag* of type $\mathbf{t} = (t_1,\dots,t_r)$ on $\mathbb{F}_q^n$ is a sequence $\mathcal{F} = (\mathcal{F}_1,\dots,\mathcal{F}_r)$ of nested subspaces $\mathcal{F}_i$ of $\mathbb{F}_q^n$ such that $\dim(\mathcal{F}_i) = t_i$, for every $1 \leqslant i \leqslant r$. The vector $(1,\dots,n-1)$ is called the *full type vector* and flags of this type are known as *full flags*.

Throughout the rest of the paper, we will write $\mathbf{t}$ to denote an arbitrary but fixed type vector $\mathbf{t} = (t_1, \dots, t_r)$. The *flag variety* $\mathcal{F}_q(\mathbf{t}, n)$ is the set of all the flags of type $\mathbf{t}$ on $\mathbb{F}_q^n$. This variety contains exactly

$$|\mathcal{F}_q(\mathbf{t}, n)| = \begin{bmatrix} n \\ t_1 \end{bmatrix}_q \begin{bmatrix} n - t_1 \\ t_2 - t_1 \end{bmatrix}_q \cdots \begin{bmatrix} n - t_{r-1} \\ n - t_r \end{bmatrix}_q \tag{8.4}$$

elements (see [15]) and it can be equipped with the *flag distance*, computed as

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i), \tag{8.5}$$

for every pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n)$.

A *flag code* $\mathcal{C}$ of type $\mathbf{t}$ on $\mathbb{F}_q^n$ is a nonempty subset of $\mathcal{F}_q(\mathbf{t}, n)$. We can naturally associate to it a family of r constant dimension codes by projection. For every $1 \leqslant i \leqslant r$, consider the map

$$p_i : \mathcal{F}_q(\mathbf{t}, n) \longrightarrow \mathcal{G}_q(t_i, n) \tag{8.6}$$

defined as $p_i((\mathcal{F}_1, \dots, \mathcal{F}_r)) = \mathcal{F}_i$, for every $(\mathcal{F}_1, \dots, \mathcal{F}_r) \in \mathcal{F}_q(\mathbf{t}, n)$. With this notation, the *i-th projected code* $\mathcal{C}_i$ of the flag code $\mathcal{C}$ is the constant dimension code $\mathcal{C}_i = p_i(\mathcal{C}) \subseteq \mathcal{G}_q(t_i, n)$, consisting of all the i-th subspaces of flags in $\mathcal{C}$.

If $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ is a flag code with $|\mathcal{C}| \geqslant 2$, its *minimum distance* is defined as

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}$$

and, if $|\mathcal{C}| = 1$, we put $d_f(\mathcal{C}) = 0$. Notice that, by means of (8.3), we can easily deduce that $d_f(\mathcal{C})$ is an even integer such that

$$0 \leqslant d_f(\mathcal{C}) \leqslant 2 \left(\sum_{t_i \leqslant \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right). \tag{8.7}$$

When working with full flag codes, the previous bound becomes

$$0 \leqslant d_f(\mathcal{C}) \leqslant \begin{cases} \frac{n^2}{2} & \text{if} \quad n \quad \text{is even}, \\ \frac{n^2-1}{2} & \text{if} \quad n \quad \text{is odd}. \end{cases} \tag{8.8}$$

In the flag codes setting, we write $A_q^f(n, d, \mathbf{t})$ to denote the maximum attainable size for a flag code in $\mathcal{F}_q(\mathbf{t}, n)$ with minimum distance equal to d. In case of working with full flags, we drop the type vector and simply write $A_q^f(n, d)$. This notation was recently introduced by Kurz in [15]. In that work, the author provided techniques to upper and lower bound these values in the full type case. Moreover, an exhaustive list of exact values of $A_q^f(n, d)$ is also given for small values of n.

3 Flag distance versus distance vectors

As seen in Section 2, the flag distance extends, in some sense, the subspace distance. However, since it is defined as a sum, a particular flag distance value might be attained by adding different combinations of subspace distances. This makes that the minimum distance of a flag code will have associated some of the possible combinations (maybe all of them). In order to clarify this fact, in this section, we introduce the concept of *distance vector* to better represent how the distance between different flags is distributed among their subspaces.

Definition 3.1. Given two different flags $\mathcal{F}, \mathcal{F}'$ of type $\mathbf{t}$ on $\mathbb{F}_q^n$, their associated *distance vector* is

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') = (d_S(\mathcal{F}_1, \mathcal{F}'_1), \dots, d_S(\mathcal{F}_r, \mathcal{F}'_r)) \in 2\mathbb{Z}^r.$$

Notice that the sum of the components of $\mathbf{d}(\mathcal{F}, \mathcal{F}')$ is the flag distance $d_f(\mathcal{F}, \mathcal{F}')$ defined in (8.5). Given a positive integer $n \geqslant 2$ and a type vector $\mathbf{t}$, we denote by $D^{(\mathbf{t},n)}$ the maximum possible value of the flag distance in $\mathcal{F}_q(\mathbf{t}, n)$ that, as a consequence of (8.7), is

$$D^{(\mathbf{t},n)} = 2\left(\sum_{t_i \leqslant \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right). \tag{8.9}$$

In particular, when working with the full type vector, we simply write

$$D^n = \begin{cases} \frac{n^2}{2} & \text{if } \ n \ \text{ is even,} \\ \frac{n^2-1}{2} & \text{if } \ n \ \text{ is odd} \end{cases} \tag{8.10}$$

to denote the maximum possible distance between full flags on $\mathbb{F}_q^n$ (see (8.8)). For technical reasons, even if we work with $n \geqslant 2$, we extend this definition to the case $n = 1$ and put $D^1 = 0$.

From now on, we write d to denote an even integer such that $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$. Observe that, under these conditions, we can always find flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n)$ such that $d_f(\mathcal{F}, \mathcal{F}') = d$. Hence, such a value d represents the possible values for the flag distance in $\mathcal{F}_q(\mathbf{t}, n)$. Let us study in which ways this distance value d can be obtained.

Definition 3.2. Let d be an even integer such that $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$. We define *the set of distance vectors associated to d for the flag variety $\mathcal{F}_q(\mathbf{t}, n)$* as

$$\mathcal{D}(d, \mathbf{t}, n) = \{\mathbf{d}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n), \ d_f(\mathcal{F}, \mathcal{F}') = d\} \subseteq 2\mathbb{Z}^r.$$

On the other hand, the set of *distance vectors for the flag variety $\mathcal{F}_q(\mathbf{t}, n)$* is

$$\mathcal{D}(\mathbf{t}, n) = \{\mathbf{d}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n)\} \subseteq 2\mathbb{Z}^r.$$

and it holds

$$\mathcal{D}(\mathbf{t}, n) = \bigcup_{d} \mathcal{D}(d, \mathbf{t}, n),$$

where d takes all the even integers between $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$. When working with the full flag variety, we drop the type vector and simply write $\mathcal{D}(d, n)$ and $\mathcal{D}(n)$, respectively.

The next result reflects that, for every choice of the type vector, the set $\mathcal{D}(\mathbf{t}, n)$ can be obtained from $\mathcal{D}(n)$ by using the projection

$$\begin{array}{rccc} \pi_{\mathbf{t}} & : & \mathbb{Z}^{n-1} & \longrightarrow & \mathbb{Z}^{r} \\ & & (v_1, \ldots, v_{n-1}) & \longmapsto & (v_{t_1}, \ldots, v_{t_r}). \end{array} \tag{8.11}$$

Proposition 3.3. *Consider a type vector* $\mathbf{t}$ *and the projection map* $\pi_{\mathbf{t}}$ *defined in (8.11). It holds*

$$\pi_{\mathbf{t}}(\mathcal{D}(n)) = \mathcal{D}(\mathbf{t}, n),$$

i.e., distance vectors for an arbitrary flag variety can be obtained by projection from (possibly several) distance vectors for the full flag variety.

Proof. Take a distance vector $\mathbf{d}(\mathcal{F}, \mathcal{F}') \in \mathcal{D}(n)$, for a pair of full flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_{n-1})$. It suffices to see that $\pi_{\mathbf{t}}(\mathbf{d}(\mathcal{F}, \mathcal{F}'))$ is the distance vector associated to the pair of flags $\bar{\mathcal{F}} = (\mathcal{F}_{t_1}, \ldots, \mathcal{F}_{t_r})$ and $\bar{\mathcal{F}}' = (\mathcal{F}'_{t_1}, \ldots, \mathcal{F}'_{t_r})$ in $\mathcal{F}_q(\mathbf{t}, n)$.

Conversely, given two flags $\bar{\mathcal{F}} = (\bar{\mathcal{F}}_1, \ldots, \bar{\mathcal{F}}_r)$ and $\bar{\mathcal{F}}' = (\bar{\mathcal{F}}'_1, \ldots, \bar{\mathcal{F}}'_r)$ in $\mathcal{F}_q(\mathbf{t}, n)$, we can consider full flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_{n-1})$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_{n-1})$ such that $\mathcal{F}_{t_i} = \bar{\mathcal{F}}_i$ and $\mathcal{F}'_{t_i} = \bar{\mathcal{F}}'_i$, for all $1 \leqslant i \leqslant r$. In this case, it holds $\pi_{\mathbf{t}}(\mathbf{d}(\mathcal{F}, \mathcal{F}')) = \mathbf{d}(\bar{\mathcal{F}}, \bar{\mathcal{F}}')$. ∎

Remark 3.4. Notice that, for every even integer d such that $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$, the set $\mathcal{D}(d, \mathbf{t}, n)$ is nonempty. Moreover, for some values of d, the set $\mathcal{D}(d, \mathbf{t}, n)$ is reduced to just one element. For instance, if we take $d = 0$, it holds $\mathcal{D}(0, \mathbf{t}, n) = \{\mathbf{0}\}$. If $d = D^{(\mathbf{t},n)}$, there is also a unique distance vector, that we denote by $\mathbf{D}^{(\mathbf{t},n)}$. For every $1 \leqslant i \leqslant r$, its i-th component $D_i^{(\mathbf{t},n)}$ is exactly

$$D_i^{(\mathbf{t},n)} = \min\{2t_i, 2(n - t_i)\}, \tag{8.12}$$

i.e., the maximum possible distance between t_i-dimensional subspaces of $\mathbb{F}_q^n$. Observe that, in particular, the distance vector $\mathbf{D}^{(\mathbf{t},n)}$ does not have any zero component. As before, when working with the full type vector, we simply write $\mathbf{D}^n = (D_1^n, \ldots, D_{n-1}^n)$ to denote the unique distance vector associated to the maximum possible flag distance D^n, given in (8.10). Its components are $D_i^n = \min\{2i, 2(n-i)\}$, for $1 \leqslant i \leqslant n-1$. In other cases, the set $\mathcal{D}(d, \mathbf{t}, n)$ might contain more than one element, as we can see in Example 3.8.

Using the projection defined in (8.11), and arguing as in Proposition 3.3, the next result follows straightforwardly.

Corollary 3.5. *Consider a positive integer n and fix a type vector $\mathbf{t}$ for $\mathbb{F}_q^n$. It holds*

$$\pi_{\mathbf{t}}(\mathbf{D}^n) = \mathbf{D}^{(\mathbf{t},n)}.$$

In the following definition we collect the subset of distance vectors of $\mathcal{D}(\mathbf{t}, n)$ that are significant for a flag code in $\mathcal{F}_q(\mathbf{t}, n)$.

Definition 3.6. Given a flag code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$, its *set of distance vectors* is

$$\mathcal{D}(\mathcal{C}) = \{\mathbf{d}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C})\}.$$

Remark 3.7. In general, given a flag code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ and a pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}')$, it holds

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') \in \mathcal{D}(\mathcal{C}) \subseteq \mathcal{D}(d_f(\mathcal{C}), \mathbf{t}, n) \subseteq \mathcal{D}(\mathbf{t}, n).$$

Example 3.8. *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard $\mathbb{F}_q$-basis of $\mathbb{F}_q^4$ and consider the following full flags on $\mathbb{F}_q^4$.*

$$\begin{array}{rcl} \mathcal{F}^1 & = & (\langle \mathbf{e}_1 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4 \rangle), \\ \mathcal{F}^2 & = & (\langle \mathbf{e}_2 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle), \\ \mathcal{F}^3 & = & (\langle \mathbf{e}_1 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle), \\ \mathcal{F}^4 & = & (\langle \mathbf{e}_2 \rangle, \ \langle \mathbf{e}_2, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle). \end{array}$$

Notice that $D^4 = 16/2 = 8$. Thus, the possible values of the flag distance for full flags on $\mathbb{F}_q^4$ are all the even integers $d \in [0, 8]$. In particular, for $d = 4$, vectors $(2,0,2), (0,2,2), (2,2,0)$ are elements in $\mathcal{D}(4,4)$ since

$$\mathbf{d}(\mathcal{F}^1, \mathcal{F}^2) = (2,0,2),\ \mathbf{d}(\mathcal{F}^1, \mathcal{F}^3) = (0,2,2) \ \text{and} \ \mathbf{d}(\mathcal{F}^2, \mathcal{F}^3) = (2,2,0).$$

On the other hand, if we take the full flag code $\mathcal{C} = \{\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^4\}$, it holds

$$\begin{array}{rcccccccl} d_f(\mathcal{F}^1, \mathcal{F}^3) & = & 0 & + & 2 & + & 2 & = & 4, \\ d_f(\mathcal{F}^1, \mathcal{F}^4) & = & 2 & + & 2 & + & 2 & = & 6, \\ d_f(\mathcal{F}^3, \mathcal{F}^4) & = & 2 & + & 2 & + & 0 & = & 4. \end{array}$$

Hence, the distance of the code is $d_f(\mathcal{C}) = 4$ and $\mathcal{D}(\mathcal{C}) = \{(0,2,2), (2,2,0)\} \subsetneq \mathcal{D}(4,4)$.

Up to now, to show that a given vector $\mathbf{v} \in 2\mathbb{Z}^r$ is a distance vector in $\mathcal{D}(\mathbf{t}, n)$, we need to exhibit a pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n)$ such that $\mathbf{v} = \mathbf{d}(\mathcal{F}, \mathcal{F}')$. We finish the section with the next result that characterizes distance vectors in terms of some properties satisfied by their components.

Theorem 3.9. *Let d be an even integer such $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$. A vector $\mathbf{v} = (v_1, \ldots, v_r)$ is a distance vector in $\mathcal{D}(d, \mathbf{t}, n)$ if, and only if, the following statements hold:*

(i) $\sum_{i=1}^{r} v_i = d$,

(ii) $v_i \in 2\mathbb{Z}$, *for all* $1 \leqslant i \leqslant r$,

(iii) $0 \leqslant v_i \leqslant \min\{2t_i, 2(n-t_i)\}$, *for every* $1 \leqslant i \leqslant r$, *and*

(iv) $|v_{i+1} - v_i| \leqslant 2(t_{i+1} - t_i)$, *for* $1 \leqslant i \leqslant r-1$.

Proof. We start assuming that $\mathbf{v} \in \mathcal{D}(d, \mathbf{t}, n)$. Statements (i), (ii) and (iii) follow from the definition of $\mathcal{D}(d, \mathbf{t}, n)$. Let us prove (iv). Since $\mathbf{v} \in \mathcal{D}(d, \mathbf{t}, n)$, there must exist flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(\mathbf{t}, n)$ such that $d = d_f(\mathcal{F}, \mathcal{F}')$ and $\mathbf{v} = \mathbf{d}(\mathcal{F}, \mathcal{F}')$, i.e., $v_i = d_S(\mathcal{F}_i, \mathcal{F}'_i) = 2(t_i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i))$, for every $1 \leqslant i \leqslant r$. Notice that, for every $1 \leqslant i \leqslant r-1$, it holds

$$2t_i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i) = \dim(\mathcal{F}_i + \mathcal{F}'_i) \leqslant \dim(\mathcal{F}_{i+1} + \mathcal{F}'_{i+1}) = 2t_{i+1} - \dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1})$$

and, as a consequence,

$$\dim(\mathcal{F}_i \cap \mathcal{F}'_i) \leqslant \dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1}) \leqslant \dim(\mathcal{F}_i \cap \mathcal{F}'_i) + 2(t_{i+1} - t_i). \tag{8.13}$$

Moreover, we have that

$$v_{i+1} - v_i = 2(t_{i+1} - t_i) - 2(\dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1}) - \dim(\mathcal{F}_i \cap \mathcal{F}'_i)). \tag{8.14}$$

Hence, by using the first inequality of (8.13), we clearly obtain $v_{i+1} - v_i \leqslant 2(t_{i+1} - t_i)$. On the other hand, combining the second inequality of (8.13) and (8.14), we get

$$v_{i+1} - v_i \geqslant 2(t_{i+1} - t_i) - 4(t_{i+1} - t_i) = -2(t_{i+1} - t_i)$$

and (iv) holds.

Let us prove the converse. To do so, assume that $\mathbf{v} = (v_1, \ldots, v_r)$ is a vector satisfying conditions (i)-(iv). We want to show that $\mathbf{v} \in \mathcal{D}(d, \mathbf{t}, n)$ or, equivalently, to find a pair of flags in $\mathcal{F}_q(\mathbf{t}, n)$ such that $\mathbf{v} = \mathbf{d}(\mathcal{F}, \mathcal{F}')$ and $d_f(\mathcal{F}, \mathcal{F}') = d$.

First, by means of (ii) and (iii), every v_i is an even integer such that $0 \leqslant v_i \leqslant \min\{2t_i, 2(n-t_i)\}$. Hence, each v_i is an admissible distance value between t_i-dimensional subspaces of $\mathbb{F}_q^n$. Moreover, we can write every $v_i = 2w_i$ for some integer w_i.

Notice that finding subspaces $\mathcal{F}_i, \mathcal{F}'_i \in \mathcal{G}_q(t_i, n)$ with distance $d_S(\mathcal{F}_i, \mathcal{F}'_i) = v_i$ is equivalent to choose them satisfying $\dim(\mathcal{F}_i \cap \mathcal{F}'_i) = t_i - w_i$. This can be clearly done for every $1 \leqslant i \leqslant r$. However, we need that the chosen subspaces form flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$. We use an inductive process in order to construct such flags. We start taking subspaces $\mathcal{F}_1, \mathcal{F}'_1 \in \mathcal{G}_q(t_1, n)$ such that $\dim(\mathcal{F}_1 \cap \mathcal{F}'_1) = t_1 - w_1$. Assume now that, for some $1 \leqslant i < r$, we have found subspaces $\mathcal{F}_j, \mathcal{F}'_j \in \mathcal{G}_q(t_j, n)$, for all $1 \leqslant j \leqslant i$, such that

$$\begin{array}{ccccccc} \mathcal{F}_1 & \subsetneq & \dots & \subsetneq & \mathcal{F}_{i-1} & \subsetneq & \mathcal{F}_i, \\ \mathcal{F}'_1 & \subsetneq & \dots & \subsetneq & \mathcal{F}'_{i-1} & \subsetneq & \mathcal{F}'_i, \end{array}$$

and $d_S(\mathcal{F}_j, \mathcal{F}'_j) = v_j = 2w_j$. Let us see that we can find suitable subspaces $\mathcal{F}_{i+1}$ and $\mathcal{F}'_{i+1}$. To do this, notice that, by using (iv) and (iii), in this order, we obtain

$$\dim(\mathcal{F}_i + \mathcal{F}'_i) = t_i + w_i \leqslant t_{i+1} + w_{i+1} \leqslant n.$$

Thus, we can consider a subspace $\mathcal{U} \in \mathcal{G}_q(t_{i+1} + w_{i+1}, n)$ such that $\mathcal{F}_i + \mathcal{F}'_i \subseteq \mathcal{U}$. It holds

$$\dim(\mathcal{U}) - \dim(\mathcal{F}_i + \mathcal{F}'_i) = t_{i+1} + w_{i+1} - (t_i + w_i) > w_{i+1} - w_i.$$

We distinguish two possible situations in terms of the value $l_i := w_{i+1} - w_i$.

• If $l_i \geqslant 0$, then we put $m_i := (t_{i+1} - t_i) - l_i$. Observe that, by means of (iv), we have that $m_i \geqslant 0$. Moreover, we have

$$\begin{array}{rcl} \dim(\mathcal{F}_i + \mathcal{F}'_i) + 2l_i + m_i & = & (t_i + w_i) + 2l_i + m_i \\ & = & (t_i + w_i) + (w_{i+1} - w_i) + (t_{i+1} - t_i) \\ & = & t_{i+1} + w_{i+1} \\ & = & \dim(\mathcal{U}). \end{array}$$

Hence, we can find linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_{l_i}, \mathbf{b}_1, \dots, \mathbf{b}_{l_i}, \mathbf{c}_1, \dots, \mathbf{c}_{m_i}$ in $\mathcal{U}$ such that this subspace can be expressed as the direct sum

$$\mathcal{U} = (\mathcal{F}_i + \mathcal{F}'_i) \oplus \langle \mathbf{a}_1, \dots, \mathbf{a}_{l_i} \rangle \oplus \langle \mathbf{b}_1, \dots, \mathbf{b}_{l_i} \rangle \oplus \langle \mathbf{c}_1, \dots, \mathbf{c}_{m_i} \rangle.$$

Now, consider the subspaces

$$\begin{array}{ccccccc} \mathcal{F}_{i+1} & := & \mathcal{F}_i & \oplus & \langle \mathbf{a}_1, \dots \mathbf{a}_{l_i} \rangle & \oplus & \langle \mathbf{c}_1, \dots, \mathbf{c}_{m_i} \rangle, \\ \mathcal{F}'_{i+1} & := & \mathcal{F}'_i & \oplus & \langle \mathbf{b}_1, \dots, \mathbf{b}_{l_i} \rangle & \oplus & \langle \mathbf{c}_1, \dots, \mathbf{c}_{m_i} \rangle, \end{array}$$

which have dimension

$$\dim(\mathcal{F}_{i+1}) = \dim(\mathcal{F}'_{i+1}) = t_i + l_i + m_i = t_{i+1}.$$

It is clear that $\mathcal{F}_i \subsetneq \mathcal{F}_{i+1}$ and $\mathcal{F}'_i \subsetneq \mathcal{F}'_{i+1}$. Moreover, observe that $\mathcal{F}_{i+1} + \mathcal{F}'_{i+1} = \mathcal{U}$. Hence, $\dim(\mathcal{F}_{i+1} + \mathcal{F}'_{i+1}) = \dim(\mathcal{U}) = t_{i+1} + w_{i+1}$ and, consequently, it holds $\dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1}) = t_{i+1} - w_{i+1}$. As a result, we obtain $d_S(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1}) = v_{i+1}$, as desired.

• If $l_i < 0$, then it holds $t_i < t_i - l_i \leqslant t_i + w_i = \dim(\mathcal{F}_i + \mathcal{F}'_i)$. Thus, we can consider $(t_i - l_i)$-dimensional subspaces $\mathcal{V}$ and $\mathcal{V}'$ such that

$$\begin{array}{ccccccc} \mathcal{F}_i & \subsetneq & \mathcal{V} & \subseteq & \mathcal{F}_i + \mathcal{F}'_i & \subseteq & \mathcal{U}, \\ \mathcal{F}'_i & \subsetneq & \mathcal{V}' & \subseteq & \mathcal{F}_i + \mathcal{F}'_i & \subseteq & \mathcal{U}. \end{array}$$

Notice that $\mathcal{V}+\mathcal{V}'=\mathcal{F}_i+\mathcal{F}'_i$. Besides, recall that

$$\dim(\mathcal{U})-\dim(\mathcal{F}_i+\mathcal{F}'_i)=(t_{i+1}+w_{i+1})-(t_i+w_i)=(t_{i+1}-t_i)+l_i\geqslant 0$$

since $\mathbf{v}$ satisfies condition (iv). Hence, there exists a subspace $\mathcal{W}\subseteq\mathcal{U}$ of dimension $(t_{i+1}-t_i)+l_i$ such that

$$\mathcal{U}=(\mathcal{F}_i+\mathcal{F}'_i)\oplus\mathcal{W}=(\mathcal{V}+\mathcal{V}')\oplus\mathcal{W}.$$

Let us consider the subspaces

$$\mathcal{F}_{i+1}=\mathcal{V}\oplus\mathcal{W} \text{ and } \mathcal{F}'_{i+1}=\mathcal{V}'\oplus\mathcal{W},$$

which have dimension

$$\dim(\mathcal{V})+\dim(\mathcal{W})=\dim(\mathcal{V}')+\dim(\mathcal{W})=(t_i-l_i)+(t_{i+1}-t_i+l_i)=t_{i+1}$$

and clearly contain $\mathcal{F}_i$ and $\mathcal{F}'_i$, respectively. Moreover, since $\mathcal{F}_{i+1}+\mathcal{F}'_{i+1}=\mathcal{U}$, we conclude that $\dim(\mathcal{F}_{i+1}\cap\mathcal{F}'_{i+1})=2t_{i+1}-\dim(\mathcal{U})=2t_{i+1}-(t_{i+1}+w_{i+1})=t_{i+1}-w_{i+1}$. This is equivalent to say that $d_S(\mathcal{F}_{i+1},\mathcal{F}'_{i+1})=2w_i=v_i$, as we wanted to prove.

In both cases, we conclude the existence of $\mathcal{F},\mathcal{F}'\in\mathcal{F}_q(\mathbf{t},n)$ such that $\mathbf{v}=\mathbf{d}(\mathcal{F},\mathcal{F}')$, which finishes the proof. ∎

Example 3.10. *Consider the full flag variety on $\mathbb{F}_q^7$. In this case, $D^7=24$ and we can consider the possible value of the distance $d=20$. According to Theorem 3.9, the set of distance vectors associated to $d=20$ is given by*

$$\mathcal{D}(20,7)=\{(2,4,4,4,4,2),(2,4,6,4,2,2),(2,2,4,6,4,2)\}.$$

Observe that, even though all the components of the vector $(2,\mathbf{2},\mathbf{6},4,4,2)$ are allowed distances between subspaces of the corresponding dimensions and they sum $d=20$, such a vector is not a distance vector in $\mathcal{D}(20,7)$. This is due to the fact that the sequence $(\mathbf{2},\mathbf{6})$ violates condition (iv) in Theorem 3.9, since $6-2=4>2=2(3-2)$.

For $n=7$ and $\mathbf{t}=(1,3,5,6)$, we have $D^{(\mathbf{t},7)}=14$. Observe that the distance $d=12$ can only be attained by distance vectors

$$\mathcal{D}(12,\mathbf{t},7)=\{(2,4,4,2),(2,6,2,2)\}.$$

In this case consecutive components $(\mathbf{2},\mathbf{6})$ in the vector$(\mathbf{2},\mathbf{6},2,2)$ are allowed, since they represent distance between nested subspaces of dimensions $t_1=1$ and $t_2=3$. Hence, the difference $6-2=4=2(t_2-t_1)$ respects the condition (iv) in Theorem 3.9.

4 Distance between flags sharing subspaces

This section is devoted to the study of the flag distance between flags in $\mathcal{F}_q(\mathbf{t}, n)$ that share subspaces. To do this, we start by analyzing the distance associated to distance vectors with a prescribed component, in particular, the ones having a component equal to zero. Then, we extend our study to distance vectors having several zeros among their components. This study will be used in Sections 5 and 6 to obtain some information about the structure of flag codes as well as bounds for their cardinality depending on their minimum distance.

4.1 Distance vectors with a fixed component

We start by describing the interval of attainable distances by distance vectors in $\mathcal{D}(\mathbf{t}, n)$ with their i-th component fixed, for some $1 \leqslant i \leqslant r$. Throughout the rest of the section, we will write v to denote an even integer $0 \leqslant v \leqslant \min\{2t_i, 2(n-t_i)\}$. In other words, the integer v represents a possible value for the distance between t_i-dimensional subspaces of $\mathbb{F}_q^n$. We focus on the set of distance vectors $\mathbf{v} = (v_1, \dots, v_r) \in \mathcal{D}(\mathbf{t}, n)$ with v as their i-th component, paying special attention to those associated to the maximum and minimum distances.

Notice that, if we require a distance vector $\mathbf{v}$ to satisfy $v_i = v$, then, by using condition (iv) in Theorem 3.9, we obtain $|v_j - v| \leqslant 2|t_i - t_j|$ for all $1 \leqslant j \leqslant r$. Moreover, by means of (ii)-(iv) in Theorem 3.9, for every $1 \leqslant j \leqslant r$, the component v_j must hold

$$\max\{0, v - 2|t_i - t_j|\} \leqslant v_j \leqslant \min\{2t_j, 2(n-t_j), v + 2|t_i - t_j|\}. \tag{8.15}$$

Definition 4.1. Given v as above, we write $d(i; v)^{(\mathbf{t},n)}$ (resp. $D(i; v)^{(\mathbf{t},n)}$) to denote the minimum (resp. maximum) distance that can be attained by distance vectors in $\mathcal{D}(\mathbf{t}, n)$ having its i-th component equal to v. According to (8.15), there exists a unique distance vector, that we denote by $\mathbf{d}(i; v)^{(\mathbf{t},n)}$ (resp. $\mathbf{D}(i; v)^{(\mathbf{t},n)}$), giving the distance $d(i; v)^{(\mathbf{t},n)}$ (resp. $D(i; v)^{(\mathbf{t},n)}$) and having v as its i-th component. For every $1 \leqslant j \leqslant r$, the j-th components of these vectors are given by

$$\begin{array}{rcl} d(i; v)_j^{(\mathbf{t},n)} & = & \max\{0, v - 2|t_i - t_j|\}, \\ D(i; v)_j^{(\mathbf{t},n)} & = & \min\{2t_j, 2(n-t_j), v + 2|t_i - t_j|\}. \end{array} \tag{8.16}$$

Consequently, the value $d(i; v)^{(\mathbf{t},n)}$ (resp. $D(i; v)^{(\mathbf{t},n)}$) is obtained as the sum of the components of $\mathbf{d}(i; v)^{(\mathbf{t},n)}$ (resp. $\mathbf{D}(i; v)^{(\mathbf{t},n)}$), given in (8.16). Notice that, by construction, these values satisfy $0 \leqslant d(i; v)^{(\mathbf{t},n)} \leqslant D(i; v)^{(\mathbf{t},n)} \leqslant D^{(\mathbf{t};n)}$. When working with the full type variety, we simply write $d(i; v)^n$, $D(i; v)^n$, $\mathbf{d}(i; v)^n$ and $\mathbf{D}(i; v)^n$.

Example 4.2. *For the full flag variety on $\mathbb{F}_q^7$, take $i = 3$ and $v = 4$, we have*

$$\mathbf{d}(3; 4)^7 = (0, 2, \mathbf{4}, 2, 0, 0) \ \textit{and} \ \mathbf{D}(3; 4)^7 = (2, 4, \mathbf{4}, 6, 4, 2)$$

and their associated distances are the values $d(3;4)^7 = 8$ *and* $D(3;4)^7 = 22$.

