DIFFERENT MOMENTS IN THE PARTICIPATORY STAGE OF THE SECONDARY SCHOOL STUDENTS' ABSTRACTION OF MATHEMATICAL CONCEPTIONS

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This study reports characteristics of participatory and anticipatory stages in the abstraction of mathematical conceptions. We carried out clinical task-based interviews with 71 secondary school students to obtain evidence of constructed mathematical conceptions and how they were used. We could distinguish both stages in different mathematical conceptions and, furthermore, two cognitive moments in the participatory stage. We argue that (a) the capacity of perceiving regularities in sets of particular cases is characteristic of reflection on activity-effect, and (b) the coordination of information provides the opportunity for changing the attention focus from the particular results to the structure of properties.

Introduction

Understanding how mathematical conceptions are constructed can help in thinking about teaching with the aim of encouraging learning. In this sense it is essential to have accurate descriptions of the processes by which mathematical knowledge is developed. This situation generates issues about what it means to know something about mathematical objects, and how the learner develops or constructs that knowledge (Dörfler, 2002). Cognitive theories based on Piagetian stances assume that mathematical conceptions reflect regularities from human actions and mental operations. In this perspective is generated the question of how to explain the way in which learners cognitively construct their mathematical conceptions. For our purposes and henceforth, "construction" refers to the emergence of a new structure through constructing actions (Monaghan, & Ozmantar, 2006; Simon, Tzur, Heinz, & Kinzel, 2004). Simon and his colleagues (Simon et al., 2004) postulate the existence of a mechanism that they call Reflection on Activity-Effect Relationship to explain this construction process. Taking into account the two phases of reflective abstraction (projection and reflection) described by Piaget (2001), Tzur & Simon (2004) point out that in the projection phase, where the actions become the objects of the reflection, learners sort activity-effect records in terms of an established goal distinguishing between records that get closer to their goal and those that do not. In the reflection phase, where a reorganization of knowledge takes place, learners reflect on the relationship between the activity and its effects.

During the resolution of a problem, the student may call-up a mathematical conception already constructed (anticipatory stage), but in the case in which this conception there isn't, student trigger some actions guided by a goal to obtain information to solve the problem (participatory stage). In this context, we adopt Simon et al.'s (2004) account of a construction process trying to provide empirical support to (i) the distinction between participatory stage and anticipatory stage in the abstraction of mathematical conceptions and (ii) a finer description of how proceeds the participatory stage.

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Methodology

Participants

511 students in the last year of compulsory education (15-16 years old) solved a questionnaire with five mathematical problems in the domains of variability, divisibility and generalization. The analysis of the replies to the problems displayed students' diverse behaviours while solving the problems from the perspective of how they used the different mathematical conceptions. These behaviours may be considered evidence of anticipatory and participatory stages in the construction of mathematical notions involved in the mathematical problems posed. To obtain further information about this phenomenon we conducted 40-minute task-based clinical interviews with 71 of these secondary students. The interviews were focused on how the mathematical conceptions were used during problem solving as a manifestation of the conception constructed. Data come from of audio-records and transcriptions of students' justifications and their written replies to the five problems. Figure 1 shows an example of the problems used.

Job offers for pizza delivery workers have appeared in a local newspaper.
Pizza takeaway A pays each delivery worker 0.6 euros for each pizza delivered and a fixed sum of 60 euros a month. Pizza takeaway B pays 0.9 euros for each pizza delivered and a fixed sum of 24 euros a month.
Which do you think is the better-paid job?
Make a decision and explain why your choice is the better one.

Figure 1. The job offer.

The interviews were carried out after the students completed the questionnaire and the researchers undertook a first analysis of their replies. The aim of the clinical interview was to get the pupils to verbalise their thought-processes used in solving the problems (Goldin, 2000) in order to obtain evidence of how they generated some abstraction processes of mathematical conceptions or used them. The interviewer had a prior interview script constructed considering the characteristics of each problem and the type of answer given by the pupils. In any case, the interviewer could modify her questions in view of the pupil's behaviour, in order to clarify or investigate more deeply the reasoning processes followed.

