Nonlinear optimal control for a 4-DOF SCARA robotic manipulator

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Abstract: SCARA robotic manipulators (Selective Compliance Articulated Robot Arms) find wide use in industry. A nonlinear optimal control approach is proposed for the dynamic model of the 4-DOF SCARA robotic manipulator. The dynamic model of the SCARA robot undergoes approximate linearization around a temporary operating point that is recomputed at each time-step of the control method. The linearization relies on Taylor series expansion and on the associated Jacobian matrices. For the linearized state-space model of the system a stabilizing optimal (H-infinity) feedback controller is designed. To compute the controller's feedback gains an algebraic Riccati equation is repetitively solved at each iteration of the control algorithm. The stability properties of the control method are proven through Lyapunov analysis. The proposed control method is advantageous because: (i) unlike the popular computed torque method for robotic manipulators, the new control approach is characterized by optimality and is also applicable when the number of control inputs is not equal to the robot's number of DOFs, (ii) it achieves fast and accurate tracking of reference setpoints under minimal energy consumption by the robot's actuators, (iii) unlike the popular Nonlinear Model Predictive Control method, the article's nonlinear optimal control scheme is of proven global stability and of ensured convergence to the optimum.

Keywords: 4-DOF SCARA robotic manipulator, industrial robots, nonlinear H-infinity control, Taylor series expansion, Jacobian matrices, Riccati equation, global stability, differential flatness properties.

1 Introduction

SCARA robots (Selective Compliance Articulated Robot Arms) are widely used in industrial tasks as well as in the teaching of robotics and in the related testing of new robot control algorithms [1-5]. The rapid development of the Computer, Communication and Consumer Electronics Industry (3C industry) has led also to the spread of the use of SCARA robots [6-10]. SCARA robots exhibit agility in assembly tasks for the 3C industry and particularly in the fabrication of electronic devices, as well as in welding, handling of objects and pick and place tasks with high speed, short time-cycle, accurate path following and in general much flexible operation [11-15]. Of course, to achieve the precise execution of such tasks SCARA robots have to be equipped with computationally powerful microprocessors and have to be also supplied with elaborated nonlinear control algorithms [16-20].

SCARA robots are high performance robotic manipulators with relatively simple structure. With three revolute joints (named as shoulder, elbow and wrist, respectively) a SCARA robot can move its end-effector horizontally, while with a prismatic joint it can move the end-effector vertically [21-26]. The configuration of the SCARA robot (Fig. 1) is outlined as follows: First it comprises a revolute joint about the vertical axis. This joint swings a rigid arm and at the end of this arm there is a second revolute joint which swings the second arm again about the vertical axis. The first two revolute joints enable to move horizontally a load picked by the robot's end-effector. A prismatic joint (tool) is mounted at the end of the second arm. This can move straight up and down. Finally, at the end of the tool there is a third revolute joint which allows for the precise positioning and orientation of the load.

In the present article, a nonlinear optimal control method is proposed for the nonlinear model of a 4-DOF SCARA robot [27-28]. The dynamic model of the 4-DOF SCARA robot undergoes first approximate linearization around a temporary operating point which is updated at each sampling instance. This operating point is defined by the present value of the robot's state vector and by the last sampled value of the control inputs vector. The linearization process relies on first-order Taylor series expansion and on the computation of the associated Jacobian matrices [29-31]. The modelling error which is due to the truncation of higher-order terms in the Taylor series expansion, is proven to be small and is asymptotically compensated by the robustness of the control algorithm. For the approximately linearized state-space description of the system a stabilizing H-infinity feedback controller is defined.

The proposed H-infinity controller achieves the solution of the optimal control problem for the SCARA robot under model uncertainty and external perturbations. Actually, it represents a min-max differential game which takes place between (i) the control inputs of the system that try to minimize a cost function comprising a quadratic term of the state vector's tracking error (ii) the model uncertainty and exogenous perturbation terms which try to maximize this cost function. To compute the staabilizing feedback gains of this controller an algebraic Riccati equation has to be also solved at each time-step of the control method [1], [32]. The global stability properties of the control scheme are proven through Lyapunov analysis. First, it is proven that the control loop satisfies the H-infinity tracking performance criterion [1], [33]. Next, it is proven that under moderate conditions, global asymptotic stability of the control loop is ensured. To implement state estimation-based control without need to measure the entire state vector of the system the H-infinity Kalman Filter is used as a robust state estimator [1]. The nonlinear optimal control method retains the advantages of linear optimal control, that is fast and accurate tracking of reference setpoints under moderate variations of the control inputs.

The article has a meaningful contribution to the area of nonlinear control. One can point out the advantages of the nonlinear optimal control method against Nonlinear Model Predictive Control (NMPC) [28]. In NMPC the stability properties of the control scheme remain unproven and the convergence of the iterative search for an optimum often depends on initialization and parameter values' selection. It is also noteworthy that the nonlinear optimal control method is applicable to a wider class of dynamical systems than approaches based on the solution of State Dependent Riccati Equations (SDRE). The SDRE approaches can be applied only to dynamical systems which can be transformed to the Linear Parameter Varying (LPV) form. Besides, the nonlinear optimal control method performs better than nonlinear optimal control schemes which use approximation of the solution of the Hamilton-Jacobi-Bellman equation by Galerkin series expansions. The stability properties of the Galerkin series expansion -based optimal control approaches are still unproven.

The structure of the paper is as follows: In Section 2 the dynamic model of the 4-DOF SCARA robot is given and the associated state-space model in the affine-in-the-input nonlinear state-space form is formulated. In Section 3 the dynamic model of the SCARA robot undergoes approximate linearization through Taylor series expansion and with the computation of the associated Jacobian matrices. In Section 4 the H-infinity optimal control problem is formulated for the dynamic model of the SCARA robot. In Section 5 the global stability properties of the H-infinity control scheme are proven through Lyapunov analysis. Besides, the H-infinity Kalman Filter is introduced as a robust state estimator. In Section 6 the accuracy of setpoints tracking by the state variables of the SCARA robot, under the nonlinear optimal control method, is further confirmed through simulation experiments. Finally, in Section 7 concluding remarks are stated.