Consider now the type vector $\mathbf{t} = (1,3,5,6)$ *on* $\mathbb{F}_q^7$. *For the same choice of* $i = 3$ *and* $v = 4$, *we have*

$$\mathbf{d}(3;4)^{(\mathbf{t},7)} = (0,0,\mathbf{4},2) \ \text{and} \ \mathbf{D}(3;4)^{(\mathbf{t},7)} = (2,6,\mathbf{4},2).$$

Hence, in this case, we have $d(3;4)^{(\mathbf{t},7)} = 6$ *and* $D(3;4)^{(\mathbf{t},7)} = 14 = D^{(\mathbf{t},7)}$.

According to the definition of $d(i;v)^{(\mathbf{t},n)}$ and $D(i;v)^{(\mathbf{t},n)}$, it is clear that if $\mathbf{v} \in \mathcal{D}(\mathbf{t},n)$ such that $v_i = v$, then its associated distance d is an even integer such that $d(i;v)^{(\mathbf{t},n)} \leqslant d \leqslant D(i;v)^{(\mathbf{t},n)}$. The next result allows us to ensure that the converse is also true.

Proposition 4.3. *Consider the type vector* $\mathbf{t}$ *on* $\mathbb{F}_q^n$, *take an index* $1 \leqslant i \leqslant r$ *and an even integer* $0 \leqslant v \leqslant \min\{2t_i, 2(n-t_i)\}$. *If* d *is an even integer such that* $d(i;v)^{(\mathbf{t},n)} \leqslant d \leqslant D(i;v)^{(\mathbf{t},n)}$, *then there exist distance vectors in* $\mathcal{D}(d,\mathbf{t},n)$ *with* v *as its* i*-th component.*

Proof. We prove the result by induction on d. For $d = d(i;v)^{(\mathbf{t},n)}$, the result holds since the vector $\mathbf{d}(i;v)^{(\mathbf{t},n)}$ satisfies the required condition. Now, assume that, for some even integer d such that $d(i;v)^{(\mathbf{t},n)} \leqslant d < D(i;v)^{(\mathbf{t},n)}$, we have found a distance vector $\mathbf{v} = (v_1,\ldots,v_r) \in \mathcal{D}(d,\mathbf{t},n)$ such that $v_i = v$. Let us use $\mathbf{v}$ to construct a suitable distance vector in $\mathcal{D}(d+2,\mathbf{t},n)$. Observe that, since $d < D(i;v)^{(\mathbf{t},n)}$, clearly the set

$$\{v_j \mid v_j < D(i;v)_j^{(\mathbf{t},n)}\}$$

is nonempty. Hence, we can consider the minimum $1 \leqslant k \leqslant r$ such that $v_k = \min\{v_j \mid v_j < D(i;v)_j^{(\mathbf{t},n)}\}$. According to this, we have that $v_k < D(i;v)_k^{(\mathbf{t},n)}$ and then $v_k + 2 \leqslant D(i;v)_k^{(\mathbf{t},n)}$ is a possible distance between t_k-dimensional subspaces of $\mathbb{F}_q^n$. Consider now the k-th canonical vector $\mathbf{e}_k \in \mathbb{Z}^r$, i.e., the vector with k-th component equal to 1 and zeros elsewhere. Since $v_i = v = D(i;v)_i^{(\mathbf{t},n)}$, it clearly holds $k \neq i$ and thus, the vector $\mathbf{v} + 2\mathbf{e}_k$ still has v as its i-th component. Hence, we just need to prove that $\mathbf{v} + 2\mathbf{e}_k$ is a distance vector in $\mathcal{D}(d+2,\mathbf{t},n)$. To so so, it only remains to check condition (iv) of Theorem 3.9 for the k-th component and the adjacent ones. In other words, we must show that the relations

$$|(v_k+2) - v_{k-1}| \leqslant 2(t_k - t_{k-1}) \ \text{ if } \ 1 < k \leqslant r \ \text{ and} \tag{8.17}$$
$$|v_{k+1} - (v_k+2)| \leqslant 2(t_{k+1} - t_k) \ \text{ if } \ 1 \leqslant k < r. \tag{8.18}$$

hold. We start by proving (8.17) in case $1 < k \leqslant r$. To do so, we distinguish two cases. First, if $v_k < v_{k-1}$, then we have

$$|(v_k+2) - v_{k-1}| = v_{k-1} - v_k - 2 = |v_k - v_{k-1}| - 2 \leqslant 2(t_k - t_{k-1}) - 2 < 2(t_k - t_{k-1}).$$

On the other hand, if $v_k \geqslant v_{k-1}$, by the minimality in the choice of k, we have that $v_{k-1} = D(i;v)_{k-1}^{(\mathbf{t},n)}$. Hence, since $\mathbf{D}(i;v)^{(\mathbf{t},n)}$ is a distance vector, it holds

$$|(v_k+2) - v_{k-1}| = (v_k+2) - v_{k-1} \leqslant D(i;v)_k^{(\mathbf{t},n)} - D(i;v)_{k-1}^{(\mathbf{t},n)} \leqslant 2(t_k - t_{k-1}).$$

The proof of (8.18), in case that $1 \leqslant k < r$, is completely analogous and we omit it. ∎

To our purposes, it will be important, in turn, to look at the behaviour of the components of distance vectors associated to a given value for the flag distance. Hence, given an even integer $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$, we consider the values

$$\bar{d}_i = \min\{v_i \mid \mathbf{v} \in \mathcal{D}(d,\mathbf{t},n)\} \text{ and } \bar{D}_i = \max\{v_i \mid \mathbf{v} \in \mathcal{D}(d,\mathbf{t},n)\}. \tag{8.19}$$

The value $\bar{d}_i$ (resp. $\bar{D}_i$) represents the minimum (resp. maximum) value that can be placed in the i-th component of a distance vector in $\mathcal{D}(d,\mathbf{t},n)$.

Remark 4.4. Notice that, the values $\bar{d}_i$ and $\bar{D}_i$ defined in (8.19) satisfy the chain of inequalities

$$d(i;\bar{d}_i)^{(\mathbf{t},n)} \leqslant d(i;\bar{D}_i)^{(\mathbf{t},n)} \leqslant d \leqslant D(i;\bar{d}_i)^{(\mathbf{t},n)} \leqslant D(i;\bar{D}_i)^{(\mathbf{t},n)}. \tag{8.20}$$

Example 4.6 illustrates this fact.

With this notation, the next result holds.

Proposition 4.5. *Let d be an even integer such that $0 \leqslant d \leqslant D^{(\mathbf{t},n)}$. Consider an index $1 \leqslant i \leqslant r$ and take an even integer v with $0 \leqslant v \leqslant \min\{2t_i, 2(n-t_i)\}$. The following statements hold:*

(1) If $v < \bar{d}_i$, then we have $D(i;v)^{(\mathbf{t},n)} < d$.

(2) If $v > \bar{D}_i$, then $d(i;v)^{(\mathbf{t},n)} > d$.

Proof. Suppose that $v < \bar{d}_i$, then, by means of (8.20), it holds

$$d(i;v)^{(\mathbf{t},n)} \leqslant d(i;\bar{d}_i)^{(\mathbf{t},n)} \leqslant d.$$

Suppose now that $d \leqslant D(i;v)^{(\mathbf{t},n)}$. In this case, by means of Proposition 4.3, there must exist a distance vector in $\mathcal{D}(d,\mathbf{t},n)$ with v as its i-th component. This leads to $\bar{d}_i \leqslant v$, which is a contradiction. Hence, it holds $d > D(i;v)^{(\mathbf{t},n)}$.

On the other hand, if $v > \bar{D}_i$, by using (8.20), we clearly have that

$$d \leqslant D(i;\bar{D}_i)^{(\mathbf{t},n)} \leqslant D(i;v)^{(\mathbf{t},n)}.$$

If we assume that $d \geqslant d(i;v)^{(\mathbf{t},n)}$, by using Proposition 4.3, we can find a distance vector in $\mathcal{D}(d,\mathbf{t},n)$ with v as its i-th component. This contradicts the fact that $v > \bar{D}_i$. As a result, we conclude that $d < d(i;v)^{(\mathbf{t},n)}$. ∎

The previous result points out the impossibility of attaining the flag distance value d when we consider subspaces distances v out of the interval $[\bar{d}_i, \bar{D}_i]$ at the i-th summand. This fact will be useful in Section 6 and it is reflected in the next example.

Example 4.6. *As said in Example 3.10, the set of distance vectors associated to $d = 20$ for the full flag variety on $\mathbb{F}_q^7$ is*

$$\mathcal{D}(20, 7) = \{(2, 4, \mathbf{4}, 4, 4, 2), (2, 4, \mathbf{6}, 4, 2, 2), (2, 2, \mathbf{4}, 6, 4, 2)\}.$$

Hence, for $d = 20$, it is clear that $\bar{d}_3 = 4$ and $\bar{D}_3 = 6$. Moreover, in this case expression (8.20) becomes

$$d(3; 4)^7 = 8 < d(3; 6)^7 = 18 < 20 < D(3; 4)^7 = 22 < D(3; 6)^7 = 24.$$

Besides, by means of Proposition 4.5, the maximum distance that can be obtained by distance vectors with third component $v_3 < 4$ is lower than 20. Indeed, that maximum distance is attained with the vector

$$\mathbf{D}(3; 2)^7 = (2, 4, \mathbf{2}, 4, 4, 2),$$

whose associated distance is $D(3; 2)^7 = 18 < 20$.

4.2 Distance vectors with prescribed zero components

In this subsection we study the set of attainable values of the flag distance by distance vectors having prescribed zero components, i.e., distance vectors associated to pairs of flags that share certain subspaces. For the sake of simplicity, we will present partial results for the full flag variety, followed by the natural general version for $\mathcal{F}_q(\mathbf{t}, n)$, deduced by using the projection map defined in (8.11).

Remark 4.7. Recall that, as pointed out in Remark 3.4, in case of working with distance vectors with no zero components, the maximum possible distance is the value $D^{(\mathbf{t},n)}$, which is attained by the vector $\mathbf{D}^{(\mathbf{t},n)}$.

Let us start our study with distance vectors with just one zero among their components by taking advantage of the results provided in Subsection 4.1. Later on, we generalize this and analyze the properties of distance vectors with several null components. Given the type vector $\mathbf{t}$ and a position $1 \leqslant i \leqslant r$, by means of (8.16), it clearly holds $d(i; 0)^{(\mathbf{t},n)} = 0$. On the other hand, now we study the value $D(i; 0)^{(\mathbf{t},n)}$ and its associated distance vector $\mathbf{D}(i; 0)^{(\mathbf{t},n)}$. Observe that, in this case, we do not need to specify the fixed component since is always zero. Hence, we will just write $D(i)^{(\mathbf{t},n)}$ and $\mathbf{D}(i)^{(\mathbf{t},n)}$. Moreover, by means of (8.16), the j-th component $\mathbf{D}(i)^{(\mathbf{t},n)}$ is given by

$$D(i)_j^{(\mathbf{t},n)} = \min\{2t_j, 2(n - t_j), 2|t_i - t_j|\} \tag{8.21}$$

for every $1 \leqslant j \leqslant r$. When working with full flags, we also drop the type vector and simply write $D(i)^n$ and $\mathbf{D}(i)^n$. In this case, expression (8.21) becomes

$$D(i)^n_j = \min\{2j, 2(n-j), 2|i-j|\}, \tag{8.22}$$

for $1 \leqslant j \leqslant n-1$. Using both (8.21) and (8.22) and the map defined in (8.11), the next result follows.

Proposition 4.8. *Given a type vector* $\mathbf{t}$ *on* $\mathbb{F}_q^n$ *and an index* $1 \leqslant i \leqslant r$, *it holds*

$$\mathbf{D}(i)^{(\mathbf{t},n)} = \pi_{\mathbf{t}}(\mathbf{D}(t_i)^n).$$

This fact allows us to restrict our study to the full type case. At the end of the section we will come back to the general flag variety $\mathcal{F}_q(\mathbf{t}, n)$.

The full type case

We start giving some properties of the value $D(i)^n$.

Proposition 4.9. *For every* $1 \leqslant i \leqslant n-1$, *we have that*

$$D(i)^n_j = D(n-i)^n_{n-j}, \ \forall j = 1, \dots, n-1.$$

In other words, to obtain $\mathbf{D}(n-i)^n$, *it suffices to read backwards the vector* $\mathbf{D}(i)^n$. *As a consequence, it holds* $D(i)^n = D(n-i)^n$.

Proof. Take an integer $1 \leqslant i \leqslant n-1$. According to (8.22), for every $1 \leqslant j \leqslant n-1$, it clearly holds

$$\begin{aligned} D(n-i)^n_{n-j} &= \min\{2(n-j), 2(n-(n-j)), 2|(n-i)-(n-j)|\} \\ &= \min\{2(n-j), 2j, 2|j-i|\} = D(i)^n_j, \end{aligned}$$

which gives the result straightforwardly. ∎

In light of this result, we just need to study the values $D(i)^n$ for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. We can also give the following nice description of $D(i)^n$ in terms of the values D^j defined in (8.10).

Proposition 4.10. *For every* $1 \leqslant i \leqslant n-1$, *it holds*

$$D(i)^n = D^i + D^{n-i}.$$

Proof. Regarding equation (8.22), we can compute the value $D(i)^n$ as

$$D(i)^n = \sum_{j=1}^{n-1} \min\{2j, 2(n-j), 2|i-j|\}.$$

Observe that, in case $i = 1$, we have

$$\begin{array}{rcl} D(1)^n & = & \sum_{j=2}^{n-1} \min\{2j, 2(n-j), 2(j-1)\} \\ & = & \sum_{j=2}^{n-1} \min\{2(j-1), 2(n-j)\} \\ & = & \sum_{k=1}^{n-2} \min\{2k, 2((n-1)-k)\} \\ & = & D^{n-1} = D^1 + D^{n-1}. \end{array}$$

Besides, by means of Proposition 4.9, the result also holds if $i = n-1$. Let us now consider the case $1 < i < n-1$. In this case, the i-th component is zero and we have

$$D(i)^n = \sum_{j=1}^{i-1} \min\{2j, 2(n-j), 2|i-j|\} + 0 + \sum_{j=i+1}^{n-1} \min\{2j, 2(n-j), 2|i-j|\}. \quad (8.23)$$

Moreover, for values of $j < i$, one have that $2|i-j| = 2(i-j) < 2(n-j)$. On the other hand, if $i < j$, it is clear that $2|i-j| = 2(j-i) < 2j$. Hence, (8.23) becomes

$$\begin{array}{rcl} D(i)^n & = & \sum_{j=1}^{i-1} \min\{2j, 2(i-j)\} + \sum_{j=i+1}^{n-1} \min\{2(n-j), 2(j-i)\} \\ & = & \sum_{j=1}^{i-1} \min\{2j, 2(i-j)\} + \sum_{k=1}^{n-i-1} \min\{2(n-i-k), 2k\} \\ & = & D^i + D^{n-i}, \end{array}$$

where the second equality comes from writing $k = j - i$. ∎

This result confirms again the fact that $D(i)^n = D(n-i)^n$. Next, we use the previous proposition together with expression (8.10) to provide an explicit formula for every $D(i)^n$.

Corollary 4.11. *For every $1 \leqslant i \leqslant n-1$, it holds*

$$D(i)^n = \begin{cases} \dfrac{i^2 + (n-i)^2}{2} & \text{if both } n \text{ and } i \text{ are even,} \\ \dfrac{i^2 + (n-i)^2 - 2}{2} & \text{if } n \text{ is even and } i \text{ is odd,} \\ \dfrac{i^2 + (n-i)^2 - 1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

This expression allows us to establish an order on the set $\{D(i)^n \mid 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\}$.

Proposition 4.12. *For every n it holds*

$$D^n > D(1)^n > D(2)^n > \cdots > D(\lfloor \frac{n}{2} \rfloor - 1)^n \geqslant D(\lfloor \frac{n}{2} \rfloor)^n,$$

and the last equality holds if, and only if, 4 divides n.

Proof. Consider any $1 \leqslant i < \lfloor \frac{n}{2} \rfloor$. We will use the expression in Corollary 4.11 in order to compare $D(i)^n$ and $D(i+1)^n$. We do so by dividing the proof into two parts, depending on the parity of n. First of all, assume that n is odd. In this case, it follows $i+1 \leqslant \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Then $2i+1 < 2i+3 \leqslant n$ and

$$\begin{array}{rcl} D(i+1)^n & = & \frac{(i+1)^2+(n-(i+1))^2-1}{2} \\ & = & \frac{i^2+2i+1+(n-i)^2-2(n-i)+1-1}{2} \\ & = & \frac{i^2+(n-i)^2-1}{2} + \frac{2(2i+1-n)}{2} \\ & = & D(i)^n - (n-(2i+1)) < D(i)^n. \end{array}$$

Now, suppose that n is an even integer and $1 \leqslant i < \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. Equivalently, $2(i+1) \leqslant n$. We distinguish two cases:

- if i is even, then $i+1$ is odd and we have

$$\begin{array}{rcl} D(i+1)^n & = & \frac{(i+1)^2+(n-(i+1))^2-2}{2} \\ & = & \frac{i^2+2i+1+(n-i)^2-2(n-i)+1-2}{2} \\ & = & \frac{i^2+(n-i)^2}{2} + \frac{2(2i-n)}{2} \\ & = & D(i)^n - (n-2i) < D(i)^n. \end{array}$$

- On the other hand, if i is odd, then $i+1$ is even and:

$$\begin{array}{rcl} D(i+1)^n & = & \frac{(i+1)^2+(n-(i+1))^2}{2} \\ & = & \frac{i^2+2i+1+(n-i)^2-2(n-i)+1}{2} \\ & = & \frac{i^2+(n-i)^2-2}{2} + \frac{2(2(i+1)-n)}{2} \\ & = & D(i)^n - (n-2(i+1)) \leqslant D(i)^n \end{array}$$

  and the last equality holds if, and only if, $n = 2(i+1)$ and then 4 divides n.

∎

The next example reflects the information given in the previous results for a specific value of n.

Example 4.13. *For $n = 7$, we have $D^7 = 24$. In this example, we compute all the values $D(i)^7$ and respective vectors $\mathbf{D}(i)^7$.*

i	$\mathbf{D}(i)^7$	$D(i)^7$	i	$\mathbf{D}(i)^7$	$D(i)^7$
1	$(0,2,4,6,4,2)$	18	6	$(2,4,6,4,2,0)$	18
2	$(2,0,2,4,4,2)$	14	5	$(2,4,4,2,0,2)$	14
3	$(2,2,0,2,4,2)$	12	4	$(2,4,2,0,2,2)$	12

Notice that, as stated in Proposition 4.9, for every $1 \leqslant i \leqslant 6$, *it holds* $D(i)^7 = D(7-i)^7$. *Moreover, the reader can see that every vector* $\mathbf{D}(i)^7$ *has the same components than* $\mathbf{D}(n-i)^7$ *but written backwards. Moreover, it is also shown that*

$$D^7 > D(1)^7 > D(2)^7 > D(3)^7.$$

We will come back to this example in Section 7.

Recall that the flag distance between full flags on $\mathbb{F}_q^n$ is an even integer in the interval $[0, D^n]$. Moreover, in light of Proposition 4.12, we can partition this interval into intervals of the form $]D(i+1)^n, D(i)^n]$ that will be used in Sections 5 and 6 to obtain useful information about full flag codes.

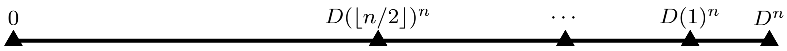

Figure 8.1: Distribution of the values $D(i)^n$.

In order to give a partition of the left interval $[0, D(\lfloor n/2 \rfloor)^n]$, let us introduce another set of relevant distances that correspond to the maximum possible flag distances associated to distance vectors with more that one zero among their components.

Definition 4.14. Consider an integer value $0 \leqslant M \leqslant n-1$ and fix dimensions $1 \leqslant i_1 < i_2 < \cdots < i_M \leqslant n-1$. We write $D(i_1, \ldots, i_M)^n$ to denote the maximum possible distance attainable by distance vectors in $\mathcal{D}(n)$ with M zeros in the positions $i_1, \ldots, i_M$. This situation corresponds uniquely to the distance vector $\mathbf{D}(i_1, \ldots, i_M)^n$, whose j-th component is given by

$$D(i_1, \ldots, i_M)_j^n = \min\{2j, 2(n-j), 2|j-i_1|, \ldots, 2|j-i_M|\}, \tag{8.24}$$

for all $1 \leqslant j \leqslant n-1$.

Notice that, in case $M = 0$, we have the distance vector $\mathbf{D}^n$ given in Remark 3.4. The case $M = 1$ corresponds to the distance vector $\mathbf{D}(i)^n$ defined in (8.22). On the other hand, if $M = n-1$, then $D(i_1, \ldots, i_M)^n = 0$ and $\mathbf{D}(i_1, \ldots, i_M)^n$ is the null vector.

Proposition 4.15. *Given indices* $1 \leqslant i_1 < \cdots < i_M \leqslant n-1$, *we have*

$$D(i_1, \ldots, i_M)^n = D^{i_1} + D^{i_2 - i_1} + \cdots + D^{i_M - i_{M-1}} + D^{n-i_M}.$$

Proof. We prove the result by induction on the number of zeros $1 \leqslant M \leqslant n-1$. Observe that, by means of Proposition 4.10, the result holds for every n and $M = 1$. Now, assume that $M > 1$ and, by induction hypothesis, that the result is true for all n and distance vectors having up to $M-1$ zeros. Let us study the case of having M zeros in the positions $i_1, \dots, i_M$. We start by considering the case where $i_1 = 1$. In this situation, according to (8.24),

$$\begin{aligned} D(1, i_2, \dots, i_M)^n &= \sum_{j=1}^{n-1} \min\{2j, 2(n-j), 2|j-1|, 2|j-i_2|, \dots, 2|j-i_M|\} \\ &= 0 + \sum_{j=2}^{n-1} \min\{2j, 2(n-j), 2|j-1|, 2|j-i_2|, \dots, 2|j-i_M|\} \\ &= 0 + \sum_{k=1}^{n-2} \min\{2((n-1)-k), 2k, 2|k-(i_2-1)|, \dots, 2|k-(i_M-1)|\} \\ &= D^1 + D(i_2-1, \dots, i_M-1)^{n-1}, \end{aligned}$$

where the third equality comes from taking $k = j-1$. Hence, the induction hypothesis leads to

$$D(1, i_2, \dots, i_M)^n = D^1 + D^{i_2-1} + D^{i_3-i_2} + \dots + D^{i_M-i_{M-1}} + D^{n-i_M},$$

as stated. Assume now that $i_1 > 1$, then we obtain

$$\begin{aligned} D(i_1, \dots, i_M)^n &= \textstyle\sum_{j=1}^{n-1} \min\{2j, 2(n-j), 2|j-i_1|, \dots, 2|j-i_M|\} \\ &= \textstyle\sum_{j=1}^{i_1-1} \min\{2j, 2(n-j), 2|j-i_1|, \dots, 2|j-i_M|\} + 0 \\ &\quad + \textstyle\sum_{j=i_1+1}^{n-1} \min\{2j, 2(n-j), 2|j-i_1|, \dots, 2|j-i_M|\}. \end{aligned}$$

Observe that, when $1 \leqslant j \leqslant i_1 - 1$, it holds $j < i_1 < i_2 < \dots < i_M < n$ and then $n - j > |j - i_l| = i_l - j \geqslant j - i_1$ for all $1 \leqslant l \leqslant M$. Hence, the first part of the last sum can be substituted by

$$\sum_{j=1}^{i_1-1} \min\{2j, 2(i_1-j)\} = D^{i_1}.$$

On the other hand, if $i_1 + 1 \leqslant j \leqslant n-1$, clearly $|j - i_1| = j - i_1 \leqslant j$ and then

$$\begin{aligned} D(i_1, \dots, i_M)^n &= D^{i_1} + \sum_{j=i_1+1}^{n-1} \min\{2(n-j), 2(j-i_1), 2|j-i_2|, \dots, 2|j-i_M|\} \\ &= D^{i_1} + \sum_{k=1}^{n-i_1-1} \min\{2(n-i_1-k), 2k, 2|k-(i_2-i_1)|, \dots, 2|k-(i_M-i_1)|\} \\ &= D^{i_1} + D(i_2-i_1, \dots, i_M-i_1)^{n-i_1}. \end{aligned}$$

Hence, by applying the induction hypothesis to $D(i_2 - i_1, \dots, i_M - i_1)^{n-i_1}$, we obtain

$$\begin{aligned} D(i_1, \dots, i_M)^n &= D^{i_1} + D^{i_2-i_1} + \dots + D^{(i_M-i_1)-(i_{M-1}-i_1)} + D^{(n-i_1)-(i_M-i_1)} \\ &= D^{i_1} + D^{i_2-i_1} + \dots + D^{i_M-i_{M-1}} + D^{n-i_M}, \end{aligned}$$

as we wanted to prove. ∎

Proposition 4.15 along with expression (8.10) allows us to compute every value $D(i_1, \ldots, i_M)^n$ depending on the parity of the positive integers $i_1, i_2 - i_1, \ldots, n - i_M$, as we see in the following example.

Example 4.16. *For $n = 7$, $M = 3$ and indices $i_1 = 1$, $i_2 = 3$ and $i_3 = 4$, by means of Proposition 4.15, we have*

$$\begin{array}{rcl} D(1,3,4)^7 & = & D^1 + D^{3-1} + D^{4-3} + D^{7-4} \\ & = & D^1 + D^2 + D^1 + D^3 \\ & = & 0 + \frac{2^2}{2} + 0 + \frac{3^2-1}{2} = 6. \end{array}$$

Remark 4.17. A *multiset* is a collection whose elements can appear more than once. The number of times that each element appears in the multiset is its *multiplicity*. We represent multisets by using double braces $\{\{\ldots\}\}$. Notice that, for any two families of M indices $1 \leqslant i_1 < \cdots < i_M \leqslant n-1$ and $1 \leqslant j_1 < \cdots < j_M \leqslant n-1$ satisfying the equality of multisets

$$\{\{i_1, i_2 - i_1, \ldots, n - i_M\}\} = \{\{j_1, j_2 - j_1, \ldots, n - j_M\}\},$$

by means of Proposition 4.15, we have the equality $D(i_1, \ldots, i_M)^n = D(j_1, \ldots, j_M)^n$. Hence, in order to compute all the values $D(i_1, \ldots, i_M)^n$, we can restrict ourselves to choices of M ordered indices $1 \leqslant i_1 < \cdots < i_M \leqslant n-1$ such that the differences

$$1 \leqslant i_1 \leqslant i_2 - i_1 \leqslant \ldots \leqslant i_M - i_{M-1} \leqslant n - M$$

are also ordered. In general, the converse is not true: there are different families of multisets as above providing the same values of the distance. It suffices to see that, with $M = 1$ and $n = 8$, it holds $D(3)^8 = 16 = D(4)^8$. However, the multisets of differences associated to indices $i = 3$ and $i = 4$ are $\{\{3, 5\}\}$ and $\{\{4, 4\}\}$, respectively.

The next result establishes that the maximum distance attainable by distance vectors in $\mathcal{D}(n)$ with M zero components is always obtained when these zeros are placed in the first M positions.

Proposition 4.18. *Given $1 \leqslant M \leqslant n-1$ and any election of indices $1 \leqslant i_1 < \cdots < i_M \leqslant n-1$, it holds*

$$D(i_1, \ldots, i_M)^n \leqslant D(1, \ldots, M)^n = D^{n-M}.$$

Proof. Notice that $D(1, \ldots, M)^n = D^{n-M}$ holds by application of Proposition 4.15. Hence, we just need to prove the first inequality. To do so, we proceed by induction on M. We start with the case $M = 1$, in which, by means of Proposition 4.12, it is clear that $D(n-i)^n = D(i)^n \leqslant D(1)^n$, for every value of n and $1 \leqslant i \leqslant n-1$.

Assume now that $M > 1$ and that the result holds for any value of n and distance vectors in $\mathcal{D}(n)$ having up to $M-1$ ceros. Let us prove that it is also true for M zeros. To do so, consider M indices $1 \leqslant i_1 < \cdots < i_M \leqslant n-1$. Notice that $M \leqslant i_M$. Moreover, if $M = i_M$, then it holds $i_j = j$, for every $j \in \{1, \dots, M\}$ and therefore $D(i_1, \dots, i_M)^n = D(1, \dots, M)^n$. Assume now that $M < i_M$. By means of Proposition 4.15 , we have

$$\begin{aligned} D(i_1, \dots, i_M)^n &= D^{i_1} + D^{i_2 - i_1} + \cdots + D^{i_M - i_{M-1}} + D^{n-i_M} \\ &= D(i_1, \dots, i_{M-1})^{i_M} + D^{n-i_M}. \end{aligned}$$

Moreover, since $M-1 < i_M - 1$, we can apply the induction hypothesis to the case of having $M-1$ zeros in the positions $i_1 < \cdots < i_{M-1}$ on $\mathbb{F}_q^{i_M}$. We obtain $D(i_1, \dots, i_{M-1})^{i_M} \leqslant D^{i_M-(M-1)}$ and Proposition 4.10 along with Proposition 4.12 gives that

$$\begin{aligned} D(i_1, \dots, i_M)^n &\leqslant D^{i_M-(M-1)} + D^{n-i_M} \\ &= D(i_M - M + 1)^{i_M - M+1+n-i_M} \\ &= D(i_M - M + 1)^{n-M+1} \\ &\leqslant D(1)^{n-M+1} = D^1 + D^{n-M+1-1} = D^{n-M}, \end{aligned}$$

as we wanted to prove.

∎

The general case

We finish this section by generalizing the previous concepts to the general flag variety $\mathcal{F}_q(\mathbf{t}, n)$ as follows. As done for the full type case in Definition 4.14, we can consider distance vectors $\mathcal{D}(\mathbf{t}, n)$ with a prescribed number of zeros, say $0 \leqslant M \leqslant r$, in the positions $1 \leqslant i_1 < \cdots < i_M \leqslant r$. We denote the corresponding maximum distance by

$$D(i_1, \dots, i_M)^{(\mathbf{t},n)}.$$

This number represents the maximum possible distance between flags in $\mathcal{F}_q(\mathbf{t}, n)$ that share simultaneously their subspaces of dimensions $t_{i_1}, \dots, t_{i_M}$. The only distance vector giving this distance and having zeros in its components $i_1, \dots, i_M$ is denoted by $\mathbf{D}(i_1, \dots, i_M)^{(\mathbf{t},n)}$ and its j-th component is given by

$$D(i_1, \dots, i_M)_j^{(\mathbf{t},n)} = \min\{2t_j, 2(n-t_j), 2|t_j - t_{i_1}|, \dots, 2|t_j - t_{i_M}|\}, \qquad (8.25)$$

for $1 \leqslant j \leqslant r$.