Data Analysis

The students' responses to the problems and the interviews were analysed from a descriptive point of view using a constant-comparative methodology (Strauss & Corbin, 1994) and taking into account the way in which each pupil set up and used elements of mathematics knowledge as tools in order to interpret the situation and then make a decision (Llinares, & Roig, 2007). Characteristics of the abstraction process generated by the students were identified through the way in which they considered the variability of the quantities, the conditions that had to be fulfilled by these quantities and the way in which discerned generalities from the registers of particular data. We interpreted these characteristics from the process involving students' goal-directed activity and the reflection process (Clement, 2000). Next, we considered the characteristics and the interpretations generated according to the stage distinction from the effect of reflection on activity-effect relationship as a coordination of the available conceptions and identified two moments in the participatory stage with similar characteristics in the different mathematical conceptions taking into account how students created records of experience, sorted and compared the records, and identified patterns in those records.

Results

Table 1 shows the results obtained from the combined analysis of the interviews and the answers of the questionnaire.

Table 1. *Percentages in Different Stages of the Abstraction Process*

	P1 n (%)	P2 n (%)	P3 n (%)	*P4* n (%)	*P5* n (%)	71x5=355 n (%)
Participatory stage	30 (42.2)	23 (32.4)	68 (95.8)	13 (18.3)	55 (77.4)	189 (53.2)
Anticipatory stage	7 (9.9)	1 (1.4)	3 (4.2)	15 (21.1)	12 (16.9)	38 (10.7)
Others	34 (47.9)	47 (66.2)	0 (0)	43 (60.6)	4 (5.7)	128 (36.1)
Total	71 (100)	71 (100)	71 (100)	71 (100)	71 (100)	355 (100)

More than 10% of students had anticipated the mathematical conception in the situation (anticipatory stage). On the other hand, 10.7% of students generated particular cases in order to obtain information about the situation (participatory stage). We identified during the interviews how some students coordinated information from particular cases and generated an answer which reflected a certain degree of generalisation which had not been present in their original written answers. This behaviour indicated a change of focus during interview lending to the generation of an abstraction that fits the reflection on activity-effect relationship mechanism, and revealed the existence of two cognitive moments in the participatory stage. We use some answers to problem 1 to explain these two moments: projection (generating a set of registers) and local anticipation (Reorganization, Identification of Regularities and Acceptance of the Generality).

Projection: Generating a Set of Registers

In nearly 20% of the total of 355 answers, the students created from the situation some type of set of registers, but had difficulty in coordinating the information available. In "The job offer" problem, 5.6% of pupils used particular cases to obtain information that might help in making a decision. A typical example of the procedure employed to create a set of registers was the following:

- For 10 pizzas delivered, Earnings A = 66€ > Earnings B = 33€ → A is better.
- For 20 pizzas delivered, Earnings A = 72€ > Earnings B = 42€ → A is better.
- …

Here the pupils centred their attention exclusively on the information provided by the set of particular cases. This kind of behaviour, using very low numbers of pizzas delivered, or focusing the attention on only some of the account in the situation, prevents the more or less explicit appearance of the existence of a change in the profitability of the offers as the number of pizzas increases. The following protocol shows an example of this kind of procedure.

E19: *What else did you do? In the end, what conclusion did you come to?*

A: *Well, I saw that in pizza takeaway A they pay better because you are guaranteed the 60 euros, so you don't have to worry about delivering one pizza more or one less.*

The consequence of using very low quantities is that in all cases job-offer A is considerably better than job-offer B. Student E19's attention was centred on the six particular cases considered instead of on the information that could have been obtained by comparing the difference in earnings as the number of pizzas delivered increased.