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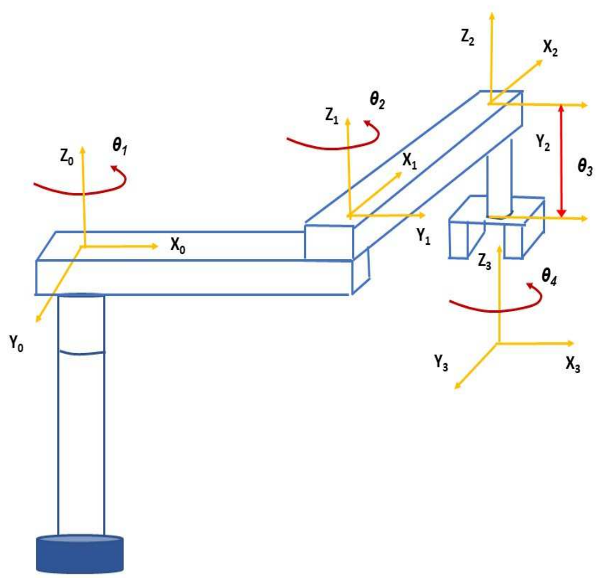

Figure 1: The 4-DOF SCARA robotic manipulator and the reference frames (coordinate systems) associated with the robot's links and joints

2 Dynamic model of the 4-DOF SCARA robotic manipulator

2.1 State-space model of the SCARA robot

The diagram of the 4-DOF SCARA robot is shown in Fig. 1. The associated state-space model of the robot's dynamics takes the form [1], [2]

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) + \tilde{d} = \tau(t) \tag{1}$$

where θ_i, $i = 1, 2, 4$ is the joints' turn angle, $\dot{\theta}_i$, $i = 1, 2, 4$ is the joints' angular speed, θ_3 is the position of the prismatic joint, $\dot{\theta}_3$ is the velocity of the prismatic joint, $\tilde{d}$ is the disturbances vector, $M(\theta)$ is the inertia matrix, $C(\theta,\dot{\theta})$ is the Coriolis and centrifugal forces matrix, and $G(\theta)$ is the gravitational forces vector/ These parameters of the robotic model are defined as follows [2]:

$$M(\theta) = \begin{pmatrix} p_1 + p_2cos(\theta_2) & p_3 + 0.5p_2cos(\theta_2) & 0 & -p_5 \\ p_2 + 0.5p_2\cos(\theta_2) & p_2 & 0 & -p_5 \\ 0 & 0 & p_4 & 0 \\ -p_5 & -p_5 0 & p_5 & \end{pmatrix} \tag{2}$$

$$C(\theta,\dot{\theta}) = \begin{pmatrix} -p_2cos(\theta_1)\dot{\theta}_2 & -0.5p_2sin(\theta_2)\dot{\theta}_2 & 0 & 0 \\ 0.5p_2sin(\theta_2)\dot{\theta}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G(\theta) = \begin{pmatrix} 0 \\ 0 \\ p_4\bar{g} \\ 0 \end{pmatrix} \quad \tilde{d} = \begin{pmatrix} b_1\dot{\theta}_1 \\ b_2\dot{\theta}_2 \\ b_3\dot{\theta}_3 \\ b_4\dot{\theta}_4 \end{pmatrix} \tag{3}$$

In the previous model τ_i are the control inputs, I_i are the moments of inertia around the centroid (the center of rotation of each link is at distance κ from the end of the link), m_i is the mass of the i-th link, d_i is the center of mass of the i-th link, l_i is the length of the i-th link, θ_i is the angle (position) i-th joint, while $\bar{g}$ is the acceleration of gravity. It holds that $I_i = m_i\kappa_i^2$, $p_1 = \sum_{i=1}^{4} I_i + m_1 d_1^2 + m_2(d_1^2 + l_1^2) + (m_3 + m_4)(l_1^2 + l_2^2)$, $p_2 = 2[l_1 d_2 m_2 + l_1 l_2(m_3 + m_4)]$, $p_3 = m_3 + m_4$, $p_4 = m_3 + m_4$ and $p_5 = I_4$.

About the elements of the inertia matrix one has: $m_{11} = p_1 + p_2 cos(\theta_2)$, $m_{12} = p_3 + 0.5p_2 cos(\theta_2)$, $m_{13} = 0$, $m_{14} = -p_5$, $m_{21} = p_2 + 0.5p_2\cos(\theta_2)$, $m_{22} = p_2$, $m_{23} = 0$, $m_{34} = -p_5$, $m_{31} = 0$, $m_{32} = 0$, $m_{33} = p_4$, $m_{34} = 0$, $m_{41} = -p_5$, $m_{42} = -p_5$, $m_{43} = 0$, $m_{44} = p_5$.

About the elements of the Coriolis matrix one has: $c_{11} = -p_2 cos(\theta_1)\dot{\theta}_2$, $c_{12} = -0.5p_2 sin(\theta_2)\dot{\theta}_2$, $c_{13} = 0$ $c_{14} = 0$, $c_{21} = 0.5p_2 sin(\theta_2)\dot{(\theta)}_1$, $c_{22} = 0$, $c_{23} = 0$, $c_{24} = 0$, $c_{31} = 0$, $c_{32} = 0$, $c_{33} = 0$, $c_{34} = 0$, $c_{41} = 0$, $c_{42} = 0$, $c_{43} = 0$, $c_{44} = 0$.

About the elements of the Gravitational forces vector one has: $g_1 = 0$, $g_2 = 0$, $g_3 = p_4\bar{g}$ and $g_4 = 0$.

About the elements of the disturbances (friction) vector one has: $d_1 = b_1\dot{\theta}_1$, $d_2 = b_2\dot{\theta}_2$, $d_3 = b_3\dot{\theta}_3$, and $d_4 = b_4\dot{\theta}_4$.

Next, the inverse of the inertia matrix M is defined as

$$N^{-1} = \tfrac{1}{detM}\begin{pmatrix} M_{11} & -M_{21} & M_{31} & -M_{41} \\ -M_{12} & M_{22} & -M_{32} & M_{42} \\ M_{13} & -M_{23} & M_{33} & -M_{43} \\ -M_{14} & M_{24} & -M_{34} & M_{44} \end{pmatrix} \tag{4}$$

where the above noted subdeterminants M_{ij} $i = 1, \cdots, 4$ and $j = 1, \cdots, 4$ are defined as

$$\begin{aligned}
M_{11} &= m_{22}(m_{33}m_{44} - m_{43}m_{34}) - m_{23}(m_{32}m_{44} - m_{43}m_{34} + m_{24})(m_{32}m_{43} - m_{42}m_{33}) \\
M_{12} &= m_{21}(m_{33}m_{44} - m_{43}m_{34}) - m_{23}(m_{31}m_{44} - m_{41}m_{34} + m_{24})(m_{31}m_{43} - m_{41}m_{33}) \\
M_{13} &= m_{21}(m_{32}m_{44} - m_{42}m_{34}) - m_{22}(m_{31}m_{44} - m_{41}m_{34} + m_{24})(m_{31}m_{43} - m_{41}m_{33}) \\
M_{14} &= m_{21}(m_{32}m_{43} - m_{42}m_{33}) - m_{22}(m_{31}m_{43} - m_{41}m_{33} + m_{23})(m_{31}m_{42} - m_{41}m_{32})
\end{aligned}$$