Using the projection map $\pi_{\mathbf{t}}$ defined in (8.11), we can give the following description of the distance vector with components as in (8.25), in terms of the vector $\mathbf{D}(t_{i_1}, \dots, t_{i_M})^n$ introduced in Definition 4.14. The next result generalizes Proposition 4.8.

Proposition 4.19. *Given a type vector* $\mathbf{t}$ *and* $1 \leqslant M \leqslant r$ *indices* $1 \leqslant i_1 < \cdots < i_M \leqslant r$*, it holds*

$$\mathbf{D}(i_1, \ldots, i_M)^{(\mathbf{t},n)} = \pi_{\mathbf{t}}(\mathbf{D}(t_{i_1}, \ldots, t_{i_M})^n).$$

Proof. Consider the type vector $\mathbf{t}$ and take $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. Notice that, for every $1 \leqslant j \leqslant r$, the j-th component of $\pi_{\mathbf{t}}(\mathbf{D}(t_{i_1}, \ldots, t_{i_M})^n)$ is exactly the t_j-th one of $\mathbf{D}(t_{i_1}, \ldots, t_{i_M})^n$ that, by (8.24), is:

$$\min\{2t_j, 2(n-t_j), 2|t_j - t_{i_1}|, \ldots, 2|t_j - t_{i_M}|\}.$$

This value corresponds to the j-th component of $\mathbf{D}(i_1, \ldots, i_M)^{(\mathbf{t},n)}$, as we wanted to prove. ∎

Next, we give a generalization of Proposition 4.15 for any arbitrary type vector $\mathbf{t} = (t_1, \ldots, t_r)$. To do so, consider $1 \leqslant M \leqslant r$ zeros in the positions $1 \leqslant i_1 < \cdots < i_M \leqslant r$. These positions allow us to split $\mathbf{t}$ into $M+1$ new type vectors, that we denote by $\mathbf{t}^1, \ldots, \mathbf{t}^{M+1}$, given by

$$\begin{cases} \mathbf{t}^1 &= (t_1, \ldots, t_{i_1 - 1}), \\ \mathbf{t}^{j+1} &= (t_{i_j+1} - t_{i_j}, \ldots, t_{i_{j+1}-1} - t_{i_j}), \text{ for } 1 \leqslant j \leqslant M-1, \\ \mathbf{t}^{M+1} &= (t_{i_M+1} - t_{i_M}, \ldots, t_r - t_{i_M}). \end{cases} \tag{8.26}$$

Using this notation, the next result holds.

Proposition 4.20. *Given a type vector* $\mathbf{t}$ *and a choice of* $1 \leqslant M \leqslant r$ *ordered indices* $1 \leqslant i_1 < \cdots < i_M \leqslant r$*, then the value* $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ *satisfies:*

$$D(i_1, \ldots, i_M)^{(\mathbf{t},n)} = D^{(\mathbf{t}^1, t_{i_1})} + D^{(\mathbf{t}^2, t_{i_2} - t_{i_1})} + \cdots + D^{(\mathbf{t}^{M+1}, n - t_{i_M})}.$$

The next example reflects this fact.

Example 4.21. *Take* $n = 12$ *and consider the type vector* $\mathbf{t} = (1, 3, 5, 6, 8, 10, 11)$ *of length* $r = 7$*. Assume that we place* $M = 2$ *zeros in the positions* $i_1 = 3$ *and* $i_2 = 5$*, i.e., the ones corresponding to the dimensions* $t_3 = 5$ *and* $t_5 = 8$*. In this case, by means of (8.26), we have*

$$\mathbf{t}^1 = (1, 3), \quad \mathbf{t}^2 = (6-5) = (1) \quad \textit{and} \quad \mathbf{t}^3 = (10-8, 11-8) = (2, 3).$$

Moreover, by (8.25), it holds

$$\mathbf{D}(3, 5)^{(\mathbf{t},12)} = (2, 4, \mathbf{0}, 2, \mathbf{0}, 4, 2).$$

Observe that the zero components of $\mathbf{D}(3, 5)^{(\mathbf{t},12)}$ *allow us to split this vector into three new ones, which are precisely*

$$\mathbf{D}^{(\mathbf{t}^1, 5)} = (2, 4), \quad \mathbf{D}^{(\mathbf{t}^2, 8-5)} = (2) \quad \textit{and} \quad \mathbf{D}^{(\mathbf{t}^3, 12-8)} = (4, 2).$$

Hence, we have

$$D(3, 5)^{(\mathbf{t},12)} = 2 + 4 + 0 + 2 + 0 + 4 + 2 = D^{(\mathbf{t}^1, 5)} + D^{(\mathbf{t}^2, 8-5)} + D^{(\mathbf{t}^3, 12-8)},$$

as stated in Proposition 4.20.

Remark 4.22. Notice that the computation of the distance $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ only depends on the flag variety $\mathcal{F}_q(\mathbf{t}, n)$ and on the choice of the indices $i_1, \ldots, i_M$. As a result, these values can be computed in advance, before considering any particular flag code, as we will see in Section 7.

In the following sections we will take advantage of this study of the values $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ in order to derive some properties related to the structure and cardinality of flag codes.

5 Disjointness in flag codes

Recall that, given a flag code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$, for every $1 \leqslant i \leqslant r$, its *i-th projected code* is the constant dimension code

$$\mathcal{C}_i = p_i(\mathcal{C}) \subseteq \mathcal{G}_q(t_i, n),$$

where p_i is the projection map defined in (8.6). As a consequence, for every $1 \leqslant i \leqslant r$, we have

$$|\mathcal{C}_i| = |p_i(\mathcal{C})| \leqslant |\mathcal{C}|$$

and the equality holds if, and only if, the projection p_i is injective when restricted to $\mathcal{C}$. If we have the equality for all $1 \leqslant i \leqslant r$, i.e., if $|\mathcal{C}| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_r|$, the flag code $\mathcal{C}$ is said to be *disjoint* (see [4]). Under the *disjointness* property, the code cardinality is completely determined by its projected codes and different flags never share a subspace. Moreover, observe that every flag code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ with $|\mathcal{C}| = 1$ is trivially disjoint and it holds $d_f(\mathcal{C}) = \sum_{i=1}^{r} d_S(\mathcal{C}_i) = d_S(\mathcal{C}_i) = 0$, for every $1 \leqslant i \leqslant r$. On the other hand, if $\mathcal{C}$ is a disjoint flag code with $|\mathcal{C}| \geqslant 2$, then

$$d_f(\mathcal{C}) \geqslant \sum_{i=1}^{r} d_S(\mathcal{C}_i) \tag{8.27}$$

and $d_S(\mathcal{C}_i) > 0$, for every $1 \leqslant i \leqslant r$. In particular, we obtain $d_f(\mathcal{C}) \geqslant 2r$.

Remark 5.1. Disjoint flag codes in $\mathcal{F}_q(\mathbf{t}, n)$ in which expression (8.27) holds with equality are called *consistent* (see [2]). It is quite easy to see that this family of disjoint flag codes is also characterized by the property of having as a unique distance vector $(d_S(\mathcal{C}_1), \ldots, d_S(\mathcal{C}_r))$. Optimum distance flag codes in $\mathcal{F}_q(\mathbf{t}, n)$ are a particular class of consistent flag codes whose associated distance vector is $\mathbf{D}^{(\mathbf{t},n)}$ defined in (8.12).

The simple structure of disjoint flag codes leads us to seek a generalization of this concept. We do so by using the next family of projections. Consider the flag variety $\mathcal{F}_q(\mathbf{t}, n)$ and take $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < i_1 < \cdots < i_M \leqslant r$. The $(i_1, \ldots, i_M)$*-projection map* is given as

$$\begin{array}{rccc} p_{(i_1,\ldots,i_M)} : & \mathcal{F}_q(\mathbf{t}, n) & \longrightarrow & \mathcal{F}_q((t_{i_1}, \ldots, t_{i_M}), n) \\ & (\mathcal{F}_1, \ldots, \mathcal{F}_r) & \longmapsto & (\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_M}) \end{array} \tag{8.28}$$

and the value M will be called the *length of the projection*. Now, given a flag code $\mathcal{C}$ in $\mathcal{F}_q(\mathbf{t}, n)$, we can define a set of flag codes of length M, naturally associated to $\mathcal{C}$, by using these projection maps.

Definition 5.2. Let $\mathcal{C} \subset \mathcal{F}_q(\mathbf{t}, n)$ be a flag code and fix $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. The set $p_{(i_1,\dots,i_M)}(\mathcal{C})$ is called *the* $(i_1, \dots, i_M)$*-projected code of* $\mathcal{C}$. The images of $\mathcal{C}$ by all the projections of length M constitute the set of the so-called *projected codes of length* M *of* $\mathcal{C}$.

Observe that in case $M = 1$, both projections p_{i_1} and $p_{(i_1)}$, defined in (8.6) and (8.28) respectively, coincide. Hence, the (i)-projected code is just the i-projected (subspace) code defined in Section 2, seen now as a flag code of length one.

Next, we use these new projected codes and we introduce two wider notions of disjointness.

Definition 5.3. Let $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ be a flag code and take $1 \leqslant M \leqslant r$ specific indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. The code $\mathcal{C}$ is said to be $(i_1, \dots, i_M)$*-disjoint* if the projection $p_{(i_1,\dots,i_M)}$ is injective when restricted to $\mathcal{C}$. If this condition holds for every choice of M indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$, we say that $\mathcal{C}$ is M*-disjoint*.

According to this definition, we provide the next geometric interpretation of $(i_1, \dots, i_M)$-disjoint flag codes.

Remark 5.4. Consider the type vector $\mathbf{t}$ and $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. A code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ is $(i_1, \dots, i_M)$-disjoint if, and only if, different flags in $\mathcal{C}$ never share simultaneously their subspaces of dimensions $t_{i_1}, \dots, t_{i_M}$. Similarly, $\mathcal{C}$ is M-disjoint if different flags $\mathcal{C}$ never have M equal subspaces.

Example 5.5. *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ *be the standard* $\mathbb{F}_q$*-basis of* $\mathbb{F}_q^5$*. We consider the full flag code* $\mathcal{C}$ *on* $\mathbb{F}_q^5$ *given by the flags*

$$\begin{array}{rcl} \mathcal{F}^1 & = & (\langle \mathbf{e}_1 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle), \\ \mathcal{F}^2 & = & (\langle \mathbf{e}_1 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle), \\ \mathcal{F}^1 & = & (\langle \mathbf{e}_1 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_3 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5 \rangle, \ \langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 \rangle). \end{array}$$

On the one hand, observe that no pair of flags in $\mathcal{C}$ *share their second and third subspaces at the same time, i.e.,* $\mathcal{C}$ *is a* $(2, 3)$*-disjoint flag code. On the other hand, it is not* (i_1, i_2)*-disjoint for any other choice of indices* $1 \leqslant i_1 < i_2 \leqslant 4$*. As a result, the code* $\mathcal{C}$ *is not* 2*-disjoint.*

Proposition 5.6. *Let* $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ *be an* $(i_1, \dots, i_M)$*-disjoint flag code for some choice of* $1 \leqslant M \leqslant r$ *indices* $1 \leqslant i_1 < \cdots < i_M \leqslant r$*. Then, for every choice of* $M \leqslant N \leqslant r$ *integers* $1 \leqslant j_1 < \cdots < j_N \leqslant r$ *such that* $\{i_1, \dots, i_M\} \subseteq \{j_1, \dots, j_N\}$*, the code* $\mathcal{C}$ *is* $(j_1, \dots, j_N)$*-disjoint. In particular, if* $\mathcal{C}$ *is* M*-disjoint, then it is* N*-disjoint as well.*

Proof. Assume that $\mathcal{C}$ is not a $(j_1, \dots, j_N)$-disjoint flag code. Hence, there exist different flags $\mathcal{F}, \mathcal{F} \in \mathcal{C}$ such that $(\mathcal{F}_{j_1}, \dots, \mathcal{F}_{j_N}) = (\mathcal{F}'_{j_1}, \dots, \mathcal{F}'_{j_N})$. Since $\{i_1, \dots, i_M\} \subseteq \{j_1, \dots, j_N\}$, then we have $(\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_M}) = (\mathcal{F}'_{i_1}, \dots, \mathcal{F}'_{i_M})$, which is a contradiction with the fact that $\mathcal{C}$ is an $(i_1, \dots, i_M)$-disjoint flag code. Similarly, assume now that $\mathcal{C}$ is not N-disjoint. The previous argument leads to different flags sharing $N \geqslant M$ subspaces at the same time. In other words, the code $\mathcal{C}$ cannot be M-disjoint. ∎

At this point, we relate the M-disjointness property of a flag code with its minimum distance. These relationships will help us to establish bounds for flag codes in Section 6. We start giving a lower bound for the distance of M-disjoint flag codes in terms of the distances of some of their projected codes of length 1.

Proposition 5.7. *Let $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ be a flag code and consider an integer $1 \leqslant M \leqslant r$. If $\mathcal{C}$ is M-disjoint, then there exist $r - (M-1)$ indices $1 \leqslant i_1 < \dots < i_{r-M+1} \leqslant r$ such that $d_S(\mathcal{C}_{i_j}) \neq 0$ and*

$$d_f(\mathcal{C}) \geqslant \sum_{j=1}^{r-M+1} d_S(\mathcal{C}_{i_j}).$$

Proof. Let $\mathcal{C} \subset \mathcal{F}_q(\mathbf{t}, n)$ be an M-disjoint flag code for some integer $1 \leqslant M \leqslant r$ and consider a pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ giving the minimum distance. The M-disjointness condition makes that $\mathcal{F}$ and $\mathcal{F}'$ cannot share more than $M-1$ subspaces. Hence, their associated distance vector, i.e., the vector

$$\mathbf{d}(\mathcal{F}, \mathcal{F}') = (d_S(\mathcal{F}_1, \mathcal{F}'_1), \dots, d_S(\mathcal{F}_r, \mathcal{F}'_r))$$

does not contain more than $M-1$ zeros. As a result, at least, $r-(M-1)$ of its r components are nonzero. Thus, there exist different indices $1 \leqslant i_1 < \dots < i_{r-M+1} \leqslant r$ such that $d_S(\mathcal{F}_{i_j}, \mathcal{F}'_{i_j}) \neq 0$. Consequently, we have $d_S(\mathcal{F}_{i_j}, \mathcal{F}'_{i_j}) \geqslant d_S(\mathcal{C}_{i_j}) > 0$ and then

$$d_f(\mathcal{C}) = d_f(\mathcal{F}, \mathcal{F}') \geqslant \sum_{j=1}^{r-(M-1)} d_S(\mathcal{F}_{i_j}, \mathcal{F}'_{i_j}) \geqslant \sum_{j=1}^{r-(M-1)} d_S(\mathcal{C}_{i_j}).$$

In other words, the distance of $\mathcal{C}$ is lower bounded by the sum of nonzero distances of $r-(M-1)$ specific projected codes of length 1. ∎

Observe that, in the previous proof, the choice of the $r-(M-1)$ indices $1 \leqslant i_1 < \dots < i_{r-M+1} \leqslant r$ strongly depends on the election of the pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ giving the minimum distance of the code. On the other hand, if $d_f(\mathcal{C}) = d_f(\bar{\mathcal{F}}, \bar{\mathcal{F}}')$, for another pair of flags $\bar{\mathcal{F}}, \bar{\mathcal{F}}' \in \mathcal{C}$, following the proof of Proposition 5.7, one might obtain another lower bound for $d_f(\mathcal{C})$ as the sum of the (positive) distances of $r-(M-1)$ different projected codes of $\mathcal{C}$.

Corollary 5.8. *Let $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ be an M-disjoint flag code for some $1 \leqslant M \leqslant r$. Then it holds*

$$d_f(\mathcal{C}) \geqslant \min \left\{ \sum_{j=1}^{r-(M-1)} d_S(\mathcal{C}_{i_j}) \;\middle|\; 1 \leqslant i_1 < \cdots < i_{r-(M-1)} \leqslant r \textit{ with } d_S(\mathcal{C}_{i_j}) \neq 0 \right\}.$$

In particular, we have that $d_f(\mathcal{C}) \geqslant 2(r-(M-1))$.

Observe that, if $\mathcal{C} \subset \mathcal{F}_q(\mathbf{t}, n)$ is a disjoint flag code, i.e, 1-disjoint in our new notation, the previous bound coincides with the one given in (8.27). On the other hand, by using the notation introduced in Section 4, we provide the following sufficient condition on the distance of a flag code to ensure some type of disjointness. More precisely, we can conclude that a given flag code is $(i_1, \dots, i_M)$-disjoint just by checking if its minimum distance is greater than the value $D(i_1, \dots, i_M)^{(\mathbf{t},n)}$. Recall that, as said in Remark 4.22, fixed the flag variety $\mathcal{F}_q(\mathbf{t}, n)$, these values only depend on the choice of the indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. Hence they are independent from any specific flag code and can be computed and stored as parameters associated to $\mathcal{F}_q(\mathbf{t}, n)$. We use these remarkable distances as follows.

Theorem 5.9. *Let $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ be a flag code such that $d_f(\mathcal{C}) > D(i_1, \dots, i_M)^{(\mathbf{t},n)}$, for some choice of $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. Then $\mathcal{C}$ is $(i_1, \dots, i_M)$-disjoint.*

Proof. Assume that $\mathcal{C}$ is not $(i_1, \dots, i_M)$-disjoint for this particular choice of indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$. Then we can find different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_{i_j} = \mathcal{F}'_{i_j}$ for every $1 \leqslant j \leqslant M$. As a result, the distance vector associated to the pair of flags $\mathcal{F}$ and $\mathcal{F}'$ has, at least, M zeros in the positions $i_1, \dots, i_M$. As a result, and according to the definition of $D(i_1, \dots, i_M)^{(\mathbf{t},n)}$, we have

$$d_f(\mathcal{C}) \leqslant d_f(\mathcal{F}, \mathcal{F}') \leqslant D(i_1, \dots, i_M)^{(\mathbf{t},n)},$$

which is a contradiction. ∎

The previous result leads to a sufficient condition for flag codes to be M-disjoint in terms of their minimum distance.

Corollary 5.10. *Let $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ be a flag code and consider an integer $1 \leqslant M \leqslant r$. If*

$$d_f(\mathcal{C}) > \max \left\{ D(i_1, \dots, i_M)^{(\mathbf{t},n)} \;\middle|\; 1 \leqslant i_1 < \cdots < i_M \leqslant r \right\},$$

then $\mathcal{C}$ is M-disjoint.

Remark 5.11. Observe that comparing the distance of a code with the maximum of the values $D(i_1, \dots, i_M)^{(\mathbf{t},n)}$ is not a big deal since, as said in Remark 4.22, for each choice of indices and type vector, this maximum value can be computed in advance. Moreover, in case of working with full flags on $\mathbb{F}_q^n$, this maximum value is explicitly computed in Proposition 4.18. Hence, we can give an easier condition to guarantee that a given full flag code is M-disjoint as follows.

Corollary 5.12. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. If $d_f(\mathcal{C}) > D^{(n-M)}$ for some $1 \leqslant M \leqslant n-1$, then $\mathcal{C}$ is M-disjoint.*

Theorem 5.9 and Corollary 5.10 state sufficient conditions to deduce some degree of disjointness in terms of the minimum distance of a flag code. The concept of disjointness and, in particular, these two results will be crucial to establish bounds for the cardinality of flag codes in $\mathcal{F}_q(\mathbf{t}, n)$ with a prescribed minimum distance.

6 Bounds for the cardinality of flag codes

This section is devoted to give upper bounds for the cardinality of flag codes from arguments introduced in both Sections 4 and 5. As said in Section 2, the value $A_q^f(n, d, \mathbf{t})$ denotes the maximum possible size for flag codes in $\mathcal{F}_q(\mathbf{t}, n)$ with distance d. In the particular case of full flags on $\mathbb{F}_q^n$, we just write $A_q^f(n, d)$. Up to the moment, bounds for $A_q^f(n, d)$ have only been studied in [15]. In that paper, the author develops techniques to determine upper bounds for the size of full flag codes and gives an exhaustive list of them for small values of n. Out of the full type case, the author also exhibits some concrete examples. The bounds in the present paper are valid for any type vector and arise from different techniques. More precisely, for each value of the distance, we apply Theorem 5.9 and Corollaries 5.10 and 5.12, in order to ensure certain degree of disjointness and derive upper bounds for $A_q^f(n, d, \mathbf{t})$, related to the size of a suitable flag variety.

From now on, we will write d to denote a possible distance between flags in $\mathcal{F}_q(\mathbf{t}, n)$, that is, an even integer $d \in [0, D^{(\mathbf{t},n)}]$. Next we will use the values $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ defined in Section 4, along with the condition of $(i_1, \ldots, i_M)$-disjointness introduced in Section 5, to derive upper bounds for $A_q^f(n, d, \mathbf{t})$.

Theorem 6.1. *If $d > D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$, for a particular choice of $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$, then*

$$A_q^f(n, d, \mathbf{t}) \leqslant |\mathcal{F}_q((t_{i_1}, \ldots, t_{i_M}), n)| = \begin{bmatrix} n \\ t_1 \end{bmatrix}_q \begin{bmatrix} n-t_1 \\ t_2-t_1 \end{bmatrix}_q \cdots \begin{bmatrix} n-t_{r-1} \\ n-t_r \end{bmatrix}_q .$$

Proof. By application of Theorem 5.9, we know that every flag code $\mathcal{C} \subseteq \mathcal{F}_q(\mathbf{t}, n)$ with distance $d > D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ must be $(i_1, \ldots, i_M)$-disjoint. Hence, it holds

$$|\mathcal{C}| = |p_{(i_1,\ldots,i_M)}(\mathcal{C})| \leqslant |\mathcal{F}_q((t_{i_1}, \ldots, t_{i_M}), n)|.$$

Consequently, every flag code in $\mathcal{F}_q(\mathbf{t}, n)$ with minimum distance $d > D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ cannot contain more flags than the flag variety $\mathcal{F}_q((t_{i_1}, \ldots, t_{i_M}), n)$. The last equality follows from (8.4). ∎

Comparing the distance d with all the possible values of $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ leads to the next result, which is a direct consequence of Theorem 6.1.

Corollary 6.2. *If $d > D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$ for every election of $1 \leqslant M \leqslant r$ indices $1 \leqslant i_1 < \cdots < i_M \leqslant r$, then*

$$A_q^f(n, d, \mathbf{t}) \leqslant \min\{|\mathcal{F}_q((t_{i_1}, \ldots, t_{i_M}), n)| \mid 1 \leqslant i_1 < \cdots < i_M \leqslant r\}.$$

Notice that, in the case that $M = 1$, Theorem 6.1 entails a bound for $A_q^f(n, d, \mathbf{t})$ in terms of the size of certain Grassmann varieties.

Corollary 6.3. *Assume that $d > D(i)^{(\mathbf{t},n)}$ for some $1 \leqslant i \leqslant r$. It holds*

$$A_q^f(n, d, \mathbf{t}) \leqslant |\mathcal{G}_q(t_i, n)|.$$

Here below, we provide a potentially tighter bound than the one in Corollary 6.3 in terms of the maximum possible size for constant dimension codes in $\mathcal{G}_q(t_i, n)$ with a suitable value of the subspace distance. Notice that, if $d > D(i)^{(\mathbf{t},n)}$, by the definition of $D(i)^{(\mathbf{t},n)}$, no distance vector in $\mathcal{D}(d, \mathbf{t}, n)$ can have a zero as its i-th component. Therefore, the value $\bar{d}_i$ defined in (8.19) satisfies $\bar{d}_i \geqslant 2$. As a consequence, it makes sense to consider the next upper bound.

Theorem 6.4. *If $d > D(i)^{(\mathbf{t},n)}$ for some $1 \leqslant i \leqslant r$, then*

$$A_q^f(n, d, \mathbf{t}) \leqslant A_q(n, \bar{d}_i, i).$$

Proof. Let $\mathcal{C}$ be a flag code in $\mathcal{F}_q(\mathbf{t}, n)$ such that $d = d_f(\mathcal{C}) > D(i)^{(\mathbf{t},n)}$ and assume that $|\mathcal{C}| > A_q(n, \bar{d}_i, i)$. By means of Theorem 5.9, we know that $\mathcal{C}$ is (i)-disjoint, i.e., $|\mathcal{C}| = |\mathcal{C}_i|$. Hence $\mathcal{C}_i$ is a code in $\mathcal{G}_q(t_i, n)$ with more than $A_q(n, \bar{d}_i, i)$ subspaces. As a result, we have that $d_S(\mathcal{C}_i) < \bar{d}_i$. Consequently, there must exist different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $d_S(\mathcal{F}_i, \mathcal{F}'_i) = d_S(\mathcal{C}_i) < \bar{d}_i$. Proposition 4.5 leads to

$$d = d_f(\mathcal{C}) \leqslant d_f(\mathcal{F}, \mathcal{F}') \leqslant D(i, d_S(\mathcal{F}_i, \mathcal{F}'_i)) < d,$$

which is a contradiction. ∎

Remark 6.5. Notice that, since $\bar{d}_i \geqslant 2$, we clearly have

$$A_q(n, \bar{d}_i, i) \leqslant A_q(n, 2, i) = |\mathcal{G}_q(i, n)|$$

and the equality holds if, and only if, $\bar{d}_i = 2$. Consequently, the upper bound for $A_q^f(n, d, \mathbf{t})$ given in Theorem 6.4 is as good as the one provided in Corollary 6.3 and it is even tighter in case $\bar{d}_i \geqslant 4$.

Let us consider now the full flag variety. To do so, from now on, we will write d to denote a feasible distance between full flags on $\mathbb{F}_q^n$, i.e., an even integer with $0 \leqslant d \leqslant D^n$. In this case, all the results in this section still hold true. However, since we have a better description of the values $D(i_1, \ldots, i_M)^n$ when we consider the full flag variety, we can give more information for this specific case. For instance, fixed $1 \leqslant M \leqslant n-1$, instead of checking the condition $d > D(i_1, \ldots, i_M)^n$ for every choice of indices as in Corollary 6.2, by means of Proposition 4.18, one just need to ascertain if $d > D^{n-M}$ holds. Moreover, when restricting to the case $M = 1$, by means of Proposition 4.9, we can restrict ourselves to indices $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.

The next result follows straightforwardly from the definition of the value $D(i)^n$ (see (8.22)) along with Propositions 4.9 and 4.12.

Lemma 6.6. *If $d > D(i)^n$ for some $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, then the values $\bar{d}_j$ defined in (8.19) satisfy*

$$\bar{d}_j \geqslant 2, \text{ for every } i \leqslant j \leqslant n-i.$$

By means of the previous lemma, and arguing as in Theorem 6.4, whenever $d > D(i)^n$ holds, we obtain the next upper bound for $A_q^f(n,d)$.

Theorem 6.7. *If $d > D(i)^n$ for a given $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, then*

$$A_q^f(n,d) \leqslant \min\{A_q(n, \bar{d}_j, j) \mid i \leqslant j \leqslant n-i\}.$$

Using this last result when working with full flags gives us a bound as good as the one given in Theorem 6.4, formulated for the general type. Moreover, in some cases, it even improves it, as we can see in the following example.

Example 6.8. *For $n = 6$ and the full type vector, consider the flag distance $d = 16$, which satisfies $d = 16 > D(1)^6 = 12$. Moreover, taking into account that $\mathcal{D}(16,6) = \{(2,4,4,4,2)\}$, it is clear that $\bar{d}_i = 2$ for $i = 1,5$ and $\bar{d}_j = 4$ for $j = 2,3,4$. Hence, Theorem 6.4, leads to*

$$A_q^f(6,16) \leqslant A_q(6,2,1) = |\mathcal{G}_q(6,1)| = q^5 + q^4 + q^3 + q^2 + q + 1$$

(see (8.1)). On the other hand, by using Theorem 6.7, we obtain

$$A_q^f(6,16) \leqslant A_q(6,4,2) = q^4 + q^2 + 1,$$

which improves the previous bound. Notice that the last equality just gives the cardinality of any 2-spread code in $\mathbb{F}_q^6$, i.e., optimal constant dimension codes (of dimension 2) having the maximum distance. These codes were introduced in [18].

7 A complete example

In this section we illustrate how to combine all the elements introduced in this paper in order to exhibit relevant information about a flag code with a prescribed minimum distance d. To do so, we compute all the values $D(i_1, \ldots, i_M)^{(\mathbf{t},n)}$, defined in Section 4 for a specific choice of n and $\mathbf{t}$.

Let us fix $n = 7$ and consider both the full type vector $(1, 2, 3, 4, 5, 6)$ and the type vector $\mathbf{t} = (t_1, t_2, t_3, t_4) = (1, 3, 5, 6)$. We start working with full flags and computing all the values $D(i_1, \ldots, i_M)^7$, for every possible choice $1 \leqslant M \leqslant 6$ and indices $1 \leqslant i_1 < \cdots < i_M \leqslant 6$. As pointed out in Section 4, these values are not only useful for the full type case but also serve to extract conclusions for any other flag variety on $\mathbb{F}_q^7$ (see Proposition 4.19).

The following table shows all these distances $D(i_1, \ldots, i_M)^7$, separated according to the number of zeros $1 \leqslant M \leqslant 6$. We also exhibit the associated distance vector $\mathbf{D}(i_1, \ldots, i_M)^7$, the choice of ordered indices $1 \leqslant i_1 < \cdots < i_M \leqslant 6$ and the multiset of differences $\{\{i_1, i_2 - i_1, \ldots, 7 - i_M\}\}$. Recall that, as stated in Remark 4.17, we can restrict ourselves to families of indices such that the differences $1 \leqslant i_1 \leqslant i_2 - i_1 \leqslant \ldots \leqslant 7 - i_M$ are also ordered.