Local Anticipation: Identifying and Using a Regularity

In the course of the interview some of the pupils coordinated the information derived from particular cases in response to prompts from the interviewer which allow them to identify a regularity. Sometimes they made inferences of a general kind from the situation, with no written trace of the activity carried out. On other occasions however the pupils wrote down registers which enabled them to investigate how to compare and relate the particular data, or generated a search for new information. In both cases they were coordinating the information.

An example of this approaching is the way in which E11 perceived during the interview the change of profitability in total earnings, basing the conclusion on a single particular case he had constructed on the written answer paper. On paper, E11 calculated the monthly earnings at each of the pizza takeaways in the case of *"20 pizzas delivered*", concluding that the better-paid job *"is the one at pizza takeaway A because you earn just over twice as much as at B"*. We had considered this kind of answer a manifestation of the Projection moment. During the interview, however, he indicated the following:

E11: *OK. Let's start with the first one. Do you remember what it was about?*

A: *Yes, here it is … you have two job offers, in one it's 6 cents for each pizza, and a fixed amount every month. In the other, the amount for … what they pay for each pizza you deliver, and then the fixed amount every month. And the other, the amount they pay for each pizza delivered is quite high, but the amount they pay every month is lower. I've given an example. I mean, imagine you have to deliver about 20 pizzas a month. So you multiply the 20 pizzas, the pizzas by 6 cents, which is the same as 12 plus 12 and then the 60 euros you get every month, that's 72 altogether. In the other case 20 by 0.9 [by 9 cents] is 18, plus 24, that's 42. So the difference is bigger. So my better offer was A. A was much better.*

E11: *You'd take A, then?*

A: *Yes.*

As the interview continued, the researcher asked him what would happen if a greater number of pizzas were delivered.

E11: *And what do you think would happen if more pizzas were sold?*

A: *Yeah, that's what I was going to tell you, that probably as the number of pizzas increased you would earn more with option B. But with the example I've given you the better offer is A. Maybe with 200 pizzas B is a better offer.*

This reply seems to show that E11 perceives the existence of a change of profitability in the offers as the number of pizzas delivered increases. To find out how he managed to perceive this change, i.e. how the abstraction was produced, the interviewer asked him to explain why he thought it might be possible to earn more in job B.

E11: *Why do you think, then, why do you think you might be able to earn more in job B?*

A: *Because … because for each pizza, eh, you get 3 cents more than at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each* *pizza. I mean, after a lot, that's more, more money […] In the end, in the end … the more pizzas you deliver you get back the difference you've got here.*

In his answer E11 refers to the difference in the money paid by each pizza takeaway for each pizza delivered, saying *"because for each pizza, eh, you get 3 cents more than at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each pizza. I mean, after a lot, that's more, more money"*. He therefore perceives that the difference between the fixed amounts offered by pizza takeaways A and B can be compensated by selling a large number of pizzas. This is possible due to the difference in payment for each pizza delivered, and E11 comes to this conclusion via a qualitative analysis of the data without having to carry out calculations for particular cases. The regularity lies in the fact that the difference between the two offers diminishes as the number of pizzas delivered increases (the earnings in A get closer and closer to those in B) and therefore there comes a point at which B is better than A (there has been a change of tendency in the profitability of the two offers). Another relevant aspect of this procedure is the way in which the identification of the regularity is triggered by the researcher's prompt *"What do you think would happen if more pizzas were sold?"*. From a theoretical viewpoint the question functioned as a prompt which moved the pupil's focus of attention from a single case of what a pizza-deliverer might earn towards a consideration of "how the difference between the two amounts earned might vary" depending on the number of pizzas delivered. We have called this change of attention-focus reflection, which makes it possible to identify the regularity by coordinating certain types of information as a consequence of the interviewer's prompts.