$$\begin{aligned}
M_{21} &= m_{12}(m_{33}m_{44} - m_{43}m_{34}) - m_{13}(m_{32}m_{44} - m_{42}m_{34} + m_{14})(m_{32}m_{43} - m_{42}m_{33}) \\
M_{22} &= m_{11}(m_{23}m_{24} - m_{43}m_{44}) - m_{13}(m_{31}m_{44} - m_{41}m_{14} + m_{14})(m_{31}m_{43} - m_{41}m_{23}) \\
M_{23} &= m_{11}(m_{32}m_{44} - m_{42}m_{44}) - m_{12}(m_{31}m_{44} - m_{41}m_{34} + m_{14})(m_{31}m_{42} - m_{41}m_{32}) \\
M_{24} &= m_{11}(m_{32}m_{43} - m_{42}m_{33}) - m_{12}(m_{31}m_{43} - m_{41}m_{33} + m_{13})(m_{31}m_{42} - m_{41}m_{32})
\end{aligned}$$

$$\begin{aligned}
M_{31} &= m_{12}(m_{23}m_{44} - m_{43}m_{24}) - m_{13}(m_{22}m_{44} - m_{42}m_{24} + m_{14})(m_{22}m_{43} - m_{42}m_{23}) \\
M_{32} &= m_{11}(m_{23}m_{44} - m_{42}m_{24}) - m_{13}(m_{12}m_{44} - m_{41}m_{24} + m_{14})(m_{12}m_{43} - m_{41}m_{23}) \\
M_{33} &= m_{11}(m_{22}m_{44} - m_{42}m_{24}) - m_{12}(m_{21}m_{44} - m_{41}m_{24} + m_{14})(m_{41}m_{22} - m_{21}m_{42}) \\
M_{34} &= m_{11}(m_{22}m_{43} - m_{42}m_{23}) - m_{12}(m_{21}m_{43} - m_{41}m_{23} + m_{13})(m_{21}m_{42} - m_{41}m_{22})
\end{aligned}$$

$$\begin{aligned}
M_{41} &= m_{12}(m_{23}m_{34} - m_{33}m_{24}) - m_{13}(m_{22}m_{34} - m_{32}m_{23} + m_{14})(m_{22}m_{33} - m_{32}m_{23}) \\
M_{42} &= m_{11}(m_{22}m_{43} - m_{33}m_{42}) - m_{31}(m_{12}m_{43} - m_{13}m_{44} + m_{41})(m_{12}m_{33} - m_{31}m_{32}) \\
M_{43} &= m_{11}(m_{22}m_{34} - m_{32}m_{24}) - m_{12}(m_{21}m_{34} - m_{31}m_{24} + m_{14})(m_{21}m_{32} - m_{31}m_{22}) \\
M_{44} &= m_{11}(m_{22}m_{33} - m_{32}m_{23}) - m_{12}(m_{21}m_{33} - m_{31}m_{23} + m_{13})(m_{21}m_{32} - m_{31}m_{22})
\end{aligned}$$

The determinant of matrix M is

$$detM = m_{11}M_{11} - m_{12}M_{12} + m_{13}M_{13} - m_{14}M_{14}$$

For the dynamic model of the SCARA robot that was initially written in the form

$$M(\theta)\ddot{\theta} + \tilde{C}(\theta, \dot{\theta})\dot{\theta} + G(\theta) + \tilde{d} = \tau \tag{5}$$

it holds that

$$\tilde{C}(\theta, \dot{\theta})\dot{\theta} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{pmatrix} \tag{6}$$

or equivalently

$$C(\theta, \dot{\theta}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} c_{11}\dot{\theta}_1 & c_{12}\dot{\theta}_2 & c_{13}\dot{\theta}_3 & c_{14}\dot{\theta}_4 \\ c_{21}\dot{\theta}_2 & c_{22}\dot{\theta}_2 & c_{23}\dot{\theta}_3 & c_{24}\dot{\theta}_4 \\ c_{31}\dot{\theta}_3 & c_{32}\dot{\theta}_2 & c_{33}\dot{\theta}_3 & c_{34}\dot{\theta}_4 \\ c_{41}\dot{\theta}_4 & c_{42}\dot{\theta}_2 & c_{43}\dot{\theta}_3 & c_{44}\dot{\theta}_4 \end{pmatrix} \tag{7}$$

Consequently, the dynamic model of the robot can be written as

$$\begin{gathered} M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + G(\theta) + \tilde{d}(\dot{\theta}) = \tau \Rightarrow \\ \ddot{\theta} + M^{-1}(\theta)C(\theta, \dot{\theta}) + M^{-1}(\theta)G(\theta) + M^{-1}(\theta))\tilde{d}(\dot{\theta}) = M^{-1}(\theta)\tau \Rightarrow \\ \ddot{\theta} = -M^{-1}(\theta)C(\theta, \dot{\theta}) - M^{-1}(\theta)G(\theta) - M^{-1}(\theta))\tilde{d}(\dot{\theta}) + M^{-1}(\theta)\tau \Rightarrow \\ \ddot{\theta} = -M^{-1}(\theta)[C(\theta, \dot{\theta}) + G(\theta) + \tilde{d}(\dot{\theta})] + M^{-1}(\theta)\tau \end{gathered} \tag{8}$$

Consequently, the dynamic model of the SCARA robot is written as

$$\begin{gathered} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{pmatrix} = -\frac{1}{detM} \begin{pmatrix} M_{11} & -M_{21} & M_{31} & -M_{41} \\ -M_{12} & M_{22} & -M_{32} & M_{42} \\ M_{13} & -M_{23} & M_{33} & -M_{43} \\ -M_{14} & M_{24} & -M_{34} & M_{44} \end{pmatrix} \begin{pmatrix} c_1 + g_1 + d_1 \\ c_2 + g_2 + d_2 \\ c_3 + g_3 + d_3 \\ c_4 + g_4 + d_4 \end{pmatrix} + \\ + \frac{1}{detM} \begin{pmatrix} M_{11} & -M_{21} & M_{31} & -M_{41} \\ -M_{12} & M_{22} & -M_{32} & M_{42} \\ M_{13} & -M_{23} & M_{33} & -M_{43} \\ -M_{14} & M_{24} & -M_{34} & M_{44} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \end{gathered} \tag{9}$$