M = 1			
i_1	Differences	$\mathbf{D}(i_1)^7$	$D(i_1)^7$
1	$\{\{1, 6\}\}$	$(0, 2, 4, 6, 4, 2)$	18
2	$\{\{2, 5\}\}$	$(2, 0, 2, 4, 4, 2)$	14
3	$\{\{3, 4\}\}$	$(2, 2, 0, 2, 4, 2)$	12
M = 2			
(i_1, i_2)	Differences	$\mathbf{D}(i_1, i_2)^7$	$D(i_1, i_2)^7$
$(1, 2)$	$\{\{1, 1, 5\}\}$	$(0, 0, 2, 4, 4, 2)$	12
$(1, 3)$	$\{\{1, 2, 4\}\}$	$(0, 2, 0, 2, 4, 2)$	10
$(1, 4)$	$\{\{1, 3, 3\}\}$	$(0, 2, 2, 0, 2, 2)$	8
$(2, 4)$	$\{\{2, 2, 3\}\}$	$(2, 0, 2, 0, 2, 2)$	8
M = 3			
(i_1, i_2, i_3)	Differences	$\mathbf{D}(i_1, i_2, i_3)^7$	$D(i_1, i_2, i_3)^7$
$(1, 2, 3)$	$\{\{1, 1, 1, 4\}\}$	$(0, 0, 0, 2, 4, 2)$	8
$(1, 2, 4)$	$\{\{1, 1, 2, 3\}\}$	$(0, 0, 2, 0, 2, 2)$	6
$(1, 3, 5)$	$\{\{1, 2, 2, 2\}\}$	$(0, 2, 0, 2, 0, 2)$	6
M = 4			
$(i_1, \ldots, i_4)$	Differences	$\mathbf{D}(i_1, \ldots, i_4)^7$	$D(i_1, \ldots, i_4)^7$
$(1, 2, 3, 4)$	$\{\{1, 1, 1, 1, 3\}\}$	$(0, 0, 0, 0, 2, 2)$	4
$(1, 2, 3, 5)$	$\{\{1, 1, 1, 2, 2\}\}$	$(0, 0, 0, 2, 0, 2)$	4
M = 5			
$(i_1, \ldots, i_5)$	Differences	$\mathbf{D}(i_1, \ldots, i_5)^7$	$D(i_1, \ldots, i_5)^7$
$(1, 2, 3, 4, 5)$	$\{\{1, 1, 1, 1, 1, 2\}\}$	$(0, 0, 0, 0, 0, 2)$	2
M = 6			
$(i_1, \ldots, i_6)$	Differences	$\mathbf{D}(i_1, \ldots, i_6)^7$	$D(i_1, \ldots, i_6)^7$
$(1, \ldots, 6)$	$\{\{1, 1, 1, 1, 1, 1, 1\}\}$	$(0, 0, 0, 0, 0, 0, 0)$	0

Table 8.1: Possible values of $D(i_1, \ldots, i_M)^7$ for every $1 \leqslant M \leqslant 6$.

Any other choice of indices $1 \leqslant i_1 < \cdots < i_M \leqslant 6$ has an associated multiset of differences $\{\{i_1, i_2 - i_1, \ldots, 7 - i_M\}\}$ that already appears in these tables. For instance, to compute the value $D(1,3,6)^7$, we just need to consider the multiset

$$\{\{1, 3-1, 6-3, 7-6\}\} = \{\{1,2,3,1\}\}$$

and order its elements as an increasing sequence $\{\{1,1,2,3\}\}$. This multiset already appears in Table 8.1, associated to the choice of indices $(1,2,4)$. Hence,

$$D(1,3,6)^7 = D(1,2,4)^7 = 6.$$

The next table contains upper bounds for $A_q^f(7,d)$, for every value of $2 \leqslant d \leqslant D^7 = 24$ and every prime power q. To compute them, we compare d with specific values $D(i_1,\ldots,i_M)^7$ provided in Table 8.1, for some $1 \leqslant M \leqslant 6$. Notice that applying Theorem 6.1 to different elections either of the integer M or of indices $i_1 < \cdots < i_M$ provides, in general, different bounds. We proceed as in Corollary 6.2 and give the tightest bound for each case.

d	$D(i_1,\ldots,i_M)^7$	Upper bound for $A_q^f(7,d)$
2	$D(1,2,3,4,5,6)^7 = 0$	$\|\mathcal{F}_q((1,2,3,4,5,6),7)\| = \dfrac{(q^7-1)\cdots(q^2-1)}{(q-1)^6}$
4	$D(1,2,3,4,5)^7 = 2$	$\|\mathcal{F}_q((1,2,3,4,5),7)\| = \dfrac{(q^7-1)\cdots(q^3-1)}{(q-1)^5}$
6	$D(1,2,3,4)^7 = 4$	$\|\mathcal{F}_q((1,2,3,4),7)\| = \dfrac{(q^7-1)\cdots(q^4-1)}{(q-1)^4}$
8	$D(1,2,4)^7 = 6$	$\|\mathcal{F}_q((1,2,4),7)\| = \dfrac{(q^7-1)(q^6-1)(q^5-1)(q^2+1)}{(q-1)^3}$
10	$D(1,4)^7 = 8$	$\|\mathcal{F}_q((1,4),7)\| = \dfrac{(q^7-1)(q^5-1)(q^3+1)(q^2+1)}{(q-1)^2}$
12	$D(1,3)^7 = 10$	$\|\mathcal{F}_q((1,3),7)\| = \dfrac{(q^7-1)(q^5-1)(q^4+q^2+1)}{(q-1)^2}$
14	$D(1,2)^7 = 12$	$\|\mathcal{F}_q((1,2),7)\| = \dfrac{(q^7-1)(q^6-1)}{(q-1)^2}$
$16-18$	$D(2)^7 = 14$	$\|\mathcal{G}_q(2,7)\| = \dfrac{(q^7-1)(q^4+q^2+1)}{(q-1)}$
$20-24$	$D(1)^7 = 18$	$\|\mathcal{G}_q(1,7)\| = \dfrac{(q^7-1)}{(q-1)}$

Table 8.2: Bounds for $A_q^f(7,d)$ obtained by using Theorem 6.1.

Moreover, observe that the restriction to the families of ordered indices in Table 8.1 is not a problem since any choice of M indices $\{i_1, \dots, i_M\}$ and $\{j_1, \dots, j_M\}$ giving equal multisets

$$\{\{i_1, i_2 - i_1, \dots, n - i_M\}\} = \{\{j_1, j_2 - j_1, \dots, n - j_M\}\}$$

also provide the same bound

$$A_q^f(n, d) \leqslant |\mathcal{F}_q((i_1, \dots, i_M), n)| = |\mathcal{F}_q((j_1, \dots, j_M), n)|$$

because the cardinality of the flag variety

$$|\mathcal{F}_q((i_1, \dots, i_M), n)| = \begin{bmatrix} n \\ i_1 \end{bmatrix}_q \begin{bmatrix} n - i_1 \\ i_2 - i_1 \end{bmatrix}_q \cdots \begin{bmatrix} n - i_{M-1} \\ n - i_M \end{bmatrix}_q$$

$$= \frac{(q^n - 1) \dots (q - 1)}{\left((q^{i_1} - 1) \dots (q - 1)\right)\left(\prod_{l=1}^M \left((q^{i_l - i_{l-1}} - 1) \dots (q - 1)\right)\right)\left((q^{n - i_M} - 1) \dots (q - 1)\right)}$$

just depends on the values $i_1, i_2 - i_1, \dots, n - i_M$.

Notice that the bounds for $A_q^f(7, d)$ in Table 8.2 do not change for distances $16 \leqslant d \leqslant 18$ or $20 \leqslant d \leqslant 24$. In Table 8.3, for each flag distance value $16 \leqslant d \leqslant 24$, we indicate the specific choice of $1 \leqslant i \leqslant 6$ and the corresponding value $\bar{d}_i$ (see (8.19)) that provide the best upper bound for $A_q^f(7, d)$ that can be obtained by means of Theorem 6.7.

d	i	$\bar{d}_i$	Upper bound for $A_q^f(7, d)$
16	2	2	$A_q(7, 2, 2) = \vert\mathcal{G}_q(2, 7)\vert = \dfrac{(q^7 - 1)(q^4 + q^2 + 1)}{(q - 1)}$
18	2	2	$A_q(7, 2, 2) = \vert\mathcal{G}_q(2, 7)\vert = \dfrac{(q^7 - 1)(q^4 + q^2 + 1)}{(q - 1)}$
20	1	2	$A_q(7, 2, 1) = \vert\mathcal{G}_q(1, 7)\vert = \dfrac{(q^7 - 1)}{(q - 1)}$
22	4	4	$A_q(7, 4, 2) \leqslant q(q^4 + q^2 + 1)$
24	3	6	$A_q(7, 6, 3) = q^4 + 1$

Table 8.3: Bounds for $A_q^f(7, d)$ obtained by using Theorem 6.7.

Notice that, as said in Remark 6.5, for those cases in which $\bar{d}_i = 2$, bounds in Tables 8.2 and 8.3 coincide. On the other hand, whenever $\bar{d}_i > 2$, Table 8.3 gives better bounds. The next example illustrates how bounds in this table have been computed.

Example 7.1. *For $d = 20$ and the full flag variety on $\mathbb{F}_q^7$, we have*

$$\mathcal{D}(20, 7) = \{(2, 4, 4, 4, 4, 2),\ (2, 2, 4, 6, 4, 2),\ (2, 4, 6, 4, 2, 2)\}.$$

As a consequence, it holds $\bar{d}_i = 2$ for $i = 1, 2, 5, 6$ and $\bar{d}_j = 4$ for $j = 3, 4$. Hence, Theorem 6.7 leads to three possible upper bounds for $A_q^f(7, 20)$:

$$\begin{array}{rcl} A_q^f(7,20) & \leqslant & A_q(7,2,1) = A_q(7,2,6) = |\mathcal{G}_q(1,7)| = \frac{q^7-1}{q-1}, \\ A_q^f(7,20) & \leqslant & A_q(7,2,2) = A_q(7,2,5) = |\mathcal{G}_q(2,7)| = \frac{(q^7-1)(q^4+q^2+1)}{(q-1)}, \\ A_q^f(7,20) & \leqslant & A_q(7,4,3) = A_q(7,4,4). \end{array}$$

Clearly the first bound is tighter than the second one. Moreover, by means of [10, Th. 3.20], we know that

$$A_q(7, 4, 3) \geqslant q^8 + q^5 + q^4 - q - 1 > q^6 + \cdots + q + 1 = \frac{q^7 - 1}{q - 1} = |\mathcal{G}_q(1, 7)|.$$

Thus, Theorem 6.7 leads to $A_q^f(7, 20) \leqslant |\mathcal{G}_q(1, 7)|$, as we see in Table 8.3.

Using similar arguments we arrive to give some upper bounds for $A_q^f(n, d)$ that coincide with the already presented in [15]. See, for instance, Propositions 2.5, 2.6, 2.7, 4.4, 4.5, 4.6, 6.1, 6.2, 6.3 and 6.4 in that paper.

Now, also for $n = 7$ but for type vector $\mathbf{t} = (1, 3, 5, 6)$, we apply the results presented in this paper with the goal of exhibiting upper bounds for the cardinality of flag codes of this specific type vector. We start computing the values $D(i_1, \ldots, i_M)^{(\mathbf{t},7)}$, for $1 \leqslant M \leqslant 4$, by applying Proposition 4.19 to the already computed values $D(t_{i_1}, \ldots, t_{i_M})^7$ in Table 8.1 and their associated vectors $\mathbf{D}(t_{i_1}, \ldots, t_{i_M})^7$.

$\mathbf{M = 1}$			
i_1	$\mathbf{D}(t_{i_1})^7$	$\mathbf{D}(t_{i_1})^{(\mathbf{t},7)}$	$D(t_{i_1})^{(\mathbf{t},7)}$
1	$(0, 2, 4, 6, 4, 2)$	$(0, 4, 4, 2)$	10
2	$(2, 2, 0, 2, 4, 2)$	$(2, 0, 4, 2)$	8
3	$(2, 4, 4, 2, 0, 2)$	$(2, 4, 0, 2)$	8
4	$(2, 4, 6, 4, 2, 0)$	$(2, 6, 2, 0)$	10
$\mathbf{M = 2}$			
(i_1, i_2)	$\mathbf{D}(t_{i_1}, t_{i_2})^7$	$\mathbf{D}(i_1, i_2)^{(\mathbf{t},7)}$	$D(i_1, i_2)^{(\mathbf{t},7)}$
$(1, 2)$	$(0, 2, 0, 2, 4, 2)$	$(0, 0, 4, 2)$	6
$(1, 3)$	$(0, 2, 4, 2, 0, 2)$	$(0, 4, 0, 2)$	6
$(1, 4)$	$(0, 2, 4, 4, 2, 0)$	$(0, 4, 2, 0)$	6
$(2, 3)$	$(2, 2, 0, 2, 0, 2)$	$(2, 0, 0, 2)$	4
$(2, 4)$	$(2, 2, 0, 2, 2, 0)$	$(2, 0, 2, 0)$	4
$(3, 4)$	$(2, 4, 4, 2, 0, 0)$	$(2, 4, 0, 0)$	6

$\mathbf{M=3}$			
(i_1,i_2,i_3)	$\mathbf{D}(t_{i_1},t_{i_2},t_{i_3})^7$	$\mathbf{D}(i_1,i_2,i_3)^{(\mathbf{t},7)}$	$D(i_1,i_2,i_3)^{(\mathbf{t},7)}$
$(1,2,3)$	$(0,2,0,2,0,2)$	$(0,0,0,2)$	2
$(1,2,4)$	$(0,2,0,2,2,0)$	$(0,0,2,0)$	2
$(1,3,4)$	$(0,2,4,2,0,0)$	$(0,4,0,0)$	4
$(2,3,4)$	$(2,2,0,2,0,0)$	$(2,0,0,0)$	2
$\mathbf{M=4}$			
(i_1,i_2,i_3,i_4)	$\mathbf{D}(t_{i_1},t_{i_2},t_{i_3},t_{i_4})^7$	$\mathbf{D}(i_1,i_2,i_3,i_4)^{(\mathbf{t},7)}$	$D(i_1,i_2,i_3,i_4)^{(\mathbf{t},7)}$
$(1,2,3,4)$	$(0,2,0,2,0,0)$	$(0,0,0,0)$	0

Table 8.4: Possible values of $D(i_1,\dots,i_M)^{(\mathbf{t},7)}$.

Using this table and applying Corollary 6.3, we obtain the next list of bounds for $A_q^f(7,d,\mathbf{t})$. As before, we provide the tightest possible upper bound for each value d. We do so by making a suitable choice of $1\leqslant M\leqslant 4$ and indices $1\leqslant i_1<\cdots<i_M\leqslant 4$. This information is collected in the next table.

d	$D(i_1,\dots,i_M)^{(\mathbf{t},7)}$	Upper bound for $A_q^f(7,d,\mathbf{t})$
2	$D(1,2,3,4)^{(\mathbf{t},7)}=0$	$\lvert\mathcal{F}_q(\mathbf{t},7)\rvert=\dfrac{(q^7-1)(q^6-1)(q^5-1)(q^3-1)(q^2+1)}{(q-1)^4}$
4	$D(1,2,4)^{(\mathbf{t},7)}=2$	$\lvert\mathcal{F}_q((1,3,6),7)\rvert=\dfrac{(q^7-1)(q^6-1)(q^5-1)(q^2+1)}{(q-1)^3}$
6	$D(2,4)^{(\mathbf{t},7)}=4$	$\lvert\mathcal{F}_q((3,6),7)\rvert=\dfrac{(q^7-1)(q^5-1)(q^3+1)(q^2+1)}{(q-1)^2}$
8	$D(3,4)^{(\mathbf{t},7)}=6$	$\lvert\mathcal{F}_q((5,6),7)\rvert=\dfrac{(q^7-1)(q^6-1)}{(q-1)^2}$
10	$D(3)^{(\mathbf{t},7)}=8$	$\lvert\mathcal{G}_q(5,7)\rvert=\dfrac{(q^7-1)(q^4+q^2+1)}{(q-1)}$
$12-14$	$D(1)^{(\mathbf{t},7)}=10$	$\lvert\mathcal{G}_q(1,7)\rvert=\dfrac{q^7-1}{q-1}$

Table 8.5: Bounds for $A_q^f(7,d,\mathbf{t})$ obtained by using Theorem 6.1.

Last, for distance $d=14=D^{(\mathbf{t},7)}$, we can improve the previous bound. Observe that

$$\mathcal{D}(14,\mathbf{t},7)=\{\mathbf{D}^{(\mathbf{t},7)}\}=\{(2,6,4,2)\}.$$

Thus, taking into account that $\bar{d}_2=6$, by using Theorem 6.4, we obtain

$$A_q^f(7,14,\mathbf{t})\leqslant A_q(7,6,t_2)=A_q(7,6,3)=q^4+1,$$

(see [10, Th. 3.43] for the last equality) which is a better bound than the one given in Table 8.5.

8 Conclusions

In this paper we have addressed an exhaustive study of the flag distance parameter. To do so, we have introduced the concept of distance vector as a tool to represent how a flag distance value can be obtained from different combinations of subspace distances. Besides, we have characterized distance vectors in terms of certain conditions satisfied by their components.

We have presented the class of $(i_1, \dots, i_M)$-disjoint flag codes, as a generalization of the notion of disjointness given in [4] and also established a connection between the property of being $(i_1, \dots, i_M)$-disjoint and the impossibility of having distance vectors with M zeros, placed in the positions $i_1, \dots, i_M$. This allows us to read some structural properties of flag codes in terms of their minimum distance and their sets of distance vectors. As a consequence of our study, we deduce upper bounds for the value $A_q^f(n, d, \mathbf{t})$ for every choice of the parameters. These bounds strongly depend on the number of subspaces that can be shared by different flags of a code in $\mathcal{F}_q(\mathbf{t}, n)$ with minimum distance d. We finish our work by explicitly computing our bounds for $A_q^f(7, d, \mathbf{t})$ and two particular type vectors when we sweep all the possible distance values in each case.

BIBLIOGRAPHY

[1] R. Ahlswede, N. Cai, R. Li and R. W. Yeung, *Network Information Flow*, IEEE Transactions on Information Theory, Vol. 46 (2000), 1204-1216.

[2] C. Alonso-González and M. A. Navarro-Pérez, *Consistent Flag Codes*, Mathematics, Vol. 8 (12) (2020), 2234.

[3] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An Orbital Construction of Optimum Distance Flag Codes*, Finite Fields and Their Applications, Vol. 73 (2021), 101861.

[4] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes from Planar Spreads in Network Coding*, Finite Fields and Their Applications, Vol. 68 (2020), 101745.

[5] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.

[6] T. Etzion and A. Vardy, *Error-Correcting Codes in Projective Spaces*, IEEE Trans. Inf. Theory, Vol. 57 (2011), 1165–1173.

[7] E. Gorla, F. Manganiello and J. Rosenthal, *An Algebraic Approach for Decoding Spread Codes*, Advances in Mathematics of Communications, Vol. 6 (4) (2012), 443-466.

[8] E. Gorla and A. Ravagnani, *Partial Spreads in Random Network Coding*, Finite Fields and Their Applications, Vol. 26 (2014), 104-115.

[9] M. Greferath, M. O. Pavčević, N. Silberstein and M. A. Vázquez-Castro, *Network Coding and Subspace Designs*, Springer, 2018.

[10] D. Heinlein, M. Kiermaier, S. Kurz and A. Wassermann *Tables of Subspace Codes*, https://arxiv.org/abs/1601.02864

[11] T. Ho, R. Kötter, M. Médard, D. Karger and M. Effros, *The Benefits of Coding over Routing in a Randomized Setting*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), Yokohama, Japan, 2003, p. 442.

[12] A.-L. Horlemann-Trautmann, *Message Encoding and Retrieval for Spread and Cyclic Orbit Codes*, Designs, Codes and Cryptography, Vol. 86 (2018), 365-386.

[13] A. Kohnert and S. Kurz, *Construction of Large Constant Dimension Codes with a Prescribed Minimum Distance*, in: J. Calmet *et al.* (Eds.), Mathematical Methods in Computer Science. Lecture Notes in Computer Science, Vol. 5393. Springer, Berlin, Heidelberg, 2008, pp. 31-42.

[14] R. Kötter and F. Kschischang, *Coding for Errors and Erasures in Random Network Coding*, in: IEEE Transactions on Information Theory, Vol. 54 (2008), 3579-3591.

[15] S. Kurz, *Bounds for Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2759–2785.

[16] S. Kurz, *Improved Upper Bounds for Partial Spreads*, Designs, Codes and Cryptography, Vol. 85 (2017), 97–106.

[17] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Network Coding with Flags*, Designs, Codes and Cryptography, Vol. 86(2) (2018), 269-284.

[18] F. Manganiello, E. Gorla and J. Rosenthal, *Spread Codes and Spread Decoding in Network Coding*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), Toronto, Canada, 2008, pp. 851-855.

[19] M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes of Maximum Distance and Constructions using Singer Groups*, https://arxiv.org/abs/2109.00270 (preprint).

[20] N. Silberstein and T. Etzion, *Codes and Designs Related to Lifted MRD Codes*, Information Theory Proceedings (ISIT) 2011 IEEE International Symposium, pp. 2288-2292, 2011.

[21] N. Silberstein and A.-L. Trautmann, *New Lower Bounds for Constant Dimension Codes*, Information Theory Proceedings (ISIT) 2013 IEEE International Symposium, 2013, pp. 514-518.

[22] A.-L. Trautmann and J. Rosenthal, *New Improvements on the Echelon Ferrers Construction*, in: 19th International Symposium on Mathematical Theory of Networks and Systems, Proceedings, Budapest, July 2010, pp. 405-408.

[23] A.-L. Trautmann and J. Rosenthal, *Constructions of Constant Dimension Codes*, in: M. Greferath et al. (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 25-42.

[24] S.-T. Xia and F.-W. Fu, *Johnson Type Bounds on Constant Dimension Codes*, Designs, Codes and Cryptography, Vol. 50 (2009), 163–172.

CHAPTER 9

A COMBINATORIAL APPROACH TO FLAG CODES

Joint work with Clementa Alonso-González.

Abstract:

In network coding, a *flag code* is a collection of *flags*, that is, sequences of nested subspaces of a vector space over a finite field. Due to its definition as the sum of the corresponding subspace distances, the *flag distance* parameter encloses a hidden combinatorial structure. To bring it to light, in this paper, we interpret flag distances by means of *distance paths* drawn in a convenient *distance support*. The shape of such a support allows us to create an *ad hoc* associated *Ferrers diagram frame* where we develop a combinatorial approach to flag codes by relating the possible realizations of their minimum distance to different partitions of appropriate integers. This novel viewpoint permits to establish noteworthy connections between the flag code parameters and the ones of its projected codes in terms of well known concepts coming from the classical partitions theory.
Keywords: Network coding, flag codes, flag codistance, Ferrers diagrams, integer partitions, Durfee squares.

1 Introduction

Random network coding, firstly introduced in [1], has proven to be the most efficient way to send data across a non-coherent network with multiple sources and sinks. Despite its efficiency, it is very susceptible to error propagation and erasures. In order to overwhelm this weakness, in [12] the authors propose an algebraic approach by simply considering subspaces of $\mathbb{F}_q^n$ as codewords and *subspace codes* as collections of subspaces. Since this pioneering paper, much research has been made in constructing large subspace codes, and also in determining their properties. In case all the subspaces in the code have the same dimension, we speak about *constant dimension codes*. To have an overview of the most important works in this subject, consult [16] and the references therein.

In [14] the authors developed techniques for using *flags* of constant type, that is, sequences of nested subspaces of prescribed dimensions, in network coding. In this context, collections of flags are denominated *flag codes* and they appear as a generalization of constant dimension codes. The recent works [2, 4, 5, 6, 13] among others, show a growing interest in this topic.

If we consider all the subspaces of a given dimension of all the flags in a flag code, we obtain a constant dimension code called *projected code*. In the study of flag codes, a central problem is the one of unraveling to what extent is it possible to get the parameters of a flag code from the ones of its projected codes and conversely. In [3, 5, 6, 15] this problem has been addressed for the family of flag codes attaining the maximum distance (*optimum distance flag codes*) whereas in [2], the authors define *consistent* flag codes as precisely those ones whose projected codes completely determine their parameters. In the paper at hand, we deal with this question for general full flag codes from an innovative combinatorial perspective.

When investigating the parameters of a flag code, one of the main difficulties lies in the definition of the distance between flags: it is obtained as the sum of their subspaces distances, which causes that many different combinations of them can give the same flag distance value. To succinctly represent these possible fluctuations, in [4], the authors introduce the notion of *distance vector* (associated to a given distance value). Here, we draw distance vectors in the *distance support* to obtain the so-called *distance paths*. This simple geometrical idea allows us to focus on the *codistance* of the flag code (the complement of the distance) and hence, naturally associate to a flag code different combinatorial objects coming from the classical theory of partitions that result very convenient for our purposes.

The remain of the paper is organized as follows. In Section 2 we remember some basics on partitions and Ferrers diagrams. We also recall some background on subspace codes and flag codes. In Section 3 we address a deep study on the flag distance parameter by defining the *distance support* of the full flag variety which allows us to graphically represent the *distance path* of a couple of flags. We analyze the properties of such paths and we define the new concept of *codistance* of a flag code. In Section 4 we translate the information that can be read in the distance support into information encoded in a combinatorial scenario. To this end, we enrich the distance support to create a *Ferrers diagram frame* where each distance path will be read as a set of *Ferrers subdiagrams*, that is, as a set of integer partitions. At the same time, we associate to each of such partitions its *underlying distribution* that gives a particular *splitting* of the corresponding codistance. In this way, we establish a one-to-one correspondence between the set of distance paths associated to a distance value and the set of splittings associated to the corresponding codistance value. Finally, in Section 5, we take advantage of the dictionary established in the previous section and, with the help of specific objects coming from the partitions world, as *Durfee rectangles*, we exhibit different results that precisely relate the parameters of a flag code to the ones of its projected codes. We finish the paper with some representative examples that illustrate our results, one of them giving rise to a combinatorial characterization of full flag codes of the maximum distance.

2 Preliminaries

In this section we briefly recall the main definitions and results on partitions, Ferrers diagrams, subspace codes and flag codes that will be needed along the paper.

2.1 Partitions and Ferrers diagrams

Let us first fix some notation on integer partitions and their representation by Ferrers diagrams. Our basic reference related to this subject is [8].

Definition 2.1. Given a positive integer s, a *partition* of s is a sequence of non-increasing positive integers $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 + \cdots + \lambda_m = s$. Each value λ_i is called a *part* of λ and we say that m is the *length* of λ.

The number of partitions, usually denoted by $p(s)$, was determined asymptotically by Hardy and Ramanujan [11]. A remarkable expansion by Rademacher that permits calculate $p(s)$ more accurately can be found in [9, Chapter 5].

Example 2.2. *Here we give the possible seven partitions of* $s = 5$*:*

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1. \end{aligned} \tag{9.1}$$

The Ferrers diagram of an integer partition provides a very useful tool for geometrically visualizing partitions and to extract relevant properties about them in some cases.

Definition 2.3. Given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$, its *associated Ferrers diagram* $\mathfrak{F}_\lambda$ is constructed by stacking right-justified m rows of dots, where the number of dots in each row corresponds to the size λ_i of the corresponding part. The dot at the top right position is called the *corner* of the Ferrers diagram.

Example 2.4. *The next picture shows the Ferrers diagrams associated to the partitions of* $s = 5$ *given in Example 2.2.*

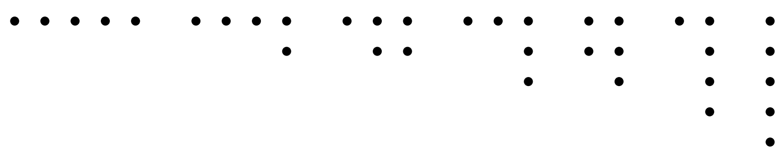

Figure 9.1: Ferrers diagrams with 5 dots.

There are other important elements naturally associated to the Ferrers diagram of a partition, as their Durfee rectangles and squares, that will be helpful for our purposes as we will see in Section 5. Let us briefly recall the precise definition.

Definition 2.5. Given the Ferrers diagram $\mathfrak{F}_\lambda$ of a partition λ, we call the *Durfee k-rectangle* associated to $\mathfrak{F}_\lambda$, and denote it by $D_k(\lambda)$, the largest-sized rectangle within $\mathfrak{F}_\lambda$ with top right vertex at the corner of $\mathfrak{F}_\lambda$ and such that its number of columns exceeds its number of rows by k. In particular, if $k = 0$, the Durfee k-rectangle will be just called the *Durfee square* associated to $\mathfrak{F}_\lambda$ and simply denoted by $D(\lambda)$.

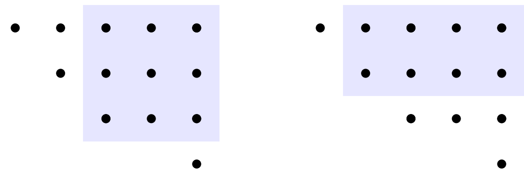

Figure 9.2: Durfee square and 2-Durfee rectangle $D_2(\lambda)$ for $\lambda = (5, 4, 3, 1)$.

In [7, 10], the reader can find more information on these objects.

2.2 Subspace codes and flag codes

Throughout the paper q will denote a fixed prime power and k, n two integers with $1 \leqslant k < n$. Consider $\mathbb{F}_q$ the finite field with q elements and denote by $\mathcal{G}_q(k, n)$ the *Grassmannian*, that is, the set of k-dimensional subspaces of $\mathbb{F}_q^n$. The set of vector subspaces of $\mathbb{F}_q^n$ can be equipped with different metrics but, in the current paper, we will use the so-called *injection distance*.

Definition 2.6. The *injection distance* between two subspaces $\mathcal{U}, \mathcal{V} \subseteq \mathbb{F}_q^n$ is defined as

$$d_I(\mathcal{U}, \mathcal{V}) = \max\{\dim(\mathcal{U}), \dim(\mathcal{V})\} - \dim(\mathcal{U} \cap \mathcal{V}). \tag{9.2}$$

In particular, if $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(k, n)$, then we have

$$d_I(\mathcal{U}, \mathcal{V}) = k - \dim(\mathcal{U} \cap \mathcal{V}). \tag{9.3}$$

Using this distance, we can define error-correcting codes in the Grassmannian as follows.

Definition 2.7. A *constant dimension code* $\mathcal{C}$ of length n and dimension k is a nonempty subset of $\mathcal{G}_q(k, n)$. The *minimum distance* of $\mathcal{C}$ is defined as

$$d_I(\mathcal{C}) = \min\{d_I(\mathcal{U}, \mathcal{V}) \mid \mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}\}$$

whenever $|\mathcal{C}| \geqslant 2$. In case $|\mathcal{C}| = 1$, we put $d_I(\mathcal{C}) = 0$.

Remark 2.8. Another frequent metric used when working with codes whose codewords are subspaces of $\mathbb{F}_q^n$ is the *subspace distance*. It is given by

$$d_S(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}). \tag{9.4}$$

Observe that, if $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(k, n)$, then

$$d_S(\mathcal{U}, \mathcal{V}) = 2(k - \dim(\mathcal{U} \cap \mathcal{V})) = 2d_I(\mathcal{U}, \mathcal{V}).$$

Hence, in the context of constant dimension codes, the injection distance and the subspace distance are equivalent metrics. Consult [16] and the references therein for more information on this class of codes.