On the other hand, once a regularity (change of profitability) has been identified it can enable the students to look for the exact number of delivered pizzas that equals both offers. In this problem the characteristic of local anticipation lies in the "adjustment" of the decision and is revealed when the pupil considers particular cases approaching 120 (which is the number of pizzas delivered that makes the two offers the same in earnings). In his written answer, E22 drew up a table showing various particular cases and the earnings corresponding to each one for both job-offers. In the interview he explains the process he followed.

A: *Look, in the first one they say there are two pizza takeaways, right? A and B, so in takeaway A they give you 60 euros a month, a fixed sum every month, and in B they give you 24, right? So if they give you more in one than in the other, but in … in the first one they give you 0.6 for every pizza you deliver, and in the second one 0.9, right? So that means that for every 10 pizzas you sell it'll be 0.6 times 10, six euros, you move the decimal point, and here it's 9 euros. So for every 10 pizzas you sell … I mean, look, it's here. From 20 to 40 that's 20, right? Well, you go on adding on, and here it says which will pay you better, right? Well, in the first one as it's 60 euros, in the first one if you don't sell many pizzas the chance is you'll get quite a bit of money, right? I mean it's quite a lot, a lot, a lot of money every month. But not in the second one. But in the second one you take more of a risk because you have to sell more pizzas. In the second one they give you more, less money every month, but they give you more money for every pizza you sell.*

E22: *Yes.*

A: *So when you get to 120 pizzas …*

E22: *What did you do? Did you keep trying it, going up and up, seeing how many deliveries…*
A: *Sure, I went 1, 2, 3, 4, 5, right? I kept on multiplying it.*
E22: *Is that the number of pizzas? [pointing to the first row in the table]*
A: *The number of pizzas sold. 5, right? But I saw it was not enough, so I went on adding more and more.*
E22: *Fine.*
A: *I went on multiplying, and here I wrote an equation, right?*
E22: *Yes.*
A: *Say x is the number of pizzas you sell at 0.60, at 0.60 cents plus the money they give you every month, then you multiply, it might only be two pizzas. Two times 0.60, 1.20 plus 60 euros maybe, and so on.*

The particular cases used are organised in a table beginning with the case of "1 pizza delivered", and increasing by one pizza at a time for the subsequent cases up to the case of "5 pizzas delivered". From 10 pizzas onwards, he uses the relation *"for every 10 pizzas you sell it'll be 0.6 times 10, six euros [Job-offer A], you move the decimal point, and here it's 9 euros [Job-offer B]"*. This regularity is perceived from the comparison between the amounts paid for each pizza delivered. As he states in his written answer:

- *"Every ten pizzas sold in A mean 6€"*
- *"Every ten pizzas sold in B mean 9€"*

The coordination of the information is revealed in the way he looks at the amounts earned for pizzas delivered (going up in tens of pizzas), together with the comparison between the fixed monthly amounts, which lead E22 to realise that job-offer B can be better than job-offer A (i.e. the regularity in the situation seen as a change of tendency). He is searching for the number of pizzas which will make the two offers the same by setting up new registers of particular cases, ten by ten. This "directed" search for the number that will indicate the change of tendency is a manifestation of the coordination of information, in which the particular cases are used as an iterative activity towards a pre-established goal. After calculating the case of 120 pizzas, E22 states that *"If you sell 120 pizzas you earn the same in both places, but if you are going to sell fewer pizzas you should choose A and if you think you will sell more you should choose B"*.

18A: And in the end I went on doing that and with 120 pizzas you earn the same in both. So if 120 pizzas are sold you would earn the same in both. So you could take either. But from 120 onwards you'd earn more in B. So …
19E: So which of the two would you choose?
20A: Personally, I'd take A because it's difficult to sell 120 pizzas. The thing is … but if you want to take a risk and you think you'll sell more, you'd take B.