Equivalently, using that the torques vector $\tau = [\tau_1, tau_2, \tau_3, \tau_4]^T$ is the control inputs vector $u = [u_1, u_2, u_3, u_4]^T$, the dynamic model of the SCARA robot is written as

$$\begin{gathered} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{pmatrix} = \begin{pmatrix} \frac{-M_{11}(c_1+g_1+d_1)+M_{21}(c_2+g_2+d_2)-M_{31}(c_3+g_3+d_3)+M_{41}(c_4+g_4+d_4)}{detM} \\ \frac{M_{12}(c_1+g_1+d_1)-M_{22}(c_2+g_2+d_2)+M_{32}(c_3+g_3+d_3)-M_{42}(c_4+g_4+d_4)}{detM} \\ \frac{-M_{13}(c_1+g_1+d_1)+M_{23}(c_2+g_2+d_2)-M_{33}(c_3+g_3+d_3)+M_{43}(c_4+g_4+d_4)}{detM} \\ \frac{M_{14}(c_1+g_1+d_1)-M_{24}(c_2+g_2+d_2)+M_{34}(c_3+g_3+d_3)-M_{44}(c_4+g_4+d_4)}{detM} \end{pmatrix} + \\ + \begin{pmatrix} \frac{M_{11}}{detM} & -\frac{M_{21}}{detM} & \frac{M_{31}}{detM} & -\frac{M_{41}}{detM} \\ -\frac{M_{12}}{detM} & \frac{M_{22}}{detM} & -\frac{M_{32}}{detM} & \frac{M_{42}}{detM} \\ \frac{M_{13}}{detM} & -\frac{M_{23}}{detM} & \frac{M_{33}}{detM} & -\frac{M_{34}}{detM} \\ -\frac{M_{14}}{detM} & \frac{M_{24}}{detM} & -\frac{M_{34}}{detM} & \frac{M_{44}}{detM} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \end{gathered} \tag{10}$$

The state vector of the SCARA robotis

$$\begin{array}{c} x=[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8]^T \Rightarrow \\ x=[\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2,\theta_3,\dot{\theta}_3,\theta_4,\dot{\theta}_4]^T \end{array} \tag{11}$$

Moreover, the following functions are defined

$$\begin{array}{ll} f_1(x)=x_2 & f_2(x)=\frac{-M_{11}(c_1+g_1+d_1)+M_{21}(c_2+g_2+d_2)-M_{31}(c_3+g_3+d_3)+M_{41}(c_4+g_4+d_4)}{detM} \\ f_3(x)=x_4 & f_4(x)=\frac{M_{12}(c_1+g_1+d_1)-M_{22}(c_2+g_2+d_2)+M_{32}(c_3+g_3+d_3)-M_{42}(c_4+g_4+d_4)}{detM} \\ f_5(x)=x_6 & f_6(x)=\frac{-M_{13}(c_1+g_1+d_1)+M_{23}(c_2+g_2+d_2)-M_{33}(c_3+g_3+d_3)+M_{43}(c_4+g_4+d_4)}{detM} \\ f_7(x)=x_8 & f_8(x)=\frac{M_{14}(c_1+g_1+d_1)-M_{24}(c_2+g_2+d_2)+M_{34}(c_3+g_3+d_3)-M_{44}(c_4+g_4+d_4)}{detM} \end{array}$$

$$\begin{array}{cccc} g_{11}(x) & g_{12}(x)=0 & g_{13}=0 & g_{14}=0 \\ g_{21}(x)=\frac{M_{11}}{detM} & g_{22}(x)=-\frac{M_{21}}{detM} & g_{23}(x)=\frac{M_{31}}{detM} & g_{24}(x)=-\frac{M_{41}}{detM} \\ g_{31}(x)=0 & g_{32}(x)=0 & g_{33}(x)=0 & g_{34}(x)=0 \\ g_{41}(x)=-\frac{M_{12}}{detM} & g_{42}(x)=\frac{M_{22}}{detM} & g_{43}(x)=-\frac{M_{32}}{detM} & g_{44}(x)=\frac{M_{42}}{detM} \\ g_{51}(x)=0 & g_{52}(x)=0 & g_{53}(x)=0 & g_{54}(x)=0 \\ g_{61}(x)=\frac{M_{13}}{detM} & g_{62}(x)=-\frac{M_{23}}{detM} & g_{63}(x)=\frac{M_{33}}{detM} & g_{64}(x)=-\frac{M_{43}}{detM} \\ g_{71}(x)=0 & g_{72}(x)=0 & g_{73}(x)=0 & g_{74}(x)=0 \\ g_{81}(x)=-\frac{M_{14}}{detM} & g_{82}(x)=\frac{M_{24}}{detM} & g_{83}(x)=-\frac{M_{34}}{detM} & g_{84}(x)=\frac{M_{44}}{detM} \end{array}$$

Thus, the state-space model of the 4-DOF SCARA robot is written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \\ f_8(x) \end{pmatrix} + \begin{pmatrix} g_{11}(x) & g_{12}(x) & g_{13}(x) & g_{14}(x) \\ g_{21}(x) & g_{22}(x) & g_{23}(x) & g_{24}(x) \\ g_{31}(x) & g_{32}(x) & g_{33}(x) & g_{34}(x) \\ g_{41}(x) & g_{42}(x) & g_{43}(x) & g_{44}(x) \\ g_{51}(x) & g_{52}(x) & g_{53}(x) & g_{54}(x) \\ g_{61}(x) & g_{62}(x) & g_{63}(x) & g_{64}(x) \\ g_{71}(x) & g_{72}(x) & g_{73}(x) & g_{74}(x) \\ g_{81}(x) & g_{82}(x) & g_{83}(x) & g_{84}(x) \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \tag{12}$$

or in concise form one has the affine-in-the-input nonlinear state-space model

$$\dot{x}=f(x)+g(x)u \tag{13}$$

where $x \in R^{8\times1}$, $f(x) \in R^{8\times1}$, $g(x) \in R^{8\times4}$ and $u \in R^{4\times1}$.

2.2 Differential flatness properties of the 4-DOF SCARA robot

The dynamic model of the 4-DOF SCARA robot is differentially flat. The flat outputs vector of the system is $Y=[y_1,y_2,y_3,y_4]^T=[x_1,x_3,x_5,x_7]^T$. Differential flatness is associated with the following two conditions: (i) all state variables of the system can be expressed as differential functions of its flat outputs, (ii) the flat outputs and their derivatives are differentially independent which signifies that they are not connected through a relation in the form of linear homogeneous differential equation [1],[32].