The concept of constant dimension code can be extended when considering flags of constant type on $\mathbb{F}_q^n$, that is, sequences of nested subspaces of $\mathbb{F}_q^n$ where the list of corresponding dimensions is fixed. The use of flags in network coding as a generalization of constant dimension codes was first proposed in [14]. Let us recall some basic background on flag codes.

Definition 2.9. A *flag* $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ on $\mathbb{F}_q^n$ is a sequence of nested $\mathbb{F}_q$-vector subspaces

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n$$

of $\mathbb{F}_q^n$. The type of $\mathcal{F}$ is the vector $(\dim(\mathcal{F}_1), \dots, \dim(\mathcal{F}_r))$. In particular, if the type vector is $(1, 2, \dots, n-1)$, we say that $\mathcal{F}$ is a *full flag*. The subspace $\mathcal{F}_i$ is called the *i-th subspace* of $\mathcal{F}$.

The set of all the flags on $\mathbb{F}_q^n$ of a fixed type vector $(t_1, \dots, t_r)$ is said to be the *flag variety* $\mathcal{F}_q((t_1, \dots, t_r), n) \subseteq \mathcal{G}_q(t_1, n) \times \cdots \times \mathcal{G}_q(t_r, n)$ and, for every $i = 1, \dots, r$, we define the *i-projection* as the map

$$\begin{array}{rccc} p_i: & \mathcal{F}_q((t_1, \dots, t_r), n) & \longrightarrow & \mathcal{G}_q(t_i, n) \\ & \mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r) & \longmapsto & p_i(\mathcal{F}) = \mathcal{F}_i. \end{array} \tag{9.5}$$

The flag variety $\mathcal{F}_q((t_1, \dots, t_r), n)$ can be endowed with a metric by a natural extension of the injection distance defined in (9.2). More precisely, given two flags $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \dots, \mathcal{F}'_r)$ in $\mathcal{F}_q((t_1, \dots, t_r), n)$, the *(injection) flag distance* between them is the value

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_I(\mathcal{F}_i, \mathcal{F}'_i). \tag{9.6}$$

Remark 2.10. Observe that the subspace distance d_S defined in (9.4) can also be extended to the flag variety. Given $\mathcal{F}$ and $\mathcal{F}'$ as above, the sum of subspace distances

$$\sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i) = 2d_f(\mathcal{F}, \mathcal{F}')$$

is an equivalent distance to d_f. Due to the approach followed in this paper, it is more convenient for us to choose the injection flag distance.

Definition 2.11. A *flag code* of type $(t_1, \dots, t_r)$ on $\mathbb{F}_q^n$ is a non-empty subset $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$. Its *minimum distance* is given by

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}.$$

when $|\mathcal{C}| \geqslant 2$. If $|\mathcal{C}| = 1$, we put $d_f(\mathcal{C}) = 0$. The *i-projected code* of $\mathcal{C}$ is the set

$$\mathcal{C}_i = \{p_i(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\} \subseteq \mathcal{G}_q(t_i, n).$$

Example 2.12. *Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the canonical $\mathbb{F}_q$-basis of $\mathbb{F}_q^6$ and consider the flag code $\mathcal{C}$ of type $(1, 3, 5)$ on $\mathbb{F}_q^6$ given by the set of flags:*

$$\begin{array}{rcl} \mathcal{F}^1 & = & (\langle e_1\rangle, \langle e_1, e_2, e_3\rangle, \langle e_1, e_2, e_3, e_4, e_5\rangle), \\ \mathcal{F}^2 & = & (\langle e_4\rangle, \langle e_4, e_5, e_6\rangle, \langle e_1, e_2, e_4, e_5, e_6\rangle), \\ \mathcal{F}^3 & = & (\langle e_5\rangle, \langle e_4, e_5, e_6\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle). \end{array}$$

Its projected codes are

$$\begin{array}{rcl} \mathcal{C}_1 & = & \{\langle e_1\rangle, \langle e_4\rangle, \langle e_5\rangle\}, \\ \mathcal{C}_2 & = & \{\langle e_1, e_2, e_3\rangle, \langle e_4, e_5, e_6\rangle\}, \\ \mathcal{C}_3 & = & \{\langle e_1, e_2, e_3, e_4, e_5\rangle, \langle e_1, e_2, e_4, e_5, e_6\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle\}, \end{array}$$

with minimum distances $d_I(\mathcal{C}_1) = d_I(\mathcal{C}_3) = 1$ and $d_I(\mathcal{C}_2) = 3$. Moreover, it holds

$$d_f(\mathcal{C}) = d_f(\mathcal{F}^2, \mathcal{F}^3) = 1 + 0 + 1 = 2.$$

Remark 2.13. Note that the i-projected code $\mathcal{C}_i$ of $\mathcal{C}$ is a constant dimension code in the Grassmannian $\mathcal{G}_q(t_i, n)$. At this point it is important to underline that, albeit the projected codes are constant dimension codes closely related to a flag code, they do not determine it at all; different flag codes can share the same set of projected codes. On the other hand, the cardinality of $|\mathcal{C}_i|$ always satisfies $|\mathcal{C}_i| \leqslant |\mathcal{C}|$, whereas concerning the distance, we can have $d_f(\mathcal{C}) > d_I(\mathcal{C}_i)$, $d_f(\mathcal{C}) = d_I(\mathcal{C}_i)$ or even $d_f(\mathcal{C}) < d_I(\mathcal{C}_i)$. It suffices to see that, if $\mathcal{C}$ is the flag code given in Example 2.12, we have $d_f(\mathcal{C}) = 2 > 1 = d_I(\mathcal{C}_i)$ for $i = 1, 3$, but $d_f(\mathcal{C}) = 2 < 3 = d_I(\mathcal{C}_2)$. This range of possibilities in the distance parameter behaviour comes from the flag distance definition itself; a fixed distance value can be obtained by adding different configurations of the subspaces distances. For instance, if we take flags $\mathcal{F}^2, \mathcal{F}^3$ as in Example 2.12 and consider

$$\mathcal{F}^4 = (\langle e_3\rangle, \langle e_3, e_5, e_6\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle),$$

then we have

$$d_f(\mathcal{F}^2, \mathcal{F}^3) = 1 + 0 + 1 = 2 = 1 + 1 + 0 = d_f(\mathcal{F}^3, \mathcal{F}^4).$$

In [4] the authors deal algebraically with this question by capturing such a variability with the so-called *distance vectors*. The *distance paths* defined in Section 3 are a geometrical version of such distance vectors.

In light of the previous remark, it naturally arises the problem of obtaining the parameters of a flag code from the ones of its projected codes and conversely. In Section 5 we tackle this problem with the help of new techniques based on the combinatorial objects that we will describe along Sections 3 and 4.

3 Flag distance and distance paths

In this section we deepen the study of the flag distance parameter describing its particular quirks from a brand-new combinatorial viewpoint. In the remain of the paper we will always work with full flags.

As said in Section 2, the flag distance defined in (9.6) extends the subspace distance given in (9.2) in the following way: the flag distance between two given flags on a vector space is exactly the sum of the distances between their subspaces. This fact implies that, contrary to what happens with subspaces distances, flag distances conceal certain complexity in the sense that a fixed value for the flag distance can be attained from different combinations of the corresponding subspace distances.

Remark 3.1. Observe that every full flag $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ of length $n-1$ can be "enlarged" to the sequence of $n+1$ nested subspaces of $\mathbb{F}_q^n$ given by

$$\bar{\mathcal{F}} = (\{0\}, \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, \mathbb{F}_q^n) \tag{9.7}$$

just by adding $\mathcal{F}_0 = \{0\}$ and $\mathcal{F}_n = \mathbb{F}_q^n$. Now, for every pair of full flags $\mathcal{F}, \mathcal{F}'$, it clearly holds

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{n-1} d_I(\mathcal{F}_i, \mathcal{F}'_i) = \sum_{i=0}^{n} d_I(\mathcal{F}_i, \mathcal{F}'_i) = d_f(\bar{\mathcal{F}}, \bar{\mathcal{F}}').$$

So that the distance between two full flags $\mathcal{F}$, $\mathcal{F}'$ does not change if we extend them respectively to $\bar{\mathcal{F}}, \bar{\mathcal{F}}'$. Taking this fact into account, and for technical reasons, our study of the flag distance parameter will be undertaken by using extended full flags as in (9.7). However, observe that, when we consider an "extended" full flag code $\mathcal{C}$, two new and trivial projected codes arise: $\mathcal{C}_0 = \{0\}$ and $\mathcal{C}_n = \{\mathbb{F}_q^n\}$. These codes do not give any relevant information about $\mathcal{C}$. Consequently, in our study, we will just take into account the projected codes $\mathcal{C}_i$ with $1 \leqslant i \leqslant n-1$, as usual.

The injection flag distance between two (extended) full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$ is an integer that satisfies $0 \leqslant d_f(\mathcal{F}, \mathcal{F}') \leqslant D^n$, where

$$D^n = \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if} \quad n \quad \text{is even,} \\ \frac{n^2-1}{4} & \text{if} \quad n \quad \text{is odd.} \end{cases} \tag{9.8}$$

This expression is a direct consequence of the possible values that the injection distance between i-dimensional subspaces of $\mathbb{F}_q^n$ can reach. For every $0 \leqslant i \leqslant n$, let us write $\mathcal{R}(i,n)$ to denote the set of attainable injection subspace distances by subspaces in $\mathcal{G}_q(i,n)$. It is clear that $\mathcal{R}(0,n) = \mathcal{R}(n,n) = \{0\}$ and

$$\mathcal{R}(i,n) = \{0, 1, \dots, \min\{i, (n-i)\}\}, \text{ for } i \in \{1, 2, \dots, n-1\}. \tag{9.9}$$

Hence, we deduce straightforwardly the next lemma.

Lemma 3.2. *The following statements hold:*

(1) $\mathcal{R}(i,n) = \mathcal{R}(n-i,n)$ *for every* $0 \leqslant i \leqslant n$.

(2) $\mathcal{R}(0,n) \subset \mathcal{R}(1,n) \subset \mathcal{R}(2,n) \subset \cdots \subset \mathcal{R}(\lfloor \frac{n}{2} \rfloor, n)$.

Using this notation, for every value of $0 \leqslant i \leqslant n$, we consider the set of points $\mathrm{S}(i,n)$ of $\mathbb{Z}^2$ given by

$$\mathrm{S}(i,n) = \{i\} \times \mathcal{R}(i,n) = \{(i,\delta) \in \mathbb{Z}^2 \mid \delta \in \mathcal{R}(i,n)\}. \tag{9.10}$$

Definition 3.3. For every dimension $0 \leqslant i \leqslant n$, the set $\mathrm{S}(i,n)$ defined in (9.10) is called *the distance support* of the Grassmannian $\mathcal{G}_q(i,n)$.

This geometrical representation can be generalized to the full flag variety as follows.

Definition 3.4. The *distance support* of the full flag variety on $\mathbb{F}_q^n$ is the set

$$\mathrm{S}(n) = \bigcup_{i=0}^{n} \mathrm{S}(i,n) \subset \mathbb{Z}^2. \tag{9.11}$$

Graphically, the distance support $\mathrm{S}(n)$ has the following representation.

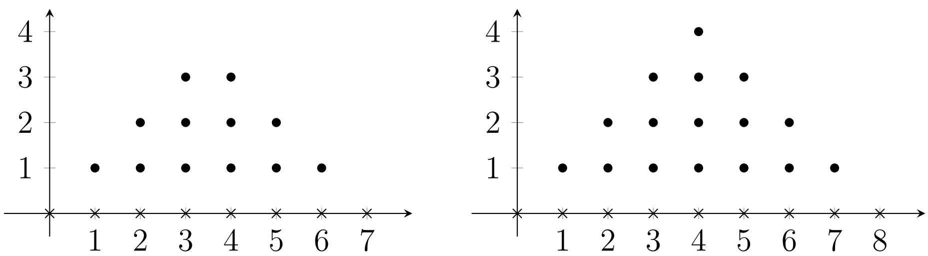

Figure 9.3: Distance supports S(7) and S(8).

Remark 3.5. The reader can appreciate that two different node styles have been used in Figure 9.3. This is due to the fact that not every point contributes equally when we use the support to represent flag distances. On one side, crossed dots denote null distances. On the other side, there are exactly D^n circle dots representing positive distances. This dichotomy will be very useful when representing distances between pairs of flags in $\mathrm{S}(n)$.

It is clear that the i-th column of the distance support $\mathrm{S}(n)$ is just the distance support $\mathrm{S}(i,n)$ of $\mathcal{G}_q(i,n)$. Hence, as a consequence of Lemma 3.2, the support $\mathrm{S}(n)$ is symmetric with respect to the vertical line $x = \frac{n}{2}$ and the columns heights grow as the dimension gets closer to $\frac{n}{2}$. Observe that there is a remarkable difference in the distance support shape depending on the parity of n: when n is even, the value $\frac{n}{2}$ is a dimension on the type vector. In this case, $\mathrm{S}(n)$ has a *peak* in this central dimension. In contrast, when n is odd, $\mathrm{S}(n)$ presents a *plateau* on its top. The higher attainable distances in this latter case are placed at dimensions $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, i.e., the closest integers (from left and right, respectively) to the value $\frac{n}{2}$.

Remark 3.6. Concerning also the distance support shape, notice that the distance support $\mathrm{S}(n-1)$ can be obtained from $\mathrm{S}(n)$ just by removing the set of points with coordinates $(i,(n-i))$ whenever $2i \geqslant n$. These points are the ones in the "right-roof" as in the next figure.

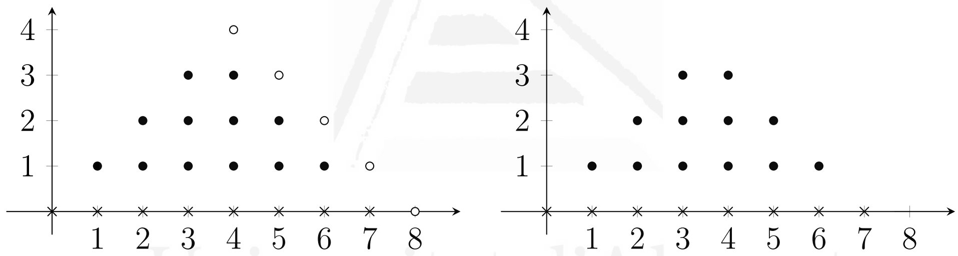

Figure 9.4: Getting S(7) (right) from S(8) (left).

At this point, if we consider two full flags on $\mathbb{F}_q^n$, their flag distance can be geometrically represented by means of a collection of $n+1$ points in the distance support $\mathrm{S}(n)$, each one of them in a different column $\mathrm{S}(i,n)$ as follows.

Definition 3.7. Given a pair of full flags $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{F}_q^n$, we define their *distance path* $\Gamma(\mathcal{F},\mathcal{F}')$ as the directed polygonal path whose vertices are the points $(i, d_I(\mathcal{F}_i,\mathcal{F}'_i))$ for every $0 \leqslant i \leqslant n$.

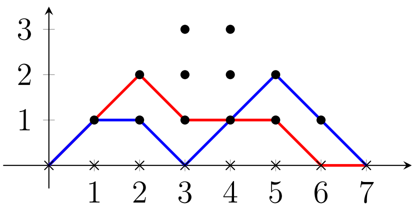

Figure 9.5: Examples of distance paths in S(7).

Similarly, given a full flag code, we can consider a collection of distance paths associated to it.

Definition 3.8. Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. The set of *distance paths of* $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}) = \{\Gamma(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'\}.$$

Notice that every distance path in $\mathrm{S}(n)$ starts at the point $(0,0)$ and arrives to $(n,0)$. Nevertheless, it is important to point out that not every polygonal path with vertices in the support $\mathrm{S}(n)$ satisfying this condition represents a potential distance path of a pair of flags. In order to characterize the polygonal paths in $\mathrm{S}(n)$ also being distance paths between a couple of flags in $\mathbb{F}_q^n$, in the following result we see that, for a given pair of full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$, the value of $d_I(\mathcal{F}_i, \mathcal{F}'_i)$ completely determines the range of possibilities for $d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1})$.

Theorem 3.9. *Consider $\mathcal{F}, \mathcal{F}'$ full flags on $\mathbb{F}_q^n$ and denote $\delta_i = d_I(\mathcal{F}_i, \mathcal{F}'_i)$ where $i \in \{0, 1, \dots, n\}$. Then, for any $0 \leqslant i < n$, it holds*

$$\delta_{i+1} \in \{\delta_i - 1, \delta_i, \delta_i + 1\}.$$

Proof. The proof is based on the flags nested structure. Consider full flags $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{F}_q^n$. For every $1 \leqslant i < n-1$, we have

$$\mathcal{F}_i \cap \mathcal{F}'_i \subseteq \mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1} \text{ and } \mathcal{F}_i + \mathcal{F}'_i \subseteq \mathcal{F}_{i+1} + \mathcal{F}'_{i+1}.$$

The second inclusion leads to the next inequality

$$2i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i) \leqslant 2(i+1) - \dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1})$$

or, equivalently,

$$\dim(\mathcal{F}_{i+1} \cap \mathcal{F}'_{i+1}) \leqslant \dim(\mathcal{F}_i \cap \mathcal{F}'_i) + 2.$$

Using this fact, and taking into account that $d_I(\mathcal{F}_j, \mathcal{F}'_j) = j - \dim(\mathcal{F}_j \cap \mathcal{F}'_j)$ for every $1 \leqslant j \leqslant n-1$, it follows that

$$i + 1 - \dim(\mathcal{F}_i \cap \mathcal{F}'_i) - 2 \leqslant d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1}) \leqslant i + 1 - \dim(\mathcal{F}_i \cap \mathcal{F}'_i),$$

which is equivalent to

$$i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i) - 1 \leqslant d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1}) \leqslant i - \dim(\mathcal{F}_i \cap \mathcal{F}'_i) + 1.$$

Hence,

$$d_I(\mathcal{F}_i, \mathcal{F}'_i) - 1 \leqslant d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1}) \leqslant d_I(\mathcal{F}_i, \mathcal{F}'_i) + 1.$$

In other words, we have that the value $\delta_{i+1} = d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1})$ is an element in $\{\delta_i - 1, \delta_i, \delta_i + 1\}$ as we wanted to prove. ∎

Example 3.10. *In view of Theorem 3.9, the next figure illustrates how a distance path is allowed to continue once we have fixed one of its points. From some points, for instance* $(3, 1)$*, we have three options to continue. On the other hand, there are just two possibilities if we fix either the point* $(0, 0)$ *or the point* $(5, 1)$*. Last, paths passing through* $(4, 3)$ *must contain the point* $(5, 2)$*.*

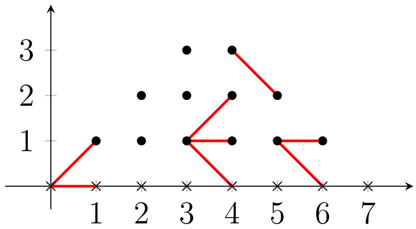

Figure 9.6: Allowed movements from some points in S(7).

In general, given the point (i, δ_i) in $\mathrm{S}(i, n)$, distance paths passing through it can only come from a point $(i-1, \delta_{i-1}) \in \mathrm{S}(i-1, n)$ with

$$\delta_{i-1} \in \{\delta_i - 1, \delta_i, \delta_i + 1\}. \tag{9.12}$$

At the same time, these paths can only continue through points $(i+1, \delta_{i+1}) \in \mathrm{S}(i+1, n)$

$$\delta_{i+1} \in \{\delta_i - 1, \delta_i, \delta_i + 1\}. \tag{9.13}$$

All in all, distance paths are, graphically, oriented polygonal paths passing through points

$$(0, 0) \to (1, \delta_1) \to (2, \delta_2) \to \cdots \to (n-1, \delta_{n-1}) \to (n, 0) \tag{9.14}$$

such that consecutive vertices (i, δ_i) and $(i+1, \delta_{i+1})$ are related according to the *trident* rules given by (9.12) and (9.13).

Remark 3.11. Observe that distance paths defined as above are the graphic representation of the notion of the *distance vector* associated to a couple flags introduced in [4]. Moreover, our Theorem 3.9 is a geometric version of Theorem 3.9 in [4], for the special case of full flags. In particular, as a consequence of that result, we can assure that, given a path Γ in $S(n)$ satisfying (9.12), (9.13) and (9.14), there is always a couple of full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$ such that $\Gamma = \Gamma(\mathcal{F}, \mathcal{F}')$.

In view of the previous remark, from now on, a path Γ in $S(n)$ described by (9.12), (9.13) and (9.14) will be said a *distance path* given that it represents the flag distance value $d_\Gamma = \sum_{i=0}^{n} \delta_i$, attained by a couple of full flags $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{F}_q^n$ such that $d_I(\mathcal{F}_i, \mathcal{F}'_i) = \delta_i$. Conversely, this tool allows us to geometrically determine how an arbitrary distance flag value d can suitably split into the $n-1$ terms that are not trivially zero (recall that $\delta_0 = \delta_n = 0$). For instance, looking again at paths Γ and Γ' in S(7) in Figure 9.5, we can say that

$$6 = 0 + 1 + 2 + 1 + 1 + 1 + 0 + 0 = 0 + 1 + 1 + 0 + 1 + 2 + 1 + 0$$

are permitted subspace distance combinations for the distance $d_\Gamma = d_{\Gamma'} = 6$. In terms of the language used in [4], these two paths correspond, respectively, to the distance vectors $(0, 1, 2, 1, 1, 1, 0, 0)$ and $(0, 1, 1, 0, 1, 2, 1, 0)$.

Note that any distance path Γ in $\mathrm{S}(n)$ consists of n segments of lines with slope equal to -1, 0 or 1. Moreover, recall that every distance path starts at the origin and ends at the point $(n, 0)$, both with null height. Hence, the number of edges with positive slope in a distance path must coincide with the one of edges with negative slope. As a result, the number of horizontal segments appearing in Γ has the same parity than n. These latter edges will play an important role in the following section. Let us define them more precisely.

Definition 3.12. A *plateau of height* δ in a distance path Γ is a sequence of two consecutive vertices $(i, \delta), (i+1, \delta)$ on it. In other words, if we consider two full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$ with $\Gamma = \Gamma(\mathcal{F}, \mathcal{F}')$, a *plateau* appears when $d_I(\mathcal{F}_i, \mathcal{F}'_i) = d_I(\mathcal{F}_{i+1}, \mathcal{F}'_{i+1})$ for some $0 \leqslant i \leqslant n-1$. We denote by p_Γ the number of *plateaus* on a given distance path Γ.

Example 3.13. *The distance path in red represented in Figure 9.5 has two plateaus of height* 1. *As said before, if n is an even (resp. odd) positive integer, then every distance path Γ in* $\mathrm{S}(n)$ *contains an even (resp. odd) number of plateaus. In particular, for odd values of n, distance paths must contain at least one plateau.*

Remark 3.14. Let us briefly come back to the support $\mathrm{S}(n)$. In the following sections it will be important to compute the number of dots associated to a given distance path. Notice that in $\mathrm{S}(i, n)$, a point (i, δ_i) leaves exactly δ_i circle dots and one crossed dot bellow it (including the point (i, δ_i) itself). Using this idea, we can compute the associated distance of a given distance path Γ by simply counting the number of circle dots on Γ or bellow Γ. Moreover, we can relate the value d_Γ to the area of the polygons determined by Γ together with the abscissa axis. In Figure 9.5, the red path on $\mathrm{S}(7)$ determines a single polygon having the points $(0, 0)$ and $(6, 0)$ as vertices whereas the path in blue determines two of them. On the other hand, the path in Figure 9.7 forms a unique polygon in $\mathrm{S}(7)$ with vertices on the points $(0, 0)$ and $(7, 0)$.

Theorem 3.15. *Let Γ be a distance path in $S(n)$ such that Γ determines a unique polygon P_Γ with the abscissa axis. Then the flag distance d_Γ is exactly the area of P_Γ.*

Proof. It is enough to point out that P_Γ is a reticulated polygon with vertices in the integer lattice $\mathbb{Z}^2$. We start assuming that the points $(0, 0)$ and $(n, 0)$ are vertices of P_Γ. In this case, if we write I and B to denote the set of lattice points in the interior and the boundary of P_Γ, respectively, then we have that I is a set of circle dots. However, B contains $n-1$ circle dots and $n+1$ crossed ones. Consequently, according to Remark 3.14, it holds $d_\Gamma = |I| + n - 1$.

By means of Pick's Theorem, the area of P_Γ can be computed in terms of I and B as

$$A(P_\Gamma) = |I| + \frac{|B|}{2} - 1 = |I| + n - 1 = d_\Gamma.$$

On the other hand, if Γ determines a unique polygon P_Γ with vertices $(i, 0)$ and $(j, 0)$, for some $0 \leqslant i < j \leqslant n$, then the result follows by interpreting P_Γ as a polygon in a smaller distance support $\mathrm{S}(j-i)$ with the points $(0,0)$ and $(j-i, 0)$ as its vertices and arguing as above. ∎

The following corollary follows then straightforwardly:

Corollary 3.16. *Let Γ be a distance path in $S(n)$ such that Γ determines the polygons $P_\Gamma^1, \ldots, P_\Gamma^k$ with the abscissa axis. Then the flag distance d_Γ is exactly the sum of the areas of $P_\Gamma^1, \ldots, P_\Gamma^k$.*

By means of Proposition 3.9 and Theorem 3.15, we can remove the coordinate axes when representing flag distances in the support $\mathrm{S}(n)$. In fact, we just need to study paths constructed by chaining the *trident* moves represented in Figure 3.10, and to count how many circle dots remain in or under such paths. Of course, different distance paths can provide the same flag distance, i.e., they leave the same amount of circle dots below them or, equivalently, above them. Next we introduce the notion of *(flag) codistance* as a complementary value associated to a flag distance which will be crucial in the remain of the paper.

Definition 3.17. Given a flag distance value d, i.e., an integer such that $0 \leqslant d \leqslant D^n$, we define its *(injection flag) codistance* as the value $\bar{d} = D^n - d$. Similarly, given a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$, we define its associated *codistance* as the value $\bar{d}_f(\mathcal{C}) = D^n - d_f(\mathcal{C})$.

Notice that both d and $\bar{d}$ provide exactly the same information since every flag distance value determines a unique codistance value and conversely. Arguing as in Remark 3.14, the codistance can be read in the distance support as follows.

Corollary 3.18. *The number of dots over a distance path Γ in $\mathrm{S}(n)$ is equal to the codistance $\bar{d}_\Gamma = D^n - d_\Gamma$ associated to it.*

Example 3.19. *Take $n = 7$ and consider the next distance path Γ in $\mathrm{S}(7)$*

$$\Gamma: \ (0,0) \to (1,1) \to (2,2) \to (3,1) \to (4,1) \to (5,2) \to (6,1) \to (7,0). \quad (9.15)$$

There are $D^7 = 12$ circle dots in the distance support $\mathrm{S}(7)$. The distance path passes exactly through 6 of them (in black) and the associated polygon P_Γ contains 2 black circle dots in its interior. The area of such a polygon is exactly 8 (see the picture below). Hence, the distance path in (9.15) represents a possible distribution to obtain the flag distance $d = 8$. Indeed, $8 = 1 + 2 + 1 + 1 + 2 + 1$.

The corresponding codistance is $12 - 8 = 4$*, which is, as said in Corollary 3.18, is the number of points over the path (white circle dots).*

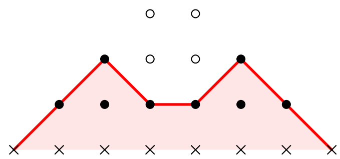

Figure 9.7: A distance path Γ with associated distance and codistance.

4 Combinatorial perspective

In this section, we introduce some combinatorial objects closely related to the distance support. To do so, we need to enrich it with an auxiliary collection of (red) points that will allow us to obtain a Ferrers diagram. With the help of such a diagram we will establish a round trip dictionary that will allows us to obtain information, in Section 5, about the distance of a flag code in terms of classical concepts related to partitions of integers.

Let us start describing the enriched version of the distance support $\mathrm{S}(n)$. We complete it by adding suitable auxiliary red points as in the next picture. The resultant two-colored set of points is called *enriched flag distance support* or just *enriched support* for short. We denote it by $\hat{\mathrm{S}}(n)$.

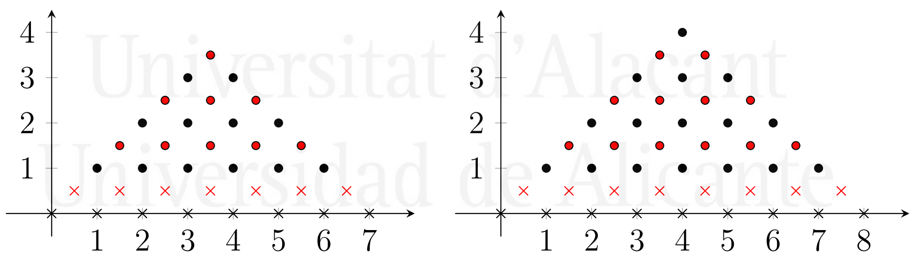

Figure 9.8: Enriched flag distance diagrams $\hat{\mathrm{S}}(7)$ and $\hat{\mathrm{S}}(8)$.

Recall that the silhouette of the distance support $\mathrm{S}(n)$ depends on the parity of n. By contrast, the enriched version $\hat{\mathrm{S}}(n)$ has always the same triangular shape. However, the position of black/red points changes depending on the parity of n. For instance, the top vertex, which has coordinates $(n/2, n)$, is black (resp. red) when n is even (resp. odd).

Remark 4.1. As stated in Remark 3.6, the distance support $\mathrm{S}(n-1)$ can be obtained from $\mathrm{S}(n)$ by deleting the set of points in the right-roof. Similarly, the two-colored diagram $\hat{\mathrm{S}}(7)$ can be constructed from $\hat{\mathrm{S}}(8)$ by performing the same operation.