E22 therefore discerns the change of tendency which occurs as the number of pizzas delivered increases, and is able to use it to discover at what number of pizzas the two job-offers pay the same. At the end of the interview he states that "Personally, I'd take A because it's difficult to sell 120 pizzas. The thing is…but if you want to take a risk and you think you'll sell more, you'd take B" (line 20). The perception of the change of tendency and the use of this insight into the structure of the situation to find the number of pizzas at which the change occurs enables the pupil to make a decision and justify it appropriately.

Discussion

The written answers and the interviews provided us with detailed information regarding different manifestations of the abstraction process and the use of mathematical conceptions in secondary-school pupils. The results obtained enabled us to zoom in describing the distinction between the participatory and the anticipatory stages as proposed by Simon et al. (2004), observing a wide range of behaviours in connection with the mechanism that Piaget called “transposing knowledge to a higher level” and “the reorganisation-reconstruction of the knowledge at this level. We identified two different moments in the participatory stage and highlighted the importance of the prompts given during the interviews to students accede to anticipation. The use of different kinds of problems in the same study, together with a broad sample of pupils and the combination of questionnaire and post-reflection interviews made possible to amplify and complement previous characterisations of the abstraction process (Ellis, 2007a; Hershkowitz, Schwartz, & Dreyfus, 2001; Sriraman, 2004). Our findings have enabled us to generate two ideas which may help to explain some aspects of the abstraction process. In the first place, the way in which activity-effect reflection reveals what route is followed from projection to local anticipation and, secondly, the two manifestations of reflective abstraction in the process of problem solving.

Progress from projection to local anticipation stage is based on the capacity to observe regularities (the effect of the activity) and coordinate information in the set of particular cases. The way in which learners use particular cases is evidence of the steps they take when they have not identified a previously-constructed mathematical structure (participatory stage). The use of particular cases is linked to the performance of cognitive actions such as comparing, relating or searching. This kind of actions leads the student to notice the effect of his/her activities and coordinate the information which in turn leads to a change in the learner’s attention-focus. Such prompted attention-changes, linked to cognitive actions, are what reflection consists of. A process of this nature has also been identified by Ellis (2007a, 2007b) via different kinds of generalisation tasks in which learners related and associated two situations or properties discernible in two situations, or used repeated acts to search for a relation. In these cases, the prompts proceed from the design of the task or from the interviewer. Our data have shown that in certain cases the existence of some kind of prompt or stimulus (made by the teacher/researcher or the task design) allow to student change through reflection and accede to anticipation (mathematical conception). These prompts favour the change of focus which is itself the beginning of the recognition of some kind of regularity in the set of data (effect of activity).

We argue that it is possible to identify different aspects of the abstraction process using problems from different mathematical domains all of which provides evidence of the general nature of this model. The relationship between the participatory and anticipatory stages in the abstraction process (Piaget, 2001) and the actions of generalisation and the characteristics of what has been generalised (Ellis, 2007b), give greater strength to this way of understanding the abstraction process when learners think mathematically, and locate the focus of attention on the relation between the learner’s mental actions while abstracting, the outcome of these acts and their subsequent use. The results obtained therefore have implications with regard to the design of tasks to encourage the construction of an abstraction and the consolidation of the construction. In the first place, the role played by prompts (in the task itself or as made by the researcher/teacher during the interview) would seem to indicate that when abstraction-centred tasks are designed they should take into account the nature of the prompts which will help the learners to coordinate the information and thus go on to local anticipation. This recommendation is compatible with that made by Tzur (2007) following a whole-class teaching experiment. Secondly, in order to give learners the opportunity change their attention-focus and begin to see a set of activity-effect registers as a unified object (the identification of the regularity and/or the general aspect) (Dörfler, 2002) it will be necessary to create opportunities for the development of language-items for the new construction. This characteristic of the task has also been considered relevant in designing tasks to consolidate a new construction (Monaghan, & Ozmantar, 2006). In any event, more research is evidently required to provide information that will be useful in reaching a clearer theoretical understanding of task-design, with all the obvious implications for the improvement of teaching methods.

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