Obviously, it holds that $x_2=\dot{x}_1$, $x_4=\dot{x}_3$, $x_6=\dot{x}_5$, and $x_8=\dot{x}_7$. This signifies that state variables x_2, x_4, x_6 and x_8 are differential functions of the system's flat outputs. Besides, using that

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_3 \\ \ddot{x}_5 \\ \ddot{x}_7 \end{pmatrix} = \begin{pmatrix} f_2(x) \\ f_4(x) \\ f_6(x) \\ f_8(x) \end{pmatrix} + \begin{pmatrix} g_{11}(x) & g_{21}(x) & g_{31}(x) & g_{41}(x) \\ g_{12}(x) & g_{22}(x) & g_{32}(x) & g_{42}(x) \\ g_{13}(x) & g_{23}(x) & g_{33}(x) & g_{43}(x) \\ g_{14}(x) & g_{24}(x) & g_{34}(x) & g_{44}(x) \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \tag{14}$$

and by solving with respect to the control inputs one obtains

$$\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\end{pmatrix} = \begin{pmatrix} g_{11}(x) & g_{21}(x) & g_{31}(x) & g_{41}(x) \\ g_{12}(x) & g_{22}(x) & g_{32}(x) & g_{42}(x) \\ g_{13}(x) & g_{23}(x) & g_{33}(x) & g_{43}(x) \\ g_{14}(x) & g_{24}(x) & g_{34}(x) & g_{44}(x) \end{pmatrix}^{-1} \left[\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_3\\ \ddot{x}_5\\ \ddot{x}_7\end{pmatrix} - \begin{pmatrix} f_2(x)\\ f_4(x)\\ f_6(x)\\ f_8(x)\end{pmatrix} \right] \tag{15}$$

The above relation signifies that the control inputs τ_1, τ_2, τ_3 and τ_4 are also differential functions of the flat outputs of the system. Consequently, the 4-DOF SCARA robot is differentially flat. The differential flatness property means also [1], [32] that (i) the robotic model is input-output linearizable, (ii) setpoints for all state variables of the robot can be defined. Actually one selects first setpoints for the state variables which coincide with the flat outputs x_1^d, x_3^d, x_5^d and x_7^d, and next defines setpoints for the rest of the state variables x_2^d, x_4^d, x_6^d and x_8^d which are asscoiated with the flat outputs through the previously explained differential relations. The differential flatness property is also an implicit proof of the system's controllability.

3 Approximate linearization of the dynamic model of the SCARA robot

The dynamic model of the 4-DOF SCARA robot being initially expressed in the state-space form

$$\dot{x} = f(x) + g(x)u \tag{16}$$

undergoes approximate linearization at each sampling instance around the temporary operating point (x^*, u^*), where x^* is the present value of the system's state vector and u^* is the last sampled value of the control inputs vector. The linearization process is based on Taylor series expansion and gives

$$\dot{x} = Ax + Bu + \tilde{d} \tag{17}$$

where $\tilde{d}$ is the cumulative disturbances vector which may ne due to truncation of higher-order terms from the Taylor series expansion (b) exogenous perturbations (c) sensor measurements noise of any distribution. Matrices A and B are Jacobian matrices of the Taylor series expansion which are defined as:

$$\begin{gathered} A = \nabla_x[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow \\ A = \nabla_x f(x) \mid_{(x^*,u^*)} + \nabla_x g_1(x)u \mid_{(x^*,u_1^*)} + \nabla_x g_2(x)u_2 \mid_{(x^*,u^*)} \\ + \nabla_x g_3(x)u_3 \mid_{(x^*,u^*)} + \nabla_x g_4(x)u_4 \mid_{(x^*,u^*)} \end{gathered} \tag{18}$$

$$B = \nabla_u[f(x) + g(x)u] \mid_{(x^*,u^*)} \Rightarrow B = g(x) \mid_{(x^*,u^*)} \tag{19}$$

where $g_i(x),\ i = 1, \cdots, 4$ are the columns of the control inputs gain matrix $g(x)$.

This linearization approach which has been followed for implementing the nonlinear optimal control scheme results into a quite accurate model of the system's dynamics. Consider again the affine-in-the-input state-space model

$$\begin{gathered} \dot{x} = f(x) + g(x)u \Rightarrow \\ \dot{x} = [f(x^*) + \nabla_x f(x) \mid_{x^*} (x - x^*)] + [g(x^*) + \nabla_x g(x) \mid_{x^*} (x - x^*)]u^* + g(x^*)u^* + g(x^*)(u - u^*) + \tilde{d}_1 \Rightarrow \\ \dot{x} = [\nabla_x f(x) \mid_{x^*} + \nabla_x g(x) \mid_{x^*} u^*]x + g(x^*)u - [\nabla_x f(x) \mid_{x^*} + \nabla_x g(x) \mid_{x^*} u^*]x^* + f(x^*) + g(x^*)u^* + \tilde{d}_1 \end{gathered} \tag{20}$$

where $\tilde{d}_1$ is the modelling error due to truncation of higher order terms in the Taylor series expansion of $f(x)$ and $g(x)$. Next, by defining $A = [\nabla_x f(x) \mid_{x^*} + \nabla_x g(x) \mid_{x^*} u^*]$, $B = g(x^*)$ one obtains

$$\dot{x} = Ax + Bu - Ax^* + f(x^*) + g(x^*)u^* + \tilde{d}_1 \tag{21}$$

Moreover by denoting $\tilde{d}=-Ax^*+f(x^*)+g(x^*)u^*+\tilde{d}_1$ about the cumulative modelling error term in the Taylor series expansion procedure one has

$$\dot{x}=Ax+Bu+\tilde{d} \tag{22}$$

which is the approximately linearized model of the dynamics of the system of Eq. (17). The term $f(x^*)+g(x^*)u^*$ is the derivative of the state vector at (x^*,u^*) which is almost annihilated by $-Ax^*$.

3.1 Computation of the Jacobian matrices

The computation of the Jacobian matrices A and B proceeds as follows:

Computation of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$:

First row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: $\frac{\partial f_1}{\partial x_1}=0$, $\frac{\partial f_1}{\partial x_2}=1$, $\frac{\partial f_1}{\partial x_3}=0$, $\frac{\partial f_1}{\partial x_4}=0$, $\frac{\partial f_1}{\partial x_5}=0$, $\frac{\partial f_1}{\partial x_6}=0$, $\frac{\partial f_1}{\partial x_7}=0$ and $\frac{\partial f_1}{\partial x_8}=0$.

Second row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: It holds that $f_2(x)=\frac{f_{2,num}}{f_{2,den}}$ with $f_{2,num}=-M_{11}(c_1+g_1+d+1)+M_{21}(c_2+g_2+d_2)-M_{31}(c_3+g_3+d_3)+M_{41}(c_4+g_4+d_4)$ and $f_{2,den}=detM$. Thus, for $i=1,2,\cdots,8$ one has

$$\begin{array}{c}\frac{\partial f_{2,num}}{\partial x_1}=-\frac{\partial M_{11}}{\partial x_i}(c_1+g_1+d_1)-M_{11}(\frac{\partial c_1}{\partial x_i}+\frac{\partial g_1}{\partial x_i}+\frac{\partial d_1}{\partial x_i})+\\ +\frac{\partial M_{21}}{\partial x_i}(c_2+g_2+d_2)+M_{21}(\frac{\partial c_2}{\partial x_i}+\frac{\partial g_2}{\partial x_i}+\frac{\partial d_2}{\partial x_i})-\\ -\frac{\partial M_{31}}{\partial x_i}(c_3+g_3+d_3)-M_{31}(\frac{\partial c_3}{\partial x_i}+\frac{\partial g_3}{\partial x_i}+\frac{\partial d_3}{\partial x_i})+\\ +\frac{\partial M_{41}}{\partial x_i}(c_4+g_4+d_4)+M_{41}(\frac{\partial c_4}{\partial x_i}+\frac{\partial g_4}{\partial x_i}+\frac{\partial d_4}{\partial x_i})\end{array} \tag{23}$$

and also

$$\frac{\partial f_{2,den}}{\partial x_i}=\frac{\partial detM}{\partial x_i} \tag{24}$$

and finally

$$\frac{\partial f_2}{\partial x_i}=\frac{\frac{\partial f_{2,num}}{\partial x_i}f_{2,den}-f_{2,num}\frac{\partial f_{2,den}}{\partial x_i}}{detM^2} \tag{25}$$