In order to give a systematic and convenient construction of the two-colored enriched support $\hat{\mathrm{S}}(n)$, we proceed as follows. First, we fix the set of points in $\mathrm{S}(n)$ and plot them in black. Next, we consider the distance support $\mathrm{S}(n-1)$ whose points we plot in red, and translate it with the vector $\left(\frac{1}{2}, \frac{1}{2}\right)$. We obtain the set

$$\mathrm{S}(n-1) + \left(\frac{1}{2}, \frac{1}{2}\right) = \left\{ \left(i + \frac{1}{2}, \delta + \frac{1}{2}\right) \;\middle|\; (i, \delta) \in \mathrm{S}(n-1) \right\}.$$

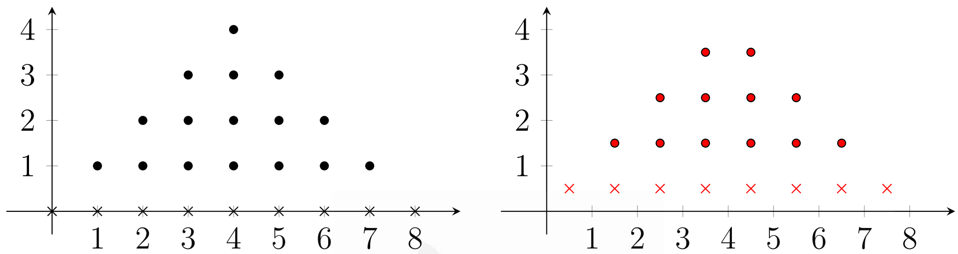

Figure 9.9: The distance support $\mathrm{S}(8)$ (left) and the set $\mathrm{S}(7) + (1/2, 1/2)$ (right).

The overlap of $\mathrm{S}(8)$ (left) and $\mathrm{S}(7) + (1/2, 1/2)$ leads to the two-colored enriched support $\hat{\mathrm{S}}(8)$ given in Figure 9.8 (right). Let us give a precise definition.

Definition 4.2. For every $n \geqslant 2$, the *enriched distance support* $\hat{\mathrm{S}}(n)$ is given by the set of points in

$$\mathrm{S}(n) \dot{\cup} \left(\mathrm{S}(n-1) + \left(\frac{1}{2}, \frac{1}{2}\right) \right).$$

Remark 4.3. One can easily compute the number of dots (both black and red) included in the enriched support $\hat{\mathrm{S}}(n)$. By Proposition 4.2 along with Remark 3.5, it is clear that $\hat{\mathrm{S}}(n)$ contains D^n circle black points and D^{n-1} red ones. Hence, using the explicit value of D^n given in formula (9.8), we conclude that for every $n \geqslant 2$, the enriched support $\hat{\mathrm{S}}(n)$ contains $\frac{n(n-1)}{2}$ circle dots. It also clearly contains $2n+1$ crossed dots.

As done with distance supports in the previous part, we can remove the axis and work with the two-colored enriched support $\hat{\mathrm{S}}(n)$ without specifying the coordinates of each point.

4.1 Associated Ferrers diagrams

This subsection is devoted to describe the flag distance between full flags on $\mathbb{F}_q^n$ through the concept of distance path, by using suitable Ferrers diagrams. To do so, fixed a positive integer n, we consider the enriched distance support $\hat{\mathrm{S}}(n)$ just introduced and rotate it around the point $(n, 0)$ it as in the next figure.

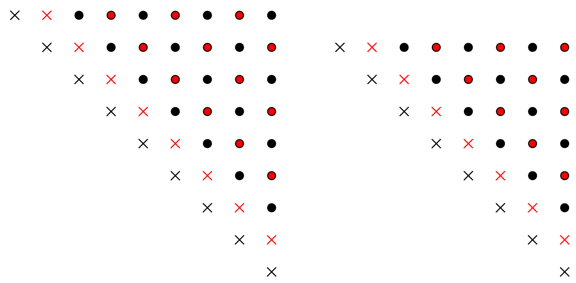

Figure 9.10: Rotated enriched supports $\hat{\mathrm{S}}(8)$ and $\hat{\mathrm{S}}(7)$.

Note that after rotation of $\hat{\mathrm{S}}(n)$ it arises a Ferrers diagram.

Definition 4.4. Given $n \geqslant 2$, the *primary Ferrers diagram frame* associated to the full flag variety on $\mathbb{F}_q^n$ is the diagram $\overline{\mathrm{FF}}(n)$ obtained from the enriched support $\hat{\mathrm{S}}(n)$ after a rotation with center $(n, 0)$ and angle $-\frac{\pi}{4}$. The set of circle dots (both black and red) in $\overline{\mathrm{FF}}(n)$ is called *Ferrers diagram frame* and denoted by $\mathrm{FF}(n)$.

Remark 4.5. Once again, the reason to distinguish these two Ferrers diagrams is the null contribution of crossed dots to the flag distance. On the other hand, observe that $\mathrm{FF}(n)$ is a Ferrers diagram associated to the partition $(n-1, n-2, \ldots, 1)$ of the integer $n(n-1)/2$ that, as said in Remark 4.3, is the exactly number of circle points in $\hat{\mathrm{S}}(n)$.

There are some partitions of positive integers $r \leqslant n(n-1)/2$ that can be represented by Ferrers diagrams contained in the Ferrers diagram frame (then in the primary one). The following definition precises this idea.

Definition 4.6. Every Ferrers diagram contained in $\mathrm{FF}(n)$ is said to be a *Ferrers subdiagram* of $\mathrm{FF}(n)$. We also say that the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of the integer $\sum_{i=1}^{m} \lambda_i$ is an *embedded partition* on $\mathrm{FF}(n)$ if

(1) $1 \leqslant m \leqslant n-1$ and,

(2) for every $1 \leqslant i \leqslant m$, it holds $\lambda_i \leqslant n-i$.

Due to technical reasons, we also consider the *empty Ferrers subdiagram* $\mathfrak{F}_0$, associated to the *null embedded partition* $\lambda = (0)$ (see the second diagram in Figure 9.11).

Observe that these embedded partitions are exactly those ones whose associated Ferrers diagram fits in $\mathrm{FF}(n)$. With this notation, we conclude directly the next result.

Proposition 4.7. *Let $\mathfrak{F}_\lambda$ be a Ferrers diagram associated to the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$. Then the following statements are equivalent.*

(1) $\mathfrak{F}_\lambda$ *is a Ferrers subdiagram of* $\mathrm{FF}(n)$.

(2) The partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ *is an embedded partition on* $\mathrm{FF}(n)$.

At this point, and with the purpose of connecting these combinatorial objects with our study on the flag distance, we define a special class of polygonal paths in the primary Ferrers diagram $\overline{\mathrm{FF}}(n)$, closely related to the set of Ferrers subdiagrams in $\mathrm{FF}(n)$.

Definition 4.8. A *staircase path* Σ on $\overline{\mathrm{FF}}(n)$ is just a polygonal directed path whose vertices are dots of $\overline{\mathrm{FF}}(n)$ (crossed or circle ones) such that:

- it starts (resp. ends) at the highest (resp. lowest) crossed black point and
- its directed edges are either vertical segments straight down or horizontal segments from left to right.

Remark 4.9. Observe that, since black and red dots are interspersed in $\overline{\mathrm{FF}}(n)$, staircase paths travel along the diagram alternating black and red points. Even more, every staircase path contains exactly $n + 1$ black dots and n red ones. Moreover, since staircase paths cannot go neither up nor to the left, the collection of points that remains at right of any staircase path satisfy the next property: the number of dots at a given row is always upper bounded by the number of dots at the previous one, that is, any staircase path is the "silhouette" of a Ferrers diagram.

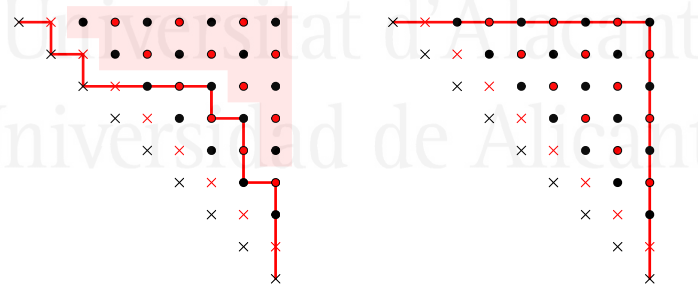

Figure 9.11: Staircase paths and corresponding Ferrers subdiagrams.

Proposition 4.10. *Given* Σ *a staircase path in* $\overline{\mathrm{FF}}(n)$*, the set of points to the right of* Σ *forms a set* $\mathfrak{F}(\Sigma)$ *that is a Ferrers subdiagram of* $\mathrm{FF}(n)$*. Conversely, a Ferrers subdiagram* $\mathfrak{F}$ *of* $\mathrm{FF}(n)$ *determines a unique staircase path* $\Sigma(\mathfrak{F})$ *in* $\overline{\mathrm{FF}}(n)$*. In this situation we say that* Σ *(resp.* $\Sigma(\mathfrak{F})$*) is the* silhouette *of the Ferrers subdiagram* $\mathfrak{F}(\Sigma)$ *(resp.* $\mathfrak{F}$*).*

Proof. Consider a staircase path Σ in $\overline{\mathrm{FF}}(n)$. Giving this path is equivalent to provide a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ such that, for every $1 \leqslant i \leqslant$

$n-1$, the value $0 \leqslant \lambda_i \leqslant n-i$ counts the number of circle dots in the i-th row of $\overline{\mathrm{FF}}(n)$ that remain to the right of Σ. As said in Remark 4.9, these values satisfy $\lambda_i \geqslant \lambda_{i+1} \geqslant 0$, for every $1 \leqslant i \leqslant n-2$. If the sequence λ does not contain zeros, or equivalently $\lambda_{n-1} \neq 0$, then it is a partition and we have that $\mathfrak{F}(\Sigma) = \mathfrak{F}_\lambda$. On the other hand, if $\lambda_1 = 0$, then $\mathfrak{F}(\Sigma) = \mathfrak{F}_0$. This case corresponds to the staircase that flows horizontally up to the corner of $\overline{\mathrm{FF}}(n)$ and then comes down vertically as in the second diagram in Figure 9.11. Last, if $\lambda = (\lambda_1, \ldots, \lambda_i, 0, \ldots, 0)$, with $\lambda_i \neq 0$, then $\mathfrak{F}(\Sigma) = \mathfrak{F}_{\lambda'}$ with $\lambda' = (\lambda_1, \ldots, \lambda_i)$.

Conversely, given a Ferrers subdiagram $\mathfrak{F}$ with associated embedded partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$, then $m \leqslant n-1$. If $m = n-1$, the partition λ' is the one corresponding to $\Sigma(\mathfrak{F})$. Otherwise, it is enough to complete λ' with $n-1-m$ extra zero components to obtain $(\lambda'_1, \ldots, \lambda'_m, 0, \ldots, 0)$ as the sequence of length $n-1$ that characterizes the silhouette $\Sigma(\mathfrak{F})$. ∎

Corollary 4.11. *The set of staircase paths in* $\overline{\mathrm{FF}}(n)$ *is in one-to-one correspondence with the set of embedded partitions in* $\mathrm{FF}(n)$*, that is, for any staircase path* Σ *there is a unique embedded partition* $\lambda(\Sigma)$ *such that* $\mathfrak{F}(\Sigma) = \mathfrak{F}_{\lambda(\Sigma)}$.

4.2 Coming back to the flag distance

In the previous subsection, we associated a couple of Ferrers diagrams to the distance support by adding a suitable collection of auxiliary red points. In that scenario, the set of staircase paths are characterized by using embedded partitions of integers. Once established these combinatorial tools, to re-connect them with our distance paths, we need to remove the auxiliary red structure somehow. This removing process will allows us describe distance paths associated to a flag distance value d in terms of appropriate summand distributions of the corresponding codistance value $\bar{d}$. Let us explain this in more detail.

Our first objective is to consistently retrieve distance paths from staircase paths. Recall that, as stated in Remark 4.9, black and red dots alternate in a staircase path. Hence, in order to study distance paths obtained by removing red dots, it is important to observe how two consecutive black dots can be connected when the intermediate red point that appears between them in a staircase path is eliminated. There are four admissible local movements for this operation:

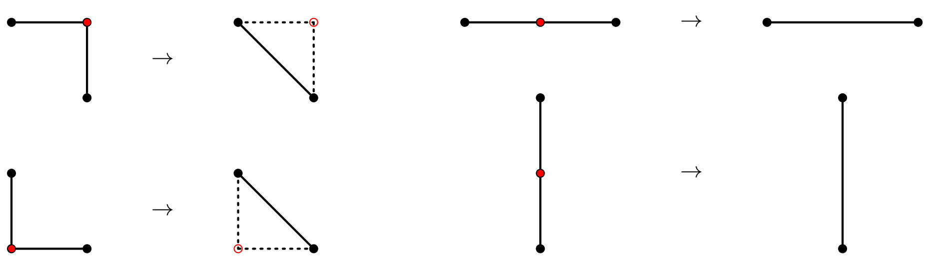

Figure 9.12: Movements to locally recover a distance path by removing a red dot.

Remark 4.12. The four motion patterns described in Figure 9.12 are also valid when they involve crossed points.

With these four movements in mind, we do the following. We consider all the possible sequences of black-red-black points that are parts of staircase paths passing through a given black dot. We remove the red intermediate point and apply the corresponding movement in Figure 9.12 and get just three possibilities, labelled as $1, 2$ and 3 in Figure 9.13, starting from the given black dot. Observe that this figure corresponds to the rotation of the trident pattern exposed in Figure 3.10 and obtained in Proposition 3.9.

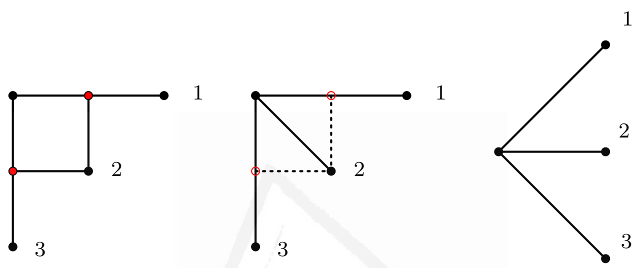

Figure 9.13: Recovering the trident pattern after deletion of red points.

As a consequence, we have straightforwardly the next result.

Proposition 4.13. *The polygonal path* $\Gamma(\Sigma)$ *obtained after removing the red dots of any staircase path* Σ *in* $\overline{\mathrm{FF}}(n)$ *by applying the movements in Figure 9.12 is a distance path in the distance support* $\mathrm{S}(n)$ *(after rotation). We call it the* skeleton distance path *associated to the staircase path* Σ.

It is clear that different staircase paths in $\overline{\mathrm{FF}}(n)$ can share the same associated skeleton distance path in $\mathrm{S}(n)$ (after rotation). Hence, we can define an equivalence relation on the set of staircase paths.

Definition 4.14. We say that two staircase paths Σ and Σ' in $\overline{\mathrm{FF}}(n)$ are *distance-equivalent* if they have the same associated skeleton distance path, i.e., $\Gamma(\Sigma) = \Gamma(\Sigma')$. Given a distance path Γ in $\mathrm{S}(n)$, we denote the set of distance-equivalent staircase paths with Γ as their skeleton distance path as $\Sigma(\Gamma)$.

Example 4.15. *In Figure 9.14, we can see a distance path* Γ *in* $\mathrm{S}(8)$ *and the same path represented in the enriched support* $\hat{\mathrm{S}}(8)$*. Below, in Figure 9.15, we represent in red the four possible staircase paths in* $\Sigma(\Gamma)$*, i.e., those staircase paths having* Γ *as their associated skeleton distance path.*

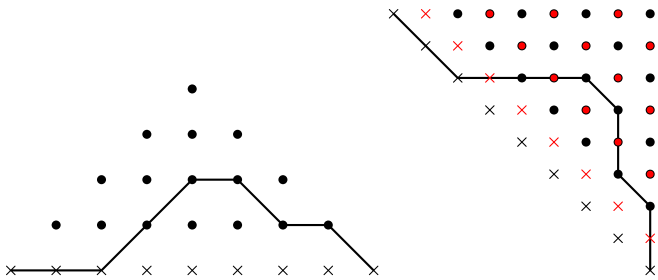

Figure 9.14: A distance path Γ in $\hat{\mathrm{S}}(8)$ (left) seen also in $\overline{\mathrm{FF}}(8)$ (right).

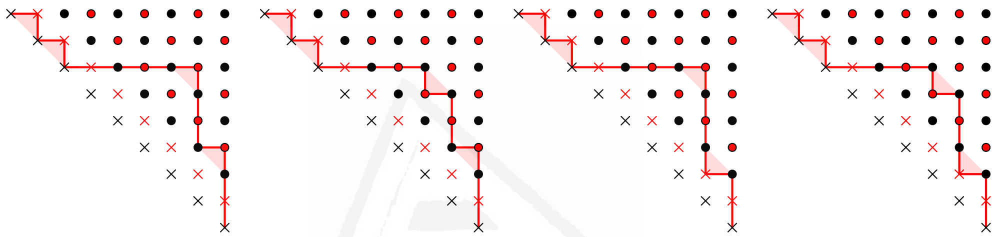

Figure 9.15: The four elements in $\Sigma(\Gamma)$.

The next result gives the exact number of different staircase paths that lie in a given coset of the distance-equivalence relationship, in terms of the number of *plateaus* of their associated skeleton distance path (recall Definition 3.12).

Proposition 4.16. *Consider a distance path Γ in the distance support $\mathrm{S}(n)$ with p plateaus of positive height. Then the number of staircase paths in $\Sigma(\Gamma)$ is exactly 2^p.*

Proof. Let Γ a the distance path in the distance support $\mathrm{S}(n)$. Consider an arbitrary edge e of Γ, connecting two consecutive vertices (i, δ_i) and $(i+1, \delta_{i+1})$, for some $0 \leqslant i \leqslant n-1$, and assume that Σ is a staircase path passing through these two black points too. Let us study the possibilities for the red point in Σ connecting these two black ones. By virtue of Proposition 3.9, we know that e has slope either $-1, 0$ or 1. If it has slope equal to -1 (resp. 1), then after rotation we obtain a vertical (resp. horizontal) segment that already determines the unique red point connecting the starting vertices, as we can see in the next figure.

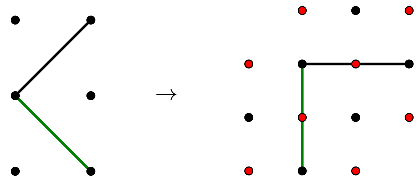

Figure 9.16: Segments of slope 1 (in black) and -1 (in green) in $\mathrm{S}(n)$ (left). The same segments seen in $\overline{\mathrm{FF}}(n)$ (right).

On the other hand, if e has slope equal to 0, i.e., it is a *plateau*, then it is transformed (after rotation) into a segment with slope -1 in $\overline{\mathrm{FF}}(n)$. It can be replaced by two sequences of movements in Σ: either right-down or down-right, marked in red and green, respectively in the picture below. These sequences correspond to use the middle red point with coordinates $(i+\frac{1}{2}, \delta_i+\frac{1}{2})$ or $(i+\frac{1}{2}, \delta_i-\frac{1}{2})$, respectively. Hence, there are two possibilities for replacing e, unless $\delta_i=\delta_{i+1}=0$. In this case, only the crossed red point $(i+\frac{1}{2}, \frac{1}{2})$ can be used. As a result, if p counts the number of *plateaus* of Γ with positive height, then there are exactly 2^p different staircase paths with Γ as their skeleton.

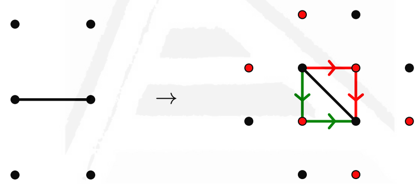

Figure 9.17: A *plateau* in $\mathrm{S}(n)$ (left) and its associated staircase paths in $\overline{\mathrm{FF}}(n)$ (right).

∎

Notice that, since distance-equivalent staircase paths have the same associated skeleton distance path, in particular, they are associated to the same flag distance value, which becomes a numerical invariant that can be assigned to staircase paths. On the other hand, a staircase path always contains $n+1$ black dots and n red ones. Using these facts along with Remark 3.14, we obtain the next result.

Corollary 4.17. *Given a staircase path Σ, the number of circle black dots in Σ or to its left is constant for all staircase paths distance-equivalent to Σ. This value is exactly $d_{\Gamma(\Sigma)}$.*

The same idea can be applied to the set of points to the right of Σ, i.e., the points in $\mathfrak{F}(\Sigma)$, the Ferrers subdiagram of $\mathrm{FF}(n)$ having Σ as their silhouette.

Definition 4.18. Let $\mathfrak{F}$ be a Ferrers subdiagram in $\mathrm{FF}(n)$. The *underlying black diagram* of $\mathfrak{F}$ is the set of black points $U_{\mathfrak{F}}$ obtained after removing the set of red points on it.

Remark 4.19. Observe that the the underlying black diagram $U_{\mathfrak{F}}$ of the Ferrers subdiagram $\mathfrak{F}$ might be empty. This happens if, and only if $\mathfrak{F}$ does not contain any circle black point. This happens if either $\mathfrak{F} = \mathfrak{F}_0$ (for every value of n) or $\mathfrak{F} = \mathfrak{F}_{(1)}$ and n is odd.

Definition 4.20. Two Ferrers subdiagrams of $\mathrm{FF}(n)$ are said to be *distance-equivalent* if they have the same underlying black diagram. Analogously, two embedded partitions λ and λ' are said to be *distance-equivalent* if their associated Ferrers subdiagrams $\mathfrak{F}_\lambda$ and $\mathfrak{F}_{\lambda'}$ are.

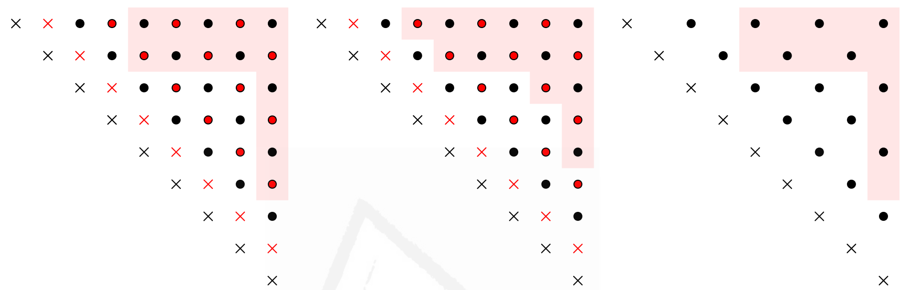

Figure 9.18: Two distance-equivalent Ferrers subdiagrams in FF(8) and their common underlying black diagram.

It is also possible to determine algebraically whether two different subdiagrams in $\mathrm{FF}(n)$ are distance-equivalent. To do so, we make use of the corresponding embedded partitions.

Remark 4.21. Observe that, given $\lambda = (\lambda_1, \ldots, \lambda_m)$ an embedded partition in $\mathrm{FF}(n)$, in order to compute the number of black (or red) dots in the i-th row of the corresponding Ferrers subdiagram $\mathfrak{F}_\lambda$, there are two possibilities depending on the parity of both i and n:

(1) In case i is even, the number of black dots in the i-th row is

$$\left\lfloor \frac{\lambda_i}{2} \right\rfloor \text{ for } n \text{ even, and } \left\lceil \frac{\lambda_i}{2} \right\rceil \text{ for } n \text{ odd.}$$

(2) In case i is odd, the number of black dots in the i-th row is

$$\left\lceil \frac{\lambda_i}{2} \right\rceil \text{ for } \ n \text{ even, and } \left\lfloor \frac{\lambda_i}{2} \right\rfloor \text{ for } \ n \ \text{ odd.}$$

Clearly, the number of red points at each row is just λ_i minus the corresponding number of black points given above.

Definition 4.22. Given $\lambda = (\lambda_1, \ldots, \lambda_m)$ an embedded partition in $\mathrm{FF}(n)$, we define its *underlying distribution* as the vector

$$U_\lambda = \begin{cases} \left(\left\lceil \frac{\lambda_1}{2} \right\rceil, \left\lfloor \frac{\lambda_2}{2} \right\rfloor, \left\lceil \frac{\lambda_3}{2} \right\rceil, \ldots\right) & \text{if } \; n \; \text{ is even,} \\ \left(\left\lfloor \frac{\lambda_1}{2} \right\rfloor, \left\lceil \frac{\lambda_2}{2} \right\rceil, \left\lfloor \frac{\lambda_3}{2} \right\rfloor, \ldots\right) & \text{if } \; n \; \text{ is odd.} \end{cases}$$

Notice that the i-th component of the underlying distribution U_λ represents the number of black points in the i-th row of the underlying black diagram $U_{\mathfrak{F}_\lambda}$. Nevertheless, the underlying distribution U_λ of a partition λ is not necessarily a partition itself as we can see in the following example.

Example 4.23. *In* $\mathrm{FF}(7)$*, for the embedded partition* $\lambda = (6, 3, 2)$*, we have* $U_\lambda = (3, 2, 1)$*, which is, in turn, a partition of* 6*. Nevertheless,* $\lambda' = (6, 5, 2, 1, 1)$ *gives us* $U_{\lambda'} = (3, 2, 1, 0, 1, 0)$*, which is not a partition. On the other side, we can have partitions with distance-equivalent Ferrers subdiagrams but with different underlying distributions. Just look Figure 9.18, where we have* $\lambda = (5, 5, 1, 1, 1, 1)$ *and* $\lambda' = (6, 5, 2, 1, 1)$*, both partitions of* 14*, such that* $U_{\mathfrak{F}_\lambda} = U_{\mathfrak{F}_{\lambda'}}$ *whereas* $U_\lambda = (3, 2, 1, 0, 1, 0) \neq U_{\lambda'} = (3, 2, 1, 0, 1)$.

In the next result we present a criterion to characterize those distance-equivalent embedded partitions also in terms of their (maybe different) associated underlying distributions.

Theorem 4.24. *Let* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *and* $\lambda' = (\lambda'_1, \ldots, \lambda'_{m'})$ *be two embedded partitions in* $\mathrm{FF}(n)$ *and assume* $m \leqslant m'$*. Then* λ *and* λ' *are distance-equivalent if, and only if, we have the following:*

(1) One of these two conditions hold:

(a) $m = m'$.

(b) $m + n$ *is odd,* $m' = m + 1$ *and* $\lambda'_{m'} = 1$.

(2) In any of the cases above,

$$\begin{cases} \left\lceil \frac{\lambda_i}{2} \right\rceil = \left\lceil \frac{\lambda'_i}{2} \right\rceil & \text{if } \; n+i \; \text{ is odd;} \\ \left\lfloor \frac{\lambda_i}{2} \right\rfloor = \left\lfloor \frac{\lambda'_i}{2} \right\rfloor & \text{if } \; n+i \; \text{ is even.} \end{cases}$$

for every $1 \leqslant i \leqslant m$.

Proof. Assume that λ and λ' are distance-equivalent, i.e., their associated Ferrers subdiagrams $\mathfrak{F}_\lambda$ and $\mathfrak{F}_{\lambda'}$ are distance-equivalent. Hence, their underlying black diagrams must coincide, that is, $U_{\mathfrak{F}_\lambda} = U_{\mathfrak{F}_{\lambda'}}$. Since one out of two rows in $\mathrm{FF}(n)$ starts with a black point, the number of rows of distance-equivalent Ferrers subdiagrams can differ by, at most, one unit. Moreover, the extra row can only contain a red point. In terms of the partitions λ and λ', we conclude that $m' \in \{m, m+1\}$. Moreover, in case $m' = m+1$, the first dot (from the right) in the $(m+1)$-th row of $\mathfrak{F}_{\lambda'}$ must be red. That happens if, and only if, $m+n$ is odd and the part $\lambda'_{m'} = 1$. In addition, for the first m rows, the number of black dots in both diagrams have to coincide. Equivalently, for every $1 \leqslant i \leqslant m$, it must satisfy

$$\begin{cases} \left\lceil \frac{\lambda_i}{2} \right\rceil = \left\lceil \frac{\lambda'_i}{2} \right\rceil & \text{if} \quad n+i \quad \text{is odd;} \\ \left\lfloor \frac{\lambda_i}{2} \right\rfloor = \left\lfloor \frac{\lambda'_i}{2} \right\rfloor & \text{if} \quad n+i \quad \text{is even.} \end{cases}$$

The converse holds immediately by using the same arguments. ∎

At this point we are able to establish in a consistent way the necessary link between embedded partitions in $\mathrm{FF}(n)$ and flag codistance values which will permit us to study properties of flag codes in terms of suitable partitions. The next result follows straightforwardly:

Proposition 4.25. *Let Σ be a staircase path with associated flag distance $d = d_{\Gamma(\Sigma)}$ and Ferrers subdiagram $\mathfrak{F}(\Sigma)$. Take $\lambda = (\lambda_1, \dots, \lambda_m)$ an embedded partition such that $\mathfrak{F}(\Sigma) = \mathfrak{F}_\lambda$. Then the codistance $\bar{d} = D^n - d$ coincides with the value u_λ where*

$$u_\lambda = \begin{cases} \left\lceil \frac{\lambda_1}{2} \right\rceil + \left\lfloor \frac{\lambda_2}{2} \right\rfloor + \left\lceil \frac{\lambda_3}{2} \right\rceil + \dots & \text{if} \quad n \quad \text{is even;} \\ \left\lfloor \frac{\lambda_1}{2} \right\rfloor + \left\lceil \frac{\lambda_2}{2} \right\rceil + \left\lfloor \frac{\lambda_3}{2} \right\rfloor + \dots & \text{if} \quad n \quad \text{is odd.} \end{cases} \tag{9.16}$$

Due to the previous proposition, we can introduce a new concept that relates embedded partitions and codistance.

Definition 4.26. Given $\lambda = (\lambda_1, \dots, \lambda_m)$ an embedded partition in $\mathrm{FF}(n)$, we say that its underlying distribution U_λ *splits* the value u_λ defined in (9.16), or that it is an *splitting* of u_λ. This value u_λ is common for $\mathfrak{F}_\lambda$ and all its distance-equivalent Ferrers subdiagrams.

Remark 4.27. By extension, if we are in the conditions of Proposition 4.25, we have that $u_\lambda = \bar{d}$, and we say that U_λ is an *splitting* of the codistance $\bar{d}$. Notice that these splittings are not *codistance vectors*. Given full flags $\mathcal{F}, \mathcal{F}'$ on $\mathbb{F}_q^n$, their codistance vector is the sequence of (subspace) codistances between their subspaces. More precisely, the i-th component of this vector is

$$\min\{i, n-i\} - d_I(\mathcal{F}_i, \mathcal{F}'_i),$$

for every $0 \leqslant i \leqslant n$. Moreover, if $d = d_f(\mathcal{F}, \mathcal{F}')$, then $\bar{d}$ is also obtained as the sum of the previous values. However, the notions of splitting and codistance vector (associated to $\bar{d}$) represent different ideas.

Finally, the next result provides the bridge to translate the information given by distance paths to the embedded partitions level and conversely.

Theorem 4.28. *Let $n \geqslant 2$ be an integer and $0 \leqslant d \leqslant D^n$ a flag distance value. Then there is a bijection between the set of distance paths of distance d in* $\mathrm{S}(n)$ *and the set of splittings of the codistance $\bar{d} = D^n - d$.*

Proof. It suffices to summarize all the results provided along this section. We start from a distance path Γ and consider the flag distance value $d = d_\Gamma$. This value is an invariant of all the distance-equivalent staircase paths in $\Sigma(\Gamma)$ by Corollary 4.17. Take now all the Ferrers subdiagrams $\mathrm{FF}(n)$ whose silhouettes are in $\Sigma(\Gamma)$, i.e., those ones of the form $\mathfrak{F}_\Sigma$ for any $\Sigma \in \Sigma(\Gamma)$. Each $\mathfrak{F}_\Sigma$ is also associated to the corresponding partition $\lambda(\Sigma)$. Notice that all these Ferrers diagrams are distance-equivalent since they have the set of black points that remains at right of Γ as their common underlying black diagram U, which has exactly $\bar{d}$ black points. Hence, all the partitions $\{\lambda(\Sigma) \mid \Sigma \in \Sigma(\Gamma)\}$ are distance-equivalent and their common underlying distribution is, by means of Proposition 4.25, a splitting of the codistance $\bar{d}$.

On the other hand, consider a splitting U_λ of the codistance $\bar{d}$, induced by an embedded partition λ (or any other distance-equivalent partition). The distribution U_λ determines the underlying black diagram $U_{\mathfrak{F}_\lambda}$ of $\mathfrak{F}_\lambda$ (and of all its distance-equivalent Ferrers subdiagrams). The silhouette $\Sigma = \Sigma(\mathfrak{F}_\lambda)$ has a skeleton $\Gamma = \Gamma(\Sigma)$ that is a distance path associated to the distance d. This skeleton Γ is common for all the staircase paths that are silhouettes of subdiagrams distance-equivalent to $\mathfrak{F}_\lambda$. Hence, every partition providing the distribution U_λ leads to the same distance path Γ. ∎

5 Applications and examples

In this section we show how this new dictionary between flag distance values and underlying distributions of certain partitions can be applied to establish connections between the parameters of a given full flag code and the ones of its projected codes. To this end, we start with a lemma that counts the number of circle black dots of a rectangular Ferrers subdiagram.

Lemma 5.1. *Let $\mathfrak{R}$ be a Ferrers subdiagram in* $\mathrm{FF}(n)$ *with rectangular shape. If $\mathfrak{R}$ has a rows and b columns, then the number of circle black dots in $\mathfrak{R}$ is*

$$\begin{cases} \left\lceil \frac{ab}{2} \right\rceil & \text{if } n \text{ is even,} \\ \left\lfloor \frac{ab}{2} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Note that, as $\mathfrak{R}$ has a rows and b columns, it is the Ferrers subdiagram associated to the partition $\lambda = (b, \overset{(a)}{\ldots}, b)$ where $\lambda = ab$. Then, the number of black points in $\mathfrak{R}$ is the value u_λ (see (9.16)), given as the sum of the components of the underlying distribution U_λ. According to Definition 4.22, the expression of U_λ is

$$U_\lambda = \begin{cases} \left(\left\lceil \frac{b}{2} \right\rceil, \left\lfloor \frac{b}{2} \right\rfloor, \left\lceil \frac{b}{2} \right\rceil, \left\lfloor \frac{b}{2} \right\rfloor, \dots \right) & \text{if } \; n \; \text{ is even and} \\ \left(\left\lfloor \frac{b}{2} \right\rfloor, \left\lceil \frac{b}{2} \right\rceil, \left\lfloor \frac{b}{2} \right\rfloor, \left\lceil \frac{b}{2} \right\rceil, \dots \right) & \text{if } \; n \; \text{ is odd.} \end{cases}$$

Hence, for even values of n, we have:

$$u_\lambda = \left\lceil \frac{a}{2} \right\rceil \cdot \left\lceil \frac{b}{2} \right\rceil + \left\lfloor \frac{a}{2} \right\rfloor \cdot \left\lfloor \frac{b}{2} \right\rfloor. \tag{9.17}$$

In general, note that for every positive integer c, we have $c = \left\lfloor \frac{c}{2} \right\rfloor + \left\lceil \frac{c}{2} \right\rceil$. Thus, in case that either a or b is even, it holds $u_\lambda = \frac{ab}{2} = \left\lceil \frac{ab}{2} \right\rceil$. On the other hand, if both a and b are odd, then expression (9.17) becomes

$$u_\lambda = \frac{a+1}{2} \cdot \frac{b+1}{2} + \frac{a-1}{2} \cdot \frac{b-1}{2} = \frac{ab+1}{2} = \left\lceil \frac{ab}{2} \right\rceil$$

and the result is true for n even. If n is odd, the result follows by using the same arguments and taking into account that

$$u_\lambda = \left\lfloor \frac{a}{2} \right\rfloor \cdot \left\lceil \frac{b}{2} \right\rceil + \left\lceil \frac{a}{2} \right\rceil \cdot \left\lfloor \frac{b}{2} \right\rfloor. \tag{9.18}$$

∎

Next we apply this lemma in order to relate the cardinality of a given flag code to the ones of its projected codes, by counting circle black dots in specific rectangles in the Ferrers diagram frame.

Theorem 5.2. *Consider a full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ with codistance $\bar{d}_f(\mathcal{C})$ and take a dimension $1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. If the codistance satisfies*

$$\bar{d}_f(\mathcal{C}) < \left\lceil \frac{i(n-i)}{2} \right\rceil, \tag{9.19}$$

then $|\mathcal{C}| = |\mathcal{C}_i| = \cdots = |\mathcal{C}_{n-i}|$.

Proof. Assume that $|\mathcal{C}| \neq |\mathcal{C}_i|$ and that $\bar{d}_f(\mathcal{C})$ satisfies (9.19). In this case, there exist different flags $\mathcal{F}, \mathcal{F}'$ in $\mathcal{C}$ with $\mathcal{F}_i = \mathcal{F}'_i$. Equivalently, the distance path $\Gamma(\mathcal{F}, \mathcal{F}') \in \Gamma(\mathcal{C})$ passes through the crossed point $(i, 0)$ of S(n), which determines, in turn, a rectangle with i rows and $n-i$ columns over it in the enriched distance support $\hat{S}(n)$. By means of Lemma 5.1, this rectangle contains exactly

$$p = \begin{cases} \left\lceil \frac{i(n-i)}{2} \right\rceil & \text{if } \; n \; \text{ is even,} \\ \left\lfloor \frac{i(n-i)}{2} \right\rfloor & \text{if } \; n \; \text{ is odd,} \end{cases}$$

circle black dots. Moreorver, notice that, if n is odd, then $i(n-i)$ is even and we can simply write

$$p = \left\lceil \frac{i(n-i)}{2} \right\rceil,$$

for every value of n. Notice that, at least all these p circle black points remain over the distance path $\Gamma(\mathcal{F}, \mathcal{F}')$, and then they do not contribute to the computation of $d_f(\mathcal{F}, \mathcal{F}')$. Hence, we have

$$d_f(\mathcal{C}) \leqslant d_f(\mathcal{F}, \mathcal{F}) \leqslant D^n - p.$$

Consequently, we obtain $\bar{d}_f(\mathcal{C}) = D^n - d_f(\mathcal{C}) \geqslant p$, which is a contradiction, and it must hold $|\mathcal{C}| = |\mathcal{C}_i|$. The same arguments, but considering a rectangle with $n-i$ rows and i columns, lead to $|\mathcal{C}| = |\mathcal{C}_{n-i}|$. On the other hand, if we take a dimension $i \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$, and we write $j = i + k$ for some integer $k \geqslant 0$, then it holds

$$j(n-j) = (i+k)(n-i-k) = i(n-i)+k(n-2i-k) \geqslant i(n-i)+k(n-2j) \geqslant i(n-i).$$

Hence, if $\bar{d}_f(\mathcal{C})$ satisfies the stated condition, in particular, we also have

$$\bar{d}_f(\mathcal{C}) < \left\lceil \frac{j(n-j)}{2} \right\rceil$$

and, arguing as above, we get $|\mathcal{C}| = |\mathcal{C}_j| = |\mathcal{C}_{n-j}|$, for every $i \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$. ∎

On the other hand, the connection between distance paths and Ferrers subdiagrams established in the previous section, enables us to associate the following combinatorial objects to a flag code $\mathcal{C}$.

Definition 5.3. Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. The set of *Ferrers subdiagrams of* $\mathcal{C}$ is

$$\mathfrak{F}(\mathcal{C}) = \{\mathfrak{F} \text{ subdiagram of } \mathrm{FF}(n) \mid \Gamma(\Sigma(\mathfrak{F})) \in \Gamma(\mathcal{C})\}.$$

In other words, subdiagrams in $\mathfrak{F}(\mathcal{C})$ are those ones whose silhouettes have a distance path in $\Gamma(\mathcal{C})$ as their distance skeleton.

Associated to this set of Ferrers subdiagrams, we can consider, in turn, their sets of Durfee rectangles as follows.

Definition 5.4. Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. For every $0 \leqslant k \leqslant n-2$, the set of *Durfee k-rectangles of* $\mathcal{C}$ is given by

$$D_k(\mathcal{C}) = \{D_k(\mathfrak{F}) \mid \mathfrak{F} \in \mathfrak{F}(\mathcal{C})\}.$$

In particular, for $k = 0$, we simply write $D(\mathcal{C}) = D_0(\mathcal{C})$ and speak about the set of *Durfee squares of* $\mathcal{C}$.

Notice that, given an embedded partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and an integer $0 \leqslant k \leqslant n-2$, the Durfee k-rectangle $D_k(\mathfrak{F}_\lambda)$ has, at least one row if, and only if, it holds $\lambda_1 \geqslant k+1$. On the other hand, for those Ferrers diagrams $\mathfrak{F}$ having no more than k points in their first row, we put $D_k(\mathfrak{F})$ as the "empty" *Durfee k-rectangle*, which has zero rows (see Figure 9.19). Considering this special case makes the set $D_k(\mathcal{C})$ contain at least one element, for every $0 \leqslant k \leqslant n-2$.

Remark 5.5. Observe that the set $D_k(\mathcal{C})$ can be encoded as a set of integers as follows. Assume that

$$D_k(\mathcal{C}) = \{\mathfrak{R}_1^k, \ldots, \mathfrak{R}_{m_k}^k\},$$

where each $\mathfrak{R}_j^k$ is a Durfee k-rectangle having $0 \leqslant r_j^k \leqslant \lfloor \frac{n-k}{2} \rfloor$ rows, thus $r_j^k \times (r_j^k + k)$ points. Then, to know $D_k(\mathcal{C})$, we just need to store the list of integers $\{r_1^k, \ldots, r_{m_k}^k\}$. Moreover, without loss of generality, we can assume that $r_1^k > r_2^k > \cdots > r_{m_k}^k \geqslant 0$ so that $\mathfrak{R}_1^k$ is the biggest Durfee k-rectangle in $D_k(\mathcal{C})$. As said before, we are also contemplating the possibility of having Durfee k-rectangles with 0 rows. Hence, in any case, the value r_1^k associated to $\mathcal{C}$ always exists.

Let us finally see how can we use Durfee rectangles to relate the parameters of $\mathcal{C}$ to the ones of its projected codes.

5.1 From the flag distance to subspace distances

By construction, a Durfee k-rectangle in $\mathrm{FF}(n)$ having r rows (and then $r+k$ columns) has its left down vertex at the point with coordinates

$$\left(\frac{n-k}{2}, \frac{n-k}{2} - (r-1)\right).$$

in the enriched distance support $\hat{\mathrm{S}}(n)$. This vertex will help us to obtain information about the projected codes of a given flag code whenever it is a point in the distance support $\mathrm{S}(n)$, i.e., if it is a circle black point. Observe also that this happens if, and only if, $\frac{n-k}{2}$ is an integer or, equivalently, when n and k have the same parity. In this case, k-rectangles give information about the projected code of dimension $\frac{n-k}{2}$. The following picture illustrates this fact.

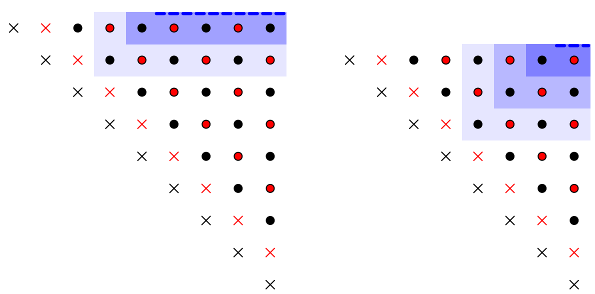

Figure 9.19: Possible Durfee 4-rectangles in FF(8) and 1-rectangles in FF(7), included those ones with zero rows (in dashed lines).

At left, for $n = 8$, we have three Durfee 4-rectangles with $0, 1$ and 2 rows, respectively. Their left down vertices, after rotating back the diagram, are in the column corresponding to the dimension $\frac{8-4}{2} = 2$. At right, different Durfee 1-rectangles on FF(7) are shown. All of them have their left down vertices in the column related to the dimension $\frac{7-1}{2} = 3$. Having this in mind, we start from our full flag code $\mathcal{C}$ on $\mathbb{F}_q^n$ and take $0 \leqslant k \leqslant n-2$ with the same parity than n. We use the set of k-rectangles $D_k(\mathcal{C})$ in order to derive information about the cardinality and minimum distance of the projected code of dimension $i = \frac{n-k}{2}$ with $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Conversely, if we are specifically interested in the projected code $\mathcal{C}_i$, where $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, then we just need to consider the set of $(n-2i)$-rectangles of $\mathcal{C}$.

Theorem 5.6. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$ and consider a dimension $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. They are equivalent:*

(i) There are Durfee $(n-2i)$-rectangles in $D_{n-2i}(\mathcal{C})$ with r rows, where $0 \leqslant r \leqslant i$.

(ii) There exist flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $d_I(\mathcal{F}_i, \mathcal{F}'_i) = i - r$.

Proof. Notice that any $(n-2i)$-Durfee rectangle in the Ferrers diagram frame has its lower left vertex in a circle black dot, corresponding to the dimension i in the distance support. Moreover, the number of rows of the rectangle coincides with the number of dots being simultaneously in the rectangle and in $\mathrm{S}(i, n)$, i.e., in the i-th column of the distance support $\mathrm{S}(n)$ (recall (9.10) and (9.11)).

Hence, a Durfee $(n-2i)$-rectangle in $D_{n-2i}(\mathcal{C})$ with r rows is determined by the existence of Ferrers subdiagrams in $\mathfrak{F}(\mathcal{C})$ whose silhouettes pass through the black circle point $(i, i-r)$. This happens if, and only if, the skeleton of each of these staircase paths is a distance path in $\Gamma(\mathcal{C})$ passing through the vertex $(i, i-r)$ as well. Equivalently, there must exist flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ with $d_I(\mathcal{F}_i, \mathcal{F}'_i) = i - r$. ∎

Now, we focus on the biggest Durfee $(n-2i)$-rectangle of $\mathcal{C}$, denoted in Remark 5.5 as $\mathfrak{R}_1^{n-2i}$, and that contains $0 \leqslant r_1^{n-2i} \leqslant i$ rows. In addition, since we will always work with rectangles in $D_{n-2i}(\mathcal{C})$, we drop the superscripts and simply write $\mathfrak{R}_1$ and r_1, respectively. In light of the previous theorem, we analyze two possible situations concluding the following results.

Theorem 5.7. *Assume that $\mathcal{C}$ is a full flag code on $\mathbb{F}_q^n$. Consider a dimension $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ and the biggest rectangle $\mathfrak{R}_1$ in $D_{n-2i}(\mathcal{C})$. The following statements are equivalent:*

(1) The number of rows of $\mathfrak{R}_1$, that is r_1, satisfies $0 \leqslant r_1 < i$.

(2) It holds $|\mathcal{C}| = |\mathcal{C}_i|$ and $d_I(\mathcal{C}_i) = i - r_1$.

Proof. By application of Theorem 5.6 and the maximality of r_1, we know that the existence of Durfee $(n-2i)$-rectangles with $0 \leqslant r_1 < i$ rows in $D_{n-2i}(\mathcal{C})$ is equivalent to say that, for any choice of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, we have

$$d_I(\mathcal{F}_i, \mathcal{F}'_i) \geqslant i - r_1 > 0$$

and the equality holds for some pair of flags in the code. In other words, $|\mathcal{C}_i| = |\mathcal{C}|$ because a couple of flags in $\mathcal{C}$ never share their i-th subspaces and, clearly, $d_I(\mathcal{C}_i) = i - r_1$. ∎

Now we study the remaining case, in which r_1 attains its maximum possible value, that is, $r_1 = i$. In that situation, we obtain valuable information about the projected code $\mathcal{C}_i$ in terms of the second largest rectangle $\mathfrak{R}_2$ in $D_{n-2i}(\mathcal{C})$, whenever it exists. More precisely:

Theorem 5.8. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. Fix a dimension $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ and assume that the largest rectangle $\mathfrak{R}_1 \in D_{n-2i}(\mathcal{C})$ has $r_1 = i$ rows. Hence, the following statements hold:*

(1) $D_{n-2i}(\mathcal{C}) = \{\mathfrak{R}_1\}$ if, and only if, $|\mathcal{C}_i| = 1$ or, equivalently, $d_I(\mathcal{C}_i) = 0$.

(2) $D_{n-2i}(\mathcal{C}) \neq \{\mathfrak{R}_1\}$ if, and only if, $|\mathcal{C}| > |\mathcal{C}_k| > 1$ and $d_I(\mathcal{C}_k) = k - r_2 > 0$, where $0 \leqslant r_2 < r_1 = i$ is the number of rows of $\mathfrak{R}_2$.

Proof. By means of Theorem 5.6, the condition $r_1 = i$ is equivalent to say that $d_I(\mathcal{F}_i, \mathcal{F}'_i) = 0$ for a pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$. This happens if, and only if, $\mathcal{F}_i = \mathcal{F}'_i$, i.e, if $|\mathcal{C}_i| < |\mathcal{C}|$. Let us distinguish two cases:

(1) If $\mathfrak{R}_1$ is the only $(n-2i)$-rectangle of the code, by Theorem 5.6, we have that $d_I(\mathcal{F}_i, \mathcal{F}'_i) = 0$ for every pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$. In other words, all the flags in $\mathcal{C}$ share their common i-th subspace. Equivalently, the projected code $\mathcal{C}_i$ consists of a single subspace and $d_I(\mathcal{C}_i) = 0$.

(2) On the other hand, if $D_{n-2i}(\mathcal{C}) \neq \{\mathfrak{R}_1\}$, we can consider the second largest $(n-2i)$-rectangle $\mathfrak{R}_2$ of $\mathcal{C}$, which has $0 \leqslant r_2 < r_1 = i$ rows. Hence, for every pair of different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ it holds either $d_I(\mathcal{F}_i, \mathcal{F}'_i) = 0$ or $d_I(\mathcal{F}_i, \mathcal{F}'_i) \geqslant i - r_2$, and the last inequality becomes an equality for some choice of flags in $\mathcal{C}$ by means of Theorem 5.6. Equivalently, $d_I(\mathcal{C}_i) = i - r_2 > i - r_1 = 0$.

∎

Remark 5.9. Observe that Theorems 5.7 and 5.8 give us information about the parameters of all the projected codes of dimensions up to $\lfloor \frac{n}{2} \rfloor$. For the remaining dimensions, it suffices to reason in the same way by simply considering maximal k-rectangles with c columns and $c + k$ rows. Moreover, it is important to point out that we do not even need to know all the rectangles in $D_{n-2i}(\mathcal{C})$; it suffices to know the largest one and, in the worst case, also the second one.

From the results stated along this subsection, and applying Lemma 5.1, we derive bounds for the minimum distance of the projected codes of $\mathcal{C}$ in terms of its codistance $\bar{d}_f(\mathcal{C})$.

Corollary 5.10. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$ with associated codistance $\bar{d}_f(\mathcal{C})$. Take a dimension $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ and consider an integer $0 \leqslant r \leqslant i$. Hence, whenever*

$$\bar{d}_f(\mathcal{C}) < \left\lceil \frac{r(r+n-2i)}{2} \right\rceil, \tag{9.20}$$

then $|\mathcal{C}_i| = |\mathcal{C}|$ and $d_I(\mathcal{C}_i) > i - r$.

Proof. It suffices to see that, by means of Lemma 5.1, the number of circle black points in a rectangle with r rows and $r+n-2i$ columns is exactly

$$\begin{cases} \left\lceil \frac{r(r+n-2i)}{2} \right\rceil & \text{if } n \text{ is even,} \\ \left\lfloor \frac{r(r+n-2i)}{2} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Morover, notice that, for odd values of n, the product $r(r+n-2i)$ is always even and then

$$\left\lfloor \frac{r(r+n-2i)}{2} \right\rfloor = \frac{r(r+n-2i)}{2} = \left\lceil \frac{r(r+n-2i)}{2} \right\rceil.$$

Hence, under condition (9.20), the number of rows of any Durfee $(n-2i)$-rectangle in $D_{n-2i}(\mathcal{C})$ is upper bounded by $r_1 < r \leqslant i$ and we conclude, by means of Theorem 5.7, that $|\mathcal{C}_i| = |\mathcal{C}|$ and $d_I(\mathcal{C}_i) = i - r_1 > i - r$. ∎

5.2 From subspace distances to the flag distance

In this subsection we deal with the converse problem: given a full flag code on $\mathbb{F}_q^n$, we obtain information of its parameters from the ones of its projected codes. As before we restrict our study to dimensions $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ since, for higher dimensions, it suffices to consider rectangles with more rows than columns.

Theorem 5.11. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$ and take $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.*

(1) If $|\mathcal{C}_i| = |\mathcal{C}|$, then

$$d_I(\mathcal{C}_i)^2 \leqslant d_f(\mathcal{C}) \leqslant D^n - \left\lceil \frac{(i-d_I(\mathcal{C}_i))(n-i-d_I(\mathcal{C}_i))}{2} \right\rceil.$$

(2) If $|\mathcal{C}_i| < |\mathcal{C}|$, then

$$0 \leqslant d_f(\mathcal{C}) \leqslant D^n - \left\lceil \frac{i(n-i)}{2} \right\rceil.$$

Proof. Let us start assuming that $|\mathcal{C}_i| = |\mathcal{C}|$. If we write $d_I(\mathcal{C}_i) = i - r$ for some $0 \leqslant r \leqslant i$, then Corollary 5.10 leads to

$$\bar{d}_f(\mathcal{C}) \geqslant \left\lceil \frac{r(r+n-2i)}{2} \right\rceil = \left\lceil \frac{(i - d_I(\mathcal{C}_i))(n - i - d_I(\mathcal{C}_i))}{2} \right\rceil.$$

As a consequence, it holds $d_f(\mathcal{C}) = D^n - \bar{d}_f(\mathcal{C}) \leqslant D^n - \left\lceil \frac{(i-d_I(\mathcal{C}_i))(n-i-d_I(\mathcal{C}_i))}{2} \right\rceil$. On the other hand, given arbitrary different flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, we have that $\mathcal{F}_i \neq \mathcal{F}'_i$ and then $d_I(\mathcal{F}_i, \mathcal{F}'_i) \geqslant d_I(\mathcal{C}_i)$. Hence, every distance path in $\Gamma(\mathcal{C})$ passes either through the vertex $(i, d_I(\mathcal{C}_i))$ (green vertex in the next figure) or above it. As a result, such a vertex determines a set of circle black dots that always remain under distance paths in $\Gamma(\mathcal{C})$. This set is exactly the triangle in red in the picture below, which top vertex is the point $(i, d_I(\mathcal{C}_i))$ and edges with slope 1 (left) and -1 (right).

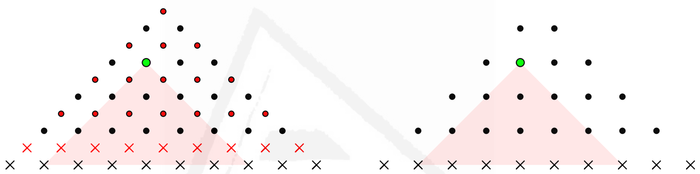

Figure 9.20: Vertex $(i, d_I(\mathcal{C}_i))$ (in green) and the triangle "under" it (in red).

The number of circle black points in such a triangle coincides with the number of circle dots in the distance support $\mathrm{S}(2d_I(\mathcal{C}_i))$, which is $d_I(\mathcal{C}_i)^2$. Hence, Remark 3.14 leads to $d_f(\mathcal{C}) \geqslant d_I(\mathcal{C}_i)^2$. Now, if we suppose that $|\mathcal{C}_i| < |\mathcal{C}|$, the result follows straightforwardly by Theorem 5.2. ∎

5.3 Some examples

We finish this section by applying the previous results to three representative situations.

Example 5.12. *For $n = 8$, the full flag distance takes values in the interval $[0, 16]$. Consider an arbitrary full flag code $\mathcal{C}$ with minimum distance $d_f(\mathcal{C}) = 12$ or, equivalently, codistance $\bar{d}_f(\mathcal{C}) = 16 - 12 = 4$. For this value of the codistance, and by application of Theorem 5.2, we can derive that $|\mathcal{C}| = |\mathcal{C}_2| = \cdots = |\mathcal{C}_6|$, since $\bar{d}_f(\mathcal{C}) = 4 < \frac{2(8-2)}{2} = 6$. Concerning the projected codes distances, for dimension 4, we look at the number of points in Durfee squares (0-rectangles). Since a Ferrers subdiagram with a Durfee square with 3 rows contains, at least $\lceil 9/2 \rceil = 5 > 4$ circle black points, by means of Corollary 5.10, we can ensure that $d_I(\mathcal{C}_4) \geqslant 2$.*

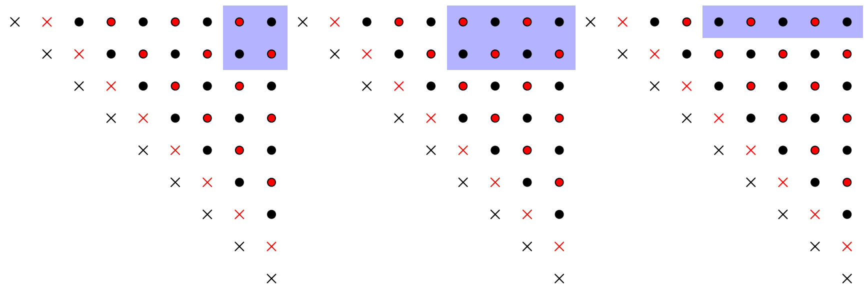

Figure 9.21: Largest k-rectangles with at most 4 black points for $k = 0, 2, 4$.

Similarly, for dimensions 2 *and* 3*, we consider* 4*-rectangles and* 2*-rectangles, respectively and obtain* $d_I(\mathcal{C}_2) \geqslant 1$ *and* $d_I(\mathcal{C}_3) \geqslant 1$*. These properties are transferred to the projected codes of dimensions* 5 *and* 6 *by symmetry of the Ferrers diagram frame. On the other hand, the knowledge of just the code codistance does not give any information about the projected codes* $\mathcal{C}_1$ *and* $\mathcal{C}_7$*. To extract more information in these last cases it is also necessary to know if* $D_6(\mathcal{C})$ *contains either one or two elements. There are three possible situations:*

$$D_6(\mathcal{C}) = \begin{cases} \{\mathfrak{R}_1\} & \textit{and} \quad r_1 = 0, \\ \{\mathfrak{R}_1\} & \textit{and} \quad r_1 = 1, \\ \{\mathfrak{R}_1, \mathfrak{R}_2\} & \textit{and} \quad r_1 = 1 > r_2 = 0. \end{cases}$$

In the first case, by means of Theorem 5.7, we know that $|\mathcal{C}_1| = |\mathcal{C}|$ *and* $d_I(\mathcal{C}_1) = 1$*. In the remaining cases, we have* $|\mathcal{C}_1| < |\mathcal{C}|$*. Moreover, Theorem 5.8 leads to* $d_I(\mathcal{C}_1) = 0$ *in the second situation and to* $d_I(\mathcal{C}_1) = 1$ *in the third one.*

Example 5.13. *Also for* $n = 8$*, now we consider a full flag code* $\mathcal{C}'$ *and assume that the projected code* $\mathcal{C}'_4$ *satisfies* $|\mathcal{C}'_4| = |\mathcal{C}'|$ *and* $d_I(\mathcal{C}'_4) = 2$*. In this case, by application of Theorem 5.11, we know that* $4 \leqslant d_f(\mathcal{C}') \leqslant 14$*. Moreover, if, in addition,* $|\mathcal{C}'_3| = |\mathcal{C}|$ *and* $d_I(\mathcal{C}'_3) = 1$*, then Theorem 5.11 leads also to* $1 \leqslant d_f(\mathcal{C}') \leqslant 12$ *and we conclude that* $4 \leqslant d_f(\mathcal{C}') \leqslant 12$*.*

In general, the more conditions on the projected codes, the more information on the flag code. However, at times we obtain redundant information. For instance, if we require the projected code $\mathcal{C}'_2$ *to fulfill* $|\mathcal{C}'_2| = |\mathcal{C}'|$ *and* $d_I(\mathcal{C}'_2) = 1$*, we obtain* $1 \leqslant d_f(\mathcal{C}') \leqslant 13$*, which does not provide any new data. Nevertheless, if* $|\mathcal{C}'_2| < |\mathcal{C}'|$*, we improve our knowledge about* $d_f(\mathcal{C}')$ *since, by virtue of Theorem 5.11, it must hold* $4 \leqslant d_f(\mathcal{C}') \leqslant 10$*.*

Remark 5.14. As pointed out in Remark 2.13, determining the exact value of $d_f(\mathcal{C}')$ from just the list of distances $\{d_I(\mathcal{C}_i)\}_{i=1}^{7}$, and conversely, is not always possible. Nevertheless, we have seen that just with the data of at most two Durfee $(n-2i)$-rectangles for any dimension i, which is considerably less than knowing the whole set of distance paths of $\mathcal{C}$, we are able to establish interesting connections between flag codes and their projected codes.

Example 5.15. *We finish this subsection by looking at the special case of full flag codes with the maximum possible distance. These codes were characterized in [5, Th. 3.11] in terms of their projected codes, which must attain the maximum possible distance for their dimensions and have the same cardinality than the flag code. Here below, we translate that result to the setting introduced in this work to state an alternative combinatorial characterization of optimum distance full flag codes.*

Theorem 5.16. *Let $\mathcal{C}$ be a full flag code on $\mathbb{F}_q^n$. They are equivalent:*

(1) $d_f(\mathcal{C}) = D^n$ (or $\bar{d}_f(\mathcal{C}) = 0$).

(2) The set $\Gamma(\mathcal{C})$ consists of the only distance path passing either through the point $(\frac{n}{2}, \frac{n}{2})$, if n is even, or through the points $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)$ and $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$, if n is odd.