Third row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: $\frac{\partial f_3}{\partial x_1}=0$, $\frac{\partial f_3}{\partial x_2}=0$, $\frac{\partial f_3}{\partial x_3}=0$, $\frac{\partial f_3}{\partial x_4}=1$, $\frac{\partial f_3}{\partial x_5}=0$, $\frac{\partial f_3}{\partial x_6}=0$, $\frac{\partial f_3}{\partial x_7}=0$ and $\frac{\partial f_3}{\partial x_8}=0$.

Fourth row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: It holds that $f_4(x)=\frac{f_{4,num}}{f_{4,den}}$ with $f_{4,num}=M_{12}(c_1+g_1+d+1)-M_{22}(c_2+g_2+d_2)+M_{32}(c_3+g_3+d_3)-M_{42}(c_4+g_4+d_4)$ and $f_{4,den}=detM$. Thus, for $i=1,2,\cdots,8$ one has

$$\begin{array}{c}\frac{\partial f_{4,num}}{\partial x_1}=\frac{\partial M_{12}}{\partial x_i}(c_1+g_1+d_1)+M_{12}(\frac{\partial c_1}{\partial x_i}+\frac{\partial g_1}{\partial x_i}+\frac{\partial d_1}{\partial x_i})+\\ -\frac{\partial M_{22}}{\partial x_i}(c_2+g_2+d_2)-M_{22}(\frac{\partial c_2}{\partial x_i}+\frac{\partial g_2}{\partial x_i}+\frac{\partial d_2}{\partial x_i})-\\ \frac{\partial M_{32}}{\partial x_i}(c_3+g_3+d_3)+M_{32}(\frac{\partial c_3}{\partial x_i}+\frac{\partial g_3}{\partial x_i}+\frac{\partial d_3}{\partial x_i})-\\ -\frac{\partial M_{42}}{\partial x_i}(c_4+g_4+d_4)-M_{42}(\frac{\partial c_4}{\partial x_i}+\frac{\partial g_4}{\partial x_i}+\frac{\partial d_4}{\partial x_i})\end{array} \tag{26}$$

and also

$$\frac{\partial f_{4,den}}{\partial x_i}=\frac{\partial detM}{\partial x_i} \tag{27}$$

and finally

$$\frac{\partial f_4}{\partial x_i} = \frac{\frac{\partial f_{4,num}}{\partial x_i} f_{4,den} - f_{4,num}\frac{\partial f_{4,den}}{\partial x_i}}{detM^2} \tag{28}$$

Fifth row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: $\frac{\partial f_5}{\partial x_1} = 0$, $\frac{\partial f_5}{\partial x_2} = 0$, $\frac{\partial f_5}{\partial x_3} = 0$, $\frac{\partial f_5}{\partial x_4} = 0$, $\frac{\partial f_5}{\partial x_5} = 0$, $\frac{\partial f_5}{\partial x_6} = 1$, $\frac{\partial f_5}{\partial x_7} = 0$ and $\frac{\partial f_5}{\partial x_8} = 0$.

Sixth row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: It holds that $f_6(x) = \frac{f_{6,num}}{f_{6,den}}$ with $f_{6,num} = -M_{13}(c_1 + g_1 + d + 1) + M_{23}(c_2 + g_2 + d_2) - M_{33}(c_3 + g_3 + d_3) + M_{43}(c_4 + g_4 + d_4)$ and $f_{6,den} = detM$. Thus, for $i = 1, 2, \cdots, 8$ one has

$$\begin{array}{c} \frac{\partial f_{6,num}}{\partial x_1} = -\frac{\partial M_{13}}{\partial x_i}(c_1 + g_1 + d_1) - M_{13}(\frac{\partial c_1}{\partial x_i} + \frac{\partial g_1}{\partial x_i} + \frac{\partial d_1}{\partial x_i}) + \\ + \frac{\partial M_{23}}{\partial x_i}(c_2 + g_2 + d_2) + M_{23}(\frac{\partial c_2}{\partial x_i} + \frac{\partial g_2}{\partial x_i} + \frac{\partial d_2}{\partial x_i}) - \\ - \frac{\partial M_{33}}{\partial x_i}(c_3 + g_3 + d_3) - M_{33}(\frac{\partial c_3}{\partial x_i} + \frac{\partial g_3}{\partial x_i} + \frac{\partial d_3}{\partial x_i}) + \\ + \frac{\partial M_{43}}{\partial x_i}(c_4 + g_4 + d_4) + M_{43}(\frac{\partial c_4}{\partial x_i} + \frac{\partial g_4}{\partial x_i} + \frac{\partial d_4}{\partial x_i}) \end{array} \tag{29}$$

and also

$$\frac{\partial f_{6,den}}{\partial x_i} = \frac{\partial detM}{\partial x_i} \tag{30}$$

and finally

$$\frac{\partial f_6}{\partial x_i} = \frac{\frac{\partial f_{6,num}}{\partial x_i} f_{6,den} - f_{6,num}\frac{\partial f_{6,den}}{\partial x_i}}{detM^2} \tag{31}$$

Seventh row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: $\frac{\partial f_7}{\partial x_1} = 0$, $\frac{\partial f_7}{\partial x_2} = 0$, $\frac{\partial f_7}{\partial x_3} = 0$, $\frac{\partial f_7}{\partial x_4} = 0$, $\frac{\partial f_7}{\partial x_5} = 0$, $\frac{\partial f_7}{\partial x_6} = 0$, $\frac{\partial f_7}{\partial x_7} = 0$ and $\frac{\partial f_7}{\partial x_8} = 1$.

Eighth row of the Jacobian matrix $\nabla_x f(x) \mid_{(x^*,u^*)}$: It holds that $f_4(x) = \frac{f_{8,num}}{f_{8,den}}$ with $f_{8,num} = M_{14}(c_1 + g_1 + d + 1) - M_{24}(c_2 + g_2 + d_2) + M_{34}(c_3 + g_3 + d_3) - M_{44}(c_4 + g_4 + d_4)$ and $f_{8,den} = detM$. Thus, for $i = 1, 2, \cdots, 8$ one has

$$\begin{array}{c} \frac{\partial f_{8,num}}{\partial x_1} = \frac{\partial M_{14}}{\partial x_i}(c_1 + g_1 + d_1) + M_{14}(\frac{\partial c_1}{\partial x_i} + \frac{\partial g_1}{\partial x_i} + \frac{\partial d_1}{\partial x_i}) + \\ - \frac{\partial M_{24}}{\partial x_i}(c_2 + g_2 + d_2) - M_{24}(\frac{\partial c_2}{\partial x_i} + \frac{\partial g_2}{\partial x_i} + \frac{\partial d_2}{\partial x_i}) - \\ \frac{\partial M_{34}}{\partial x_i}(c_3 + g_3 + d_3) + M_{34}(\frac{\partial c_3}{\partial x_i} + \frac{\partial g_3}{\partial x_i} + \frac{\partial d_3}{\partial x_i}) - \\ - \frac{\partial M_{44}}{\partial x_i}(c_4 + g_4 + d_4) - M_{44}(\frac{\partial c_4}{\partial x_i} + \frac{\partial g_4}{\partial x_i} + \frac{\partial d_4}{\partial x_i}) \end{array} \tag{32}$$

and also

$$\frac{\partial f_{8,den}}{\partial x_i} = \frac{\partial detM}{\partial x_i} \tag{33}$$

and finally

$$\frac{\partial f_8}{\partial x_i} = \frac{\frac{\partial f_{8,num}}{\partial x_i} f_{8,den} - f_{8,num}\frac{\partial f_{8,den}}{\partial x_i}}{detM^2} \tag{34}$$

Computation of the Jacobian matrix $\nabla_x g_1(x) \mid_{(x^*,u^*)}$.

First row of the Jacobian matrix $\nabla_x g_1(x) \mid_{(x^*,u^*)}$: $\frac{\partial g_{11}(x)}{\partial x_1} = 0$ for $i = 1, 2, \cdots, 8$.

Second row of the Jacobian matrix $\nabla_x g_1(x) \mid_{(x^*,u^*)}$: $\frac{\partial g_{21}(x)}{\partial x_i} = \frac{\frac{\partial M_{11}}{\partial x_i} detM - M_{11}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Third row of the Jacobian matrix $\nabla_x g_1(x) \mid_{(x^*,u^*)}$: $\frac{\partial g_{31}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Fourth row of the Jacobian matrix $\nabla_x g_1(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{41}(x)}{\partial x_i} = \frac{-\frac{\partial M_{12}}{\partial x_i} detM + M_{12}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Fifth row of the Jacobian matrix $\nabla_x g_1(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{51}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Sixth row of the Jacobian matrix $\nabla_x g_1(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{61}(x)}{\partial x_i} = \frac{\frac{\partial M_{13}}{\partial x_i} detM - M_{13}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Seventh row of the Jacobian matrix $\nabla_x g_1(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{71}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Eighth row of the Jacobian matrix $\nabla_x g_1(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{81}(x)}{\partial x_i} = \frac{-\frac{\partial M_{14}}{\partial x_i} detM + M_{14}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Computation of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$.

First row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{12}(x)}{\partial x_1} = 0$ for $i = 1, 2, \cdots, 8$.

Second row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{22}(x)}{\partial x_i} = \frac{-\frac{\partial M_{21}}{\partial x_i} detM + M_{21}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Third row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{32}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Fourth row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{42}(x)}{\partial x_i} = \frac{\frac{\partial M_{22}}{\partial x_i} detM - M_{22}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Fifth row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{52}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Sixth row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{62}(x)}{\partial x_i} = \frac{-\frac{\partial M_{23}}{\partial x_i} detM + M_{23}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Seventh row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{72}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Eighth row of the Jacobian matrix $\nabla_x g_2(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{82}(x)}{\partial x_i} = \frac{\frac{\partial M_{24}}{\partial x_i} detM - M_{24}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Computation of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$.

First row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{13}(x)}{\partial x_1} = 0$ for $i = 1, 2, \cdots, 8$.

Second row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{23}(x)}{\partial x_i} = \frac{\frac{\partial M_{31}}{\partial x_i} detM - M_{31}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Third row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{33}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Fourth row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{43}(x)}{\partial x_i} = \frac{-\frac{\partial M_{32}}{\partial x_i} detM + M_{32}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Fifth row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{53}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Sixth row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{63}(x)}{\partial x_i} = \frac{\frac{\partial M_{33}}{\partial x_i} detM - M_{33}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Seventh row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{73}(x)}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Eighth row of the Jacobian matrix $\nabla_x g_3(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{83}(x)}{\partial x_i} = \frac{-\frac{\partial M_{34}}{\partial x_i} detM + M_{34}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i = 1, 2, \cdots, 8$

Computation of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$.

First row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{14}(x)}{\partial x_1}=0$ for $i=1,2,\cdots,8$.

Second row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{24}(x)}{\partial x_i}=\frac{-\frac{\partial M_{41}}{\partial x_i}detM+M_{41}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i=1,2,\cdots,8$

Third row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{34}(x)}{\partial x_i}=0$, for $i=1,2,\cdots,8$.

Fourth row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{44}(x)}{\partial x_i}=\frac{\frac{\partial M_{42}}{\partial x_i}detM-M_{42}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i=1,2,\cdots,8$

Fifth row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{54}(x)}{\partial x_i}=0$, for $i=1,2,\cdots,8$.

Sixth row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{64}(x)}{\partial x_i}=\frac{-\frac{\partial M_{33}}{\partial x_i}detM+M_{33}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i=1,2,\cdots,8$

Seventh row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{74}(x)}{\partial x_i}=0$, for $i=1,2,\cdots,8$.

Eighth row of the Jacobian matrix $\nabla_x g_4(x)\mid_{(x^*,u^*)}$: $\frac{\partial g_{84}(x)}{\partial x_i}=\frac{\frac{\partial M_{44}}{\partial x_i}detM-M_{44}\frac{\partial detM}{\partial x_i}}{detM^2}$, for $i=1,2,\cdots,8$

Next, one computes the partial derivatives of the sub-determinants M_{ij} and of the determinant $detM$:

$$\begin{array}{c}
\frac{\partial M_{11}}{\partial x_i}=\frac{\partial m_{22}}{\partial x_i}(m_{33}m_{44}-m_{43}m_{34})+\\
+m_{22}(\frac{\partial m_{33}}{\partial x_i}m_{44}+m_{33}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{43}}{\partial x_i}m_{34}-m_{43}\frac{\partial m_{34}}{\partial x_i})-\\
-\frac{\partial m_{23}}{\partial x_i}(m_{32}m_{44}-m_{42}m_{34}-)\\
-m_{23}(\frac{\partial m_{33}}{\partial x_i}m_{44}+m_{32}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{42}}{\partial x_i}m_{34}-m_{42}\frac{\partial m_{34}}{\partial x_i})+\\
+\frac{\partial m_{24}}{\partial x_i}(m_{32}m_{43}-m_{42}m_{33})+\\
+m_{24}(\frac{\partial m_{32}}{\partial x_i}m_{43}+m_{32}\frac{\partial m_{43}}{\partial x_i}-\frac{\partial m_{42}}{\partial x_i}m_{33}-m_{42}\frac{\partial m_{33}}{\partial x_i})
\end{array}\tag{35}$$