(3) The set of Ferrers subdiagrams associated to $\mathcal{C}$ is

$$\mathfrak{F}(\mathcal{C}) = \begin{cases} \{\mathfrak{F}_0\} & \text{if } n \text{ is even or} \\ \{\mathfrak{F}_0, \mathfrak{F}_{(1)}\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Observe that, by means of Corollary 3.18, the condition $\bar{d}_f(\mathcal{C}) = 0$ holds if, and only if, distance paths in $\Gamma(\mathcal{C})$ leave no point above them. In other words, $\mathcal{C}$ has a unique distance path: the one that passes trough the points of $S(n)$ having coordinates $(i, \min\{i, n-i\})$, for all $0 \leqslant i \leqslant n$ (see the picture below). By means of the trident rules (9.12) and (9.13), this is exactly the only distance path passing through $(\frac{n}{2}, \frac{n}{2})$, if n is even, or through $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)$ and $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$, if n is odd.

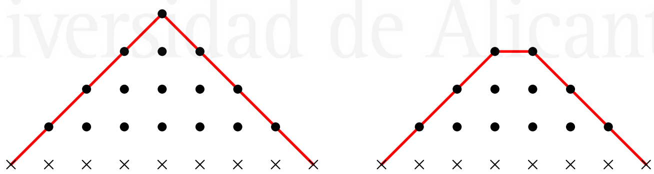

Figure 9.22: Maximum distance paths with n even (left) and n odd (right).

As a result, statements (1) and (2) are equivalent. Moreover, assuming (2), and according to Definition 5.3, we conclude that only Ferrers diagrams $\mathfrak{F}_0$ and $\mathfrak{F}_{(1)}$ (if n is odd) appear in $\mathfrak{F}(\mathcal{C})$. On the other hand, if condition (3) holds, the underlying black diagrams associated to the flag code $\mathcal{C}$ are empty and then, the associated value of the codistance $\bar{d}_f(\mathcal{C})$ is zero, which finishes the proof. ∎

Remark 5.17. The result given in [5, Th. 3.11] can also be proved in our new combinatorial terms. Observe that condition (3) in the previous theorem is equivalent to say that, for every $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, the set $D_{n-2i}(\mathcal{C})$ contains just one element: the Durfee $(n-2i)$-rectangle with zero rows. Hence, by means of Theorem 5.7, we conclude that every projected code of dimension up to $\lfloor \frac{n}{2} \rfloor$ attains the maximum possible distance, i.e., $d_I(\mathcal{C}_i) = i$, and has size $|\mathcal{C}_i| = |\mathcal{C}|$. For those projected codes of higher dimensions, as stated in Remark 5.9, it suffices to consider rectangles with more columns than rows.

6 Conclusions and future work

In this paper, we have undertaken a detailed study of the flag distance in terms of different combinatorial objects. To this end, we have first devised a nice way to graphically represent this numerical parameter through distance paths drawn in a distance support whose particular shape has led us to design an associated Ferrers diagram frame. Hence, we have established a one-to-one correspondence between the set of distance paths (associated to a precise distance value d) and the set of underlying distributions of Ferrers subdiagrams having exactly $\bar{d}$ (the corresponding codistance value) black points. This fact allows us to perfectly translate properties related to the flag distance into the language of integer partitions. Moreover, we take advantage of this dictionary to suitably associate a family of Ferrers subdiagrams (and their corresponding Durfee rectangles) to a given full flag code. Finally, we show how these objects result very useful to make connections between the parameters of the flag code (minimum distance and cardinality) and the ones of the corresponding projected codes.

In future research, we want apply the dictionary established in this paper to derive new results concerning specific families of full flag codes. Moreover, we would also like to adapt the ideas in the current work to the flag variety of general type vector, where the distance support loses its triangular shape.

BIBLIOGRAPHY

[1] R. Ahlswede, N. Cai, R. Li and R. W. Yeung, *Network Informatin Flow*, IEEE Transactions on Information Theory, Vol. 46 (2000), 1204-1216.

[2] C. Alonso-González and M. A. Navarro-Pérez, *Consistent Flag Codes*, Mathematics, Vol. 8(12) (2020), 2243.

[3] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An orbital construction of Optimum Distance Flag Codes*, Finite Fields and Their Applications, Vol. 73 (2021), 101861.

[4] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes: Distance Vectors and Cardinality Bounds*, https://arxiv.org/abs/2111.00910 (preprint).

[5] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes from Planar Spreads in Network Coding*, Finite Fields and Their Applications, Vol. 68 (2020), 101745.

[6] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.

[7] G. E. Andrews, *Generalizations of the Durfee Square*, Journal of the London Mathematical Society, Vol. 3(2) (1971), 563-570.

[8] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, (1984).

[9] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, NY, (1990).

[10] B. Gordon and L. Houten, *Notes on Plane Partitions II*, Journal of Combinatorial Theory, Vol. 4(1) (1968), 81-89.

[11] G. H. Hardy and R. Ramanujan, *Asymptotic Formulae in Combinatory Analysis*, Proceedings London Mathematical Society, Vol. 17(2) (1918), 75-115.

[12] R. Koetter and F. Kschischang, *Coding for Errors and Erasures in Random Network Coding*, IEEE Transactions on Information Theory, Vol. 54 (2008), 3579-3591.

[13] S. Kurz, *Bounds for Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2759–2785.

[14] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Network Coding with Flags*, Designs, Codes and Cryptography, Vol. 86(2) (2018), 269-284.

[15] M. A. Navarro-Pérez and Soler-Escrivà, *Flag Codes of Maximum Distance and Constructions using Singer Groups*, https://arxiv.org/abs/2109.00270 (preprint).

[16] A.-L. Trautmann and J. Rosenthal, *Constructions of Constant Dimension Codes*, in: M. Greferath *et al.* (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 25-42.

Conclusiones

(English version on page 231)

Para la elaboración de esta tesis, nos hemos sumergido en la teoría de códigos *flag* en codificación de red. A lo largo de nuestro camino, hemos abordado distintos problemas desde diferentes puntos de vista, utilizando herramientas provenientes de distintas áreas de las Matemáticas, como el Álgebra Lineal, la Geometría Finita, la Teoría de Grafos, la Teoría de Grupos o la Combinatoria. Estos enfoques han dado lugar a los distintos trabajos presentados en los Capítulos 1-9 de esta memoria. En la mayoría de ellos, hemos puesto el acento en la relación entre los códigos flag y sus códigos proyectados. Este hecho nos ha permitido obtener información muy valiosa acerca de los códigos *flag* en términos de ciertos códigos de dimension constante. A continuación, recapitulamos las contribuciones más relevantes recogidas en esta tesis, presentada como el compendio de los trabajos anteriormente mencionados.

Gran parte de nuestra investigación se centra en el estudio de códigos *flag* de distancia óptima. En el Capítulo 1, presentamos esta familia de códigos y los caracterizamos como aquellos códigos *flag* disjuntos con códigos proyectados de distancia máxima (Capítulo 1, Teorema 3.11). Más tarde, obtenemos un resultado similar, pero que solo involucra, a lo sumo, a dos códigos proyectados (Capítulo 6, Teorema 4.8). De entre todos los códigos *flag* de distancia óptima, probamos que aquellos con un *spread* como código proyectado alcanzan el mejor cardinal para su vector tipo que, por contra, queda restringido por la presencia del *spread*. Para todas las posibles elecciones de los parámetros, hemos dado construcciones sistemáticas de códigos *flag* de distancia óptima con un *spread* como proyectado en los Capítulos 1 y 2. Además, la construcción del primer capítulo va acompañada de un algoritmo de decodificación sobre el canal de borrado que puede ser fácilmente adaptado a la construcción del Capítulo 2, donde empleamos argumentos combinatorios relacionados con problemas clásicos de emparejamientos en grafos.

Inspirados por el estudio de códigos (de subespacio) orbitales, y siguiendo el enfoque orbital urilizado por Liebhold *et al.* para el estudio de códigos *flag*, también nos hemos interesado en aquellos códigos *flag* obtenidos como órbitas bajo la acción (de subgrupos) del grupo general lineal. En los Capítulos 3 y 6, el lector puede encontrar construcciones orbitales de códigos *flag* de distancia óptima con un *spread* entre sus códigos proyectados. Además, en el Capítulo 6 presentamos una construcción de tipo completo con distancia y cardinal óptimos, partiendo de un *spread* parcial. Todas estas construcciones han sido obtenidas utilizando la acción de un subgrupo de Singer adecuado sobre la variedad de *flags*. Por otra parte, y en virtud del isomorfismo $\mathbb{F}_q$-lineal entre el espacio vectorial $\mathbb{F}_q^n$ y el cuerpo $\mathbb{F}_{q^n}$, sabemos que la acción de los grupos de Singer del grupo general lineal puede traducirse como la acción multiplicativa de $\mathbb{F}_{q^n}^*$. Este es el planteamiento adoptado en los Capítulos 4 y 7. En el primero de ellos, introducimos el concepto de código *flag* orbital (β-)cíclico (generado por cierto *flag* $\mathcal{F}$ en el cuerpo $\mathbb{F}_{q^n}$)

y estudiamos propiedades generales de esta familia de códigos *flag* en función del mejor amigo del *flag* generador. También dedicamos parte de este trabajo al estudio de los códigos *flag* de Galois, esto es, aquellos generados por una sucesión de subcuerpos de $\mathbb{F}_{q^n}$ encajados, y probamos que la distancia mínima de estos códigos solo puede tomar unos valores muy concretos. Más aún, damos una correspondencia entre este conjunto de distancias admisibles y los subgrupos de $\mathbb{F}_{q^n}^*$ para los que se alcanzan (Capítulo 4, Teorema 4.14).

En el Capítulo 7, generalizamos este estudio a la familia de códigos orbitales cíclicos generados por *flags* con algún subcuerpo entre sus subespacios (pero no todos ellos) a los que llamamos *flags* de Galois generalizados. En este trabajo, probamos que la presencia de determinados subcuerpos en el *flag* generador descarta ciertos valores de la distancia de los códigos cíclicos que este genera. Sin embargo, nos planteamos si, tal y como ocurre para los códigos *flag* de Galois, todos aquellos valores de la distancia compatibles con la estructura encajada de subcuerpos –o, equivalentemente, con la cadena de *spread* (parciales) entre los códigos proyectados– son realmente alcanzables. En la segunda parte de este capítulo, presentamos una construcción concreta de códigos de Galois generalizados que nos permite responder negativamente a nuestra conjetura.

Volviendo sobre la relación entre los códigos *flag* y sus códigos proyectados, en el Capítulo 5, nos centramos en una familia de códigos *flag* cuyos parámetros quedan perfectamente determinados por los de sus códigos proyectados: los códigos *flag* consistentes, caracterizados como códigos *flag* disjuntos con distancia mínima igual a la suma de las de sus proyectados (Capítulo 5, Teorema 1). Además de esta relación entre los parámetros, probamos que algunas propiedades estructurales, como la de ser equidistante o un girasol, se transfieren perfectamente del código *flag* a sus proyectados y viceversa. Por último, explotamos la condición de consistencia para elaborar un algoritmo de decodificación sobre el canal de borrado.

Otra peculiaridad que observamos al trabajar con códigos *flag* consistentes es que la distancia mínima se obtiene siempre utilizando la misma combinación de distancias de subespacio: las distancias de sus proyectados. Sin embargo, en general, cada valor de la distancia de *flags* puede alcanzarse de varias formas. Es por ello que la distancia mínima de un código *flag* no siempre proporciona suficiente información. Por esta razón, en los Capítulos 8 y 9 profundizamos en el estudio de la distancia de *flags*. En el primero de ellos, utilizamos un punto de vista algebraico e introducimos el concepto de vector distancia. Caracterizamos este nuevo objeto y determinamos una serie de valores señalados de la distancia para la variedad de *flags* de cualquier tipo. Al comparar la distancia mínima de un código *flag* con dichas distancias, obtenemos información sobre el máximo número de subespacios que pueden compartir dos *flags* distintos del código. Razonando de esta forma, acotamos el cardinal de cualquier código *flag* con una distancia prefijada, para cualquier elección del vector tipo.

Por otra parte, en el Capítulo 9 utilizamos un enfoque combinatorio en el que relacionamos la distancia entre *flags* completos con elementos relacionados con particiones de enteros y diagramas de Ferrers. Más concretamente, establecemos una biyección entre los caminos de distancia y el conjunto de distribuciones subyacentes de ciertas particiones que asociamos a la variedad de *flags* completos. Este prisma nos permite exhibir la relación entre los parámetros de un código *flag* completo y los de sus proyectados, a través del conteo de puntos en subdiagramas de Ferrers adecuados.

Part IV
Conclusions

For the preparation of this thesis, we have dived deep into the theory of flag codes in network coding. In our way, we have undertaken diverse problems related to this class of codes and we have addressed them by using concepts coming from distinct areas in Mathematics, such as Linear Algebra, Finite Geometry, Graph Theory, Group Theory or Combinatorics. These different approaches have led to the list of works provided in Chapters 1-9. Throughout the vast majority of them, the accent has been put on the connection of flag codes with their corresponding projected codes. This fact has allowed us to obtain valuable information about flag codes in terms of a list of constant dimension codes. Here below we summarize the most relevant contributions in this thesis, presented as the compendium of the mentioned works.

Big part of our investigation is devoted to the study of optimum distance flag codes. In Chapter 1, we introduced this class of flag codes and characterized them as disjoint flag codes with projected codes with maximum distance (Chapter 1, Theorem 3.11). A similar result, but just involving, at most, two projected codes can be found in Chapter 6 (Theorem 4.8). Among flag codes with maximum distance, we have shown that those having a spread as a projected code attain the maximum possible size for their type vector that, on the negative side, must satisfy a restriction. For all the admissible choices of the parameters, we have provided systematic constructions of optimum distance flag codes with a spread as a projected code in Chapters 1 and 2. In addition, the construction in Chapter 1 is accompanied by a decoding algorithm over the erasure channel that can be easily adapted to the construction given in Chapter 2, based on combinatorial arguments related to matching problems on graphs.

Inspired by the class of orbit (subspace) codes, and following the orbital approach for flag codes introduced by Liebhold *et al.*, we have also studied those flag codes arising as orbits of subgroups of the general linear group. Using this viewpoint, in Chapters 3 and 6, we have also tackled the challenge of obtaining optimum distance flag codes with a spread among the projected codes as orbits of certain groups. Moreover, in Chapter 6, we have come up with a construction of full flag codes with both maximum distance and optimal size, starting from a partial spread. All these orbit constructions have been achieved by using the action of suitable Singer subgroups on the corresponding flag variety. On the other hand, and by virtue of the $\mathbb{F}_q$-linear isomorphism between the vector space $\mathbb{F}_q^n$ and the extension field $\mathbb{F}_{q^n}^*$, the action of Singer subgroups of the general linear group can be appropriately translated into the multiplicative action of $\mathbb{F}_{q^n}^*$ on flags. This is the approach used in Chapters 4 and 7. In the first one, we introduce the notion of (β-)cyclic orbit flag code generated by certain flag $\mathcal{F}$ on the subfield $\mathbb{F}_{q^n}$ and study general properties of this new family of flag codes in terms of the best friend of the flag. Moreover, we dedicate part of the paper to the study of Galois flag codes, i.e., those generated by sequences of nested subfields of $\mathbb{F}_{q^n}$. For these codes, we show that only a limited set of distances can be attained and, in addition, we establish a correspondence between them and the set of subgroups of $\mathbb{F}_{q^n}^*$ (Chapter 4, Theorem 4.14). In Chapter 7, we study those cyclic orbit flag codes generated by flags containing subfields among their subspaces (but not only subfields). We call them generalized Galois flags and show that the presence of certain subfields in the generating flag (its underlying Galois subflag) is not compatible with many values of the distance, which are automatically discarded of our study. However, we wonder if those values of the distance that are compatible with having certain subfields among the subspaces of the generating flag –or, equivalently, with having a prescribed list of (partial) spreads among the projected codes– can actually be obtained as the minimum distance of a generalized Galois flag code. We use a specific construction of generalized Galois flag code in order to answer negatively to our conjecture.

Concerning the link between a flag code and its projected codes, in Chapter 5, we focus on a special family of flag codes in which the parameters of the flag code are completely determined by the ones of its projected codes: the class of consistent flag codes. More precisely, we characterize them as disjoint flag codes with minimum distance equal to the sum of the ones of their projected codes (Chapter 5, Theorem 1). Apart from this nice relations of the parameters, we see that when we consider consistent flag codes, some properties (such as being equidistant or a sunflower) are perfectly transferred from the flag code to its projected codes and vice versa. In addition, the consistency condition is also exploited in order to obtain a decoding algorithm over the erasure channel.

Another particularity of consistent flag codes is that the minimum distance of any is always obtained as the sum of the minimum distances of the corresponding projected codes. However, in general, a given value of the flag distance (which is defined as a sum of subspace distances) might be obtained by many different combinations of subspace distances. Due to this fact, the minimum distance of a flag code does not always give enough information. For this reason, in Chapters 8 and 9 we deepen the study on the flag distance parameter. In the first one, we use an algebraic approach and introduce the notion of distance vector. We characterize distance vectors and use them to determine a list of remarkable values of the distance for the corresponding flag variety. Comparing the minimum distance of a flag code with the ones in the mentioned list gives us information about the maximum number of common subspaces that different flags in the code can have. This study entails new bounds for the cardinality of flag codes with prescribed distance, for every type vector. On the other hand, in Chapter 9 we address the study on the flag distance by using combinatorial elements related to partitions of integers and Ferrers diagrams. More precisely, we work with full flags and establish a one-to-one correspondence between distance paths (the graphic representation of distance vectors) and the set of underlying distributions of a list of partitions associated to the full flag variety. This research enables us to obtain information about the parameters of a given full flag code from the ones of its projected codes and vice versa, by counting points in a suitable Ferrers diagram.

Here below, we attach the general list of references of the thesis.

BIBLIOGRAPHY

[1] R. Ahlswede, N. Cai, R. Li and R. W. Yeung, *Network Information Flow*, IEEE Transactions on Information Theory, Vol. 46 (2000), 1204-1216.

[2] C. Alonso-González and M. A. Navarro-Pérez, *A Combinatorial Approach to Flag Codes*, https://arxiv.org/abs/2111.15388 (preprint).

[3] C. Alonso-González and M. A. Navarro-Pérez, *Consistent Flag Codes*, Mathematics, Vol. 8(12) (2020), 2234.

[4] C. Alonso-González and M. A. Navarro-Pérez, *Cyclic Orbit Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2331-2356.

[5] C. Alonso-González and M. A. Navarro-Pérez, *On Generalized Galois Cyclic Orbit Flag Codes*, https://arxiv.org/abs/2111.09615 (preprint).

[6] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *An Orbital Construction of Optimum Distance Flag Codes*, Finite Fields and their Applications, Vol. 73 (2021), 101861.

[7] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag codes: Distance Vectors and Cardinality Bounds*, https://arxiv.org/abs/2111.00910 (preprint).

[8] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes from Planar Spreads in Network Coding*, Finite Fields and their Applications, Vol. 68 (2020), 101745.

[9] C. Alonso-González, M. A. Navarro-Pérez and X. Soler-Escrivà, *Optimum Distance Flag Codes from Spreads via Perfect Matchings in Graphs*, Journal of Algebraic Combinatorics (2021), https://doi.org/10.1007/s10801-021-01086-y.

[10] J. L. Alperin and R. B. Bell, *Groups and Representations.* Graduate Texts in Mathematics, Vol. 162. Springer, New York, 1995.

[11] G. E. Andrews, *Generalizations of the Durfee Square*, Journal of the London Mathematical Society, Vol. 3(2) (1971), 563-570.

[12] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, (1984).

[13] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, NY, (1990).

[14] E. Ben-Sasson, T. Etzion, A. Gabizon, and N. Raviv, *Subspace Polynomials and Cyclic Subspace Codes*, IEEE Transactions on Information Theory, Vol. 62 (2016), 1157–1165.

[15] Th. Beth, D. Jungnickel and H. Lenz, *Desing Theory*, Cambridge University Press, Cambridge, (1999).

[16] B. Chen and H. Liu, *Constructions of Cyclic Constant Dimension Codes*, Designs, Codes and Cryptography, Vol. 86(6) (2018), 1267–1279.

[17] R. Diestel, *Graph Theory*, Third Edition, Springer-Verlag Heidelberg, New York, 2005.

[18] K. Drudge, *On the Orbits of Singer Groups and Their Subgroups*, The Electronic Journal of Combinatorics, Vol. 9 (2002), Research Paper 15.

[19] J. Edmonds, *Paths, Trees, and Flowers*, Canadian Journal of Mathematics, Vol. 17 (1965), 449-467.

[20] E. Egerváry, *On Combinatorial Properties of Matrices*, Matematikai és Fizikai Lapok, Vol. 38 (1931), 16-28.

[21] T. Etzion and N. Raviv. *Equidistant Codes in the Grassmannian,* Discrete Applied Mathematics, Vol. 186 (2015), 87-97.

[22] T. Etzion and A. Vardy, *Error-Correcting Codes in Projective Spaces*, IEEE Trans. Inf. Theory, Vol. 57 (2011), 1165–1173.

[23] G. Fourier and G. Nebe *Degenerate Flag Varieties in Network Coding*, to appear in Advances in Mathematics of Communications.

[24] H. Gluesing-Luerssen and H. Lehmann, *Distance Distributions of Cyclic Orbit Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 447–470.

[25] H. Gluesing-Luerssen, K. Morrison and C. Troha, *Cyclic Orbit Codes and Stabilizer Subfields*, Advances in Mathematics of Communications, Vol. 9(2) (2015), 177-197.

[26] A. Goel, M. Kapralov and S. Khanna, *Perfect Matchings in* $O(n \log n)$ *Time in Regular Bipartite Graphs*, SIAM Journal on Computing, Vol. 42(3) (2013), 1392-1404.

[27] B. Gordon and L. Houten, *Notes on Plane Partitions II*, Journal of Combinatorial Theory, Vol. 4(1) (1968), 81-89.

[28] E. Gorla, F. Manganiello and J. Rosenthal, *An Algebraic Approach for Decoding Spread Codes*, Advances in Mathematics of Communications, Vol. 6 (4) (2012), 443-466.

[29] E. Gorla and A. Ravagnani. *Equidistant Subspace Codes*, Linear Algebra and its Applications, Vol. 490 (2016), 48-65.

[30] E. Gorla and A. Ravagnani, *Partial Spreads in Random Network Coding*, Finite Fields and Their Applications, Vol. 26 (2014), 104-115.

[31] M. Greferath, M. O. Pavčević, N. Silberstein and M. A. Vazquez-Castro, *Network Coding and Subspace Designs*, Springer, 2018.

[32] M. Hall Jr., *An Algorithm for Distinct Representatives,* The American Mathematical Monthly, Vol. 63 (10) (1956), 716-717.

[33] P. Hall, *On Representatives of Subsets*, Journal of the London Mathematical Society, Vol. 10 (1935), 26-30.

[34] G. H. Hardy and R. Ramanujan, *Asymptotic Formulae in Combinatory Analysis*, Proceedings London Mathematical Society, Vol. 17(2) (1918), 75-115.

[35] D. Heinlein, M. Kiermaier, S. Kurz and A. Wassermann *Tables of Subspace Codes*, https://arxiv.org/abs/1601.02864.

[36] M. D. Hestenes, *Singer Groups*, Canadian Journal of Mathematics, Vol. XXII(3) (1970), 492-513.

[37] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[38] T. Ho, R. Kötter, M. Médard, D. Karger and M. Effros, *The Benefits of Coding over Routing in a Randomized Setting*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), Yokohama, Japan, 2003, p. 442.

[39] T. Ho, M. Médard, R. Kötter, D.R. Karger, M. Effros, J. Shi and B. Leong, *A Random Linear Network Coding Approach to Multicast*, IEEE Transactions on Information Theory, Vol. 52 (2006), 4413-4430.

[40] J. E. Hopcroft and R. M. Karp. *An $n^{5/2}$ Algorithm for Maximum Matchings in Bipartite Graphs*, SIAM Journal on Computing, Vol. 2(4) (1973), 225–231.

[41] A.-L. Horlemann-Trautmann, *Message Encoding and Retrieval for Spread and Cyclic Orbit Codes*, Designs, Codes and Cryptography, Vol. 86 (2018), 365-386.

[42] B. Huppert, *Endliche Gruppen*, Vol. I, Springer-Verlag, New York, 1967.

[43] A. Kohnert and S. Kurz, *Construction of Large Constant Dimension Codes with a Prescribed Minimum Distance*, in: J. Calmet *et al.* (Eds.), Mathematical Methods in Computer Science. Lecture Notes in Computer Science, Vol. 5393. Springer, Berlin, Heidelberg, 2008, pp. 31-42.

[44] D. König, *Graphs and Matrices*, Matematikai és Fizikai Lapok, Vol. 38 (1931), 116-119.

[45] R. Kötter and F. Kschischang, *Coding for Errors and Erasures in Random Network Coding*, in: IEEE Transactions on Information Theory, Vol. 54 (2008), 3579-3591.

[46] H. W. Kuhn, *The Hungarian Method for the Assignment Problem*, Naval Research Logistics Quarterly , Vol. 2 (1955), 83-97.

[47] S. Kurz, *Bounds for Flag Codes*, Designs, Codes and Cryptography, Vol. 89 (2021), 2759–2785.

[48] S. Kurz, *Improved Upper Bounds for Partial Spreads*, Designs, Codes and Cryptography, Vol. 85 (2017), 97–106.

[49] M. Lavrauw, *Scattered Spaces with respect to Spreads and Eggs in Finite Projective Spaces*, PhD Dissertation, Eindhoven University of Technology, Eindhoven, 2001.

[50] M. Lavrauw and G. Van de Voorde, *Field Reduction and Linear Sets in Finite Geometry*, Contemporary Mathematics, Vol. 632 (2015), 271-293.

[51] R. Lidl and H. Niederreiter, *Introduction to Finite Fields and their Applications*, Cambridge University Press, Cambridge, London, 1994. Revised edition.

[52] D. Liebhold, *Flag Codes with Application to Network Coding*, PhD Dissertation, RWTH Aachen University, Aachen, 2019.

[53] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Generalizing Subspace Codes to Flag Codes Using Group Actions*, in: M. Greferath *et al.* (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 67-89.

[54] D. Liebhold, G. Nebe and A. Vazquez-Castro, *Network Coding with Flags*, Designs, Codes and Cryptography, Vol. 86(2) (2018), 269-284.

[55] L. Lovász and M. D. Plummer, *Matching Theory*, Annals of Discrete Mathematics, Vol. 29, North-Holland (1986).

[56] F. Manganiello, E. Gorla and J. Rosenthal, *Spread Codes and Spread Decoding in Network Coding*, in: IEEE International Symposium on Information Theory, Proceedings (ISIT), Toronto, Canada, 2008, pp. 851-855.

[57] F. Manganiello and A.-L. Trautmann, *Spread Decoding in Extension Fields*, Finite Fields and Their Applications, Vol. 25 (2014), 94-105.

[58] F. Manganiello, A.-L. Trautmann and J. Rosenthal, *On Conjugacy Classes of Subgroups of the General Linear Group and Cyclic Orbit Codes*, in: Proceedings of the 2011 IEEE International Symposium on Information Theory (ISIT), Saint Pettersburg, 2011, pp. 1916–1920.

[59] M. A. Navarro-Pérez and X. Soler-Escrivà, *Flag Codes of Maximum Distance and Constructions using Singer Groups*, https://arxiv.org/abs/2109.00270 (preprint).

[60] R. W. Nóbrega and B. F. Uchôa-Filho, *Multishot Codes for Network Coding: Bounds and a Multilevel Construction*, in: 2009 IEEE International Symposium on Information Theory, Proceedings (ISIT), Seoul, South Korea, 2009, pp. 428-432.

[61] R. W. Nóbrega, B. F. Uchôa-Filho, *Multishot Codes for Network Coding Using Rank-Metric Codes*, in: 2010 Third IEEE International Workshop on Wireless Network Coding, Boston, USA, 2010, pp. 1–6.

[62] K. Otal and F. Ozbudak, *Cyclic Subspace Codes via Subspace Polynomials*, Designs, Codes and Cryptography, Vol. 85(2) (2017), 191-204.

[63] J. Rosenthal and A.-L. Trautmann, *A Complete Characterization of Irreducible Cyclic Orbit Codes and their Plücker Embedding*, Designs, Codes and Cryptography, Vol. 66 (2013), 275–289.

[64] R. M. Roth, N. Raviv and I. Tamo, *Construction of Sidon Spaces with Applications to Coding*, IEEE Transactions on Information Theory, Vol. 64(6) (2018), 4412-4422.

[65] B. Segre, *Teoria di Galois, Fibrazioni Proiettive e Geometrie non Desarguesiane*, Annali di Matematica Pura ed Applicata, Vol. 64 (1964), 1-76.

[66] N. Silberstein and T. Etzion, *Codes and Designs Related to Lifted MRD Codes*, Information Theory Proceedings (ISIT) 2011 IEEE International Symposium, pp. 2288-2292, 2011.

[67] N. Silberstein and A.-L. Trautmann, *New Lower Bounds for Constant Dimension Codes*, Information Theory Proceedings (ISIT) 2013 IEEE International Symposium, pp. 514-518, 2013.

[68] A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal, *Cyclic Orbit Codes*, IEEE Transactions on Information Theory, Vol. 59(11) (2013), 7386-7404.

[69] A.-L. Trautmann, F. Manganiello, and J. Rosenthal, *Orbit Codes: A New Concept in the Area of Network Coding*, in: Proceedings of IEEE Information Theory Workshop, Dublin, Ireland, 2010, pp. 1–4.

[70] A.-L. Trautmann and J. Rosenthal, *New Improvements on the Echelon Ferrers Construction*, in: Proc. Int. Symp. on Math. Theory of Networks and Systems, pp. 405–408, July 2010.

[71] A.-L. Trautmann and J. Rosenthal, *Constructions of Constant Dimension Codes*, in: M. Greferath *et al.* (Eds.), Network Coding and Subspace Designs, E-Springer International Publishing AG, 2018, pp. 25-42.

[72] G. Van de Voorde, *Desarguesian Spreads and Field Reduction for Elements of the Semilinear Group*, Linear Algebra and its Applications, Vol. 507 (2016), 96-120.

[73] S.-T. Xia and F.-W. Fu, *Johnson Type Bounds on Constant Dimension Codes*, Designs, Codes and Cryptography, Vol. 50 (2009), 163–172.

[74] W. Zhao and X. Tang. *A Characterization of Cyclic Subspace Codes via Subspace Polynomials*, Finite Fields and Their Applications, Vol. 57 (2019), 1–12.