Equivalently, one has

$$\begin{array}{c}
\frac{\partial M_{12}}{\partial x_i}=\frac{\partial m_{21}}{\partial x_i}(m_{33}m_{44}-m_{43}m_{34})+\\
+m_{21}(\frac{\partial m_{13}}{\partial x_i}m_{44}+m_{13}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{43}}{\partial x_i}m_{34}-m_{43}\frac{\partial m_{34}}{\partial x_i})-\\
-\frac{\partial m_{23}}{\partial x_i}(m_{31}m_{44}-m_{41}m_{34}-)\\
-m_{23}(\frac{\partial m_{31}}{\partial x_i}m_{44}+m_{31}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{41}}{\partial x_i}m_{34}-m_{41}\frac{\partial m_{34}}{\partial x_i})+\\
+\frac{\partial m_{24}}{\partial x_i}(m_{31}m_{43}-m_{41}m_{33})+\\
+m_{24}(\frac{\partial m_{31}}{\partial x_i}m_{43}+m_{31}\frac{\partial m_{43}}{\partial x_i}-\frac{\partial m_{41}}{\partial x_i}m_{33}-m_{41}\frac{\partial m_{33}}{\partial x_i})
\end{array}\tag{36}$$

Moreover, it holds that

$$\begin{array}{c}
\frac{\partial M_{13}}{\partial x_i}=\frac{\partial m_{21}}{\partial x_i}(m_{32}m_{44}-m_{42}m_{34})+\\
+m_{21}(\frac{\partial m_{32}}{\partial x_i}m_{44}+m_{32}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{43}}{\partial x_i}m_{34}-m_{43}\frac{\partial m_{34}}{\partial x_i})-\\
-\frac{\partial m_{23}}{\partial x_i}(m_{31}m_{44}-m_{41}m_{34}-)\\
-m_{23}(\frac{\partial m_{31}}{\partial x_i}m_{44}+m_{31}\frac{\partial m_{44}}{\partial x_i}-\frac{\partial m_{41}}{\partial x_i}m_{34}-m_{41}\frac{\partial m_{34}}{\partial x_i})+\\
+\frac{\partial m_{24}}{\partial x_i}(m_{31}m_{42}-m_{41}m_{32})+\\
+m_{24}(\frac{\partial m_{31}}{\partial x_i}m_{42}+m_{31}\frac{\partial m_{42}}{\partial x_i}-\frac{\partial m_{41}}{\partial x_i}m_{32}-m_{41}\frac{\partial m_{32}}{\partial x_i})
\end{array}\tag{37}$$

Additionally, it holds that

$$
\begin{gathered}
\frac{\partial M_{14}}{\partial x_i} = \frac{\partial m_{21}}{\partial x_i}(m_{32}m_{43} - m_{42}m_{33}) + \\
+m_{21}(\frac{\partial m_{32}}{\partial x_i}m_{43} + m_{32}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{44}}{\partial x_i}m_{33} - m_{42}\frac{\partial m_{32}}{\partial x_i}) - \\
-\frac{\partial m_{22}}{\partial x_i}(m_{31}m_{43} - m_{41}m_{33}-) \\
-m_{22}(\frac{\partial m_{31}}{\partial x_i}m_{43} + m_{31}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{33}}{\partial x_i}m_{33} - m_{41}\frac{\partial m_{33}}{\partial x_i}) + \\
+\frac{\partial m_{23}}{\partial x_i}(m_{31}m_{42} - m_{41}m_{32}) + \\
+m_{23}(\frac{\partial m_{31}}{\partial x_i}m_{42} + m_{31}\frac{\partial m_{42}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{32} - m_{41}\frac{\partial m_{32}}{\partial x_i})
\end{gathered}
\tag{38}
$$

In a similar manner one obtains

$$
\begin{gathered}
\frac{\partial M_{21}}{\partial x_i} = \frac{\partial m_{12}}{\partial x_i}(m_{33}m_{44} - m_{43}m_{34}) + \\
+m_{12}(\frac{\partial m_{33}}{\partial x_i}m_{44} + m_{33}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{43}}{\partial x_i}m_{33} - m_{43}\frac{\partial m_{34}}{\partial x_i}) - \\
-\frac{\partial m_{13}}{\partial x_i}(m_{32}m_{44} - m_{42}m_{34}-) \\
-m_{13}(\frac{\partial m_{32}}{\partial x_i}m_{44} + m_{32}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{34} - m_{42}\frac{\partial m_{34}}{\partial x_i}) + \\
+\frac{\partial m_{14}}{\partial x_i}(m_{32}m_{43} - m_{42}m_{33}) + \\
+m_{14}(\frac{\partial m_{32}}{\partial x_i}m_{43} + m_{32}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{33} - m_{42}\frac{\partial m_{33}}{\partial x_i})
\end{gathered}
\tag{39}
$$

Equivalently one has

$$
\begin{gathered}
\frac{\partial M_{22}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{23}m_{24} - m_{43}m_{44}) + \\
+m_{11}(\frac{\partial m_{23}}{\partial x_i}m_{24} + m_{23}\frac{\partial m_{24}}{\partial x_i} - \frac{\partial m_{43}}{\partial x_i}m_{44} - m_{43}\frac{\partial m_{44}}{\partial x_i}) - \\
-\frac{\partial m_{13}}{\partial x_i}(m_{31}m_{44} - m_{41}m_{24}-) \\
-m_{13}(\frac{\partial m_{31}}{\partial x_i}m_{44} + m_{31}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{24} - m_{41}\frac{\partial m_{24}}{\partial x_i}) + \\
+\frac{\partial m_{14}}{\partial x_i}(m_{31}m_{43} - m_{41}m_{23}) + \\
+m_{14}(\frac{\partial m_{31}}{\partial x_i}m_{43} + m_{31}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{23} - m_{41}\frac{\partial m_{23}}{\partial x_i})
\end{gathered}
\tag{40}
$$

Following this procedure one gets

$$
\begin{gathered}
\frac{\partial M_{23}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{32}m_{44} - m_{42}m_{34}) + \\
+m_{11}(\frac{\partial m_{32}}{\partial x_i}m_{44} + m_{32}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{34} - m_{42}\frac{\partial m_{34}}{\partial x_i}) - \\
-\frac{\partial m_{12}}{\partial x_i}(m_{31}m_{43} - m_{41}m_{33}-) \\
-m_{12}(\frac{\partial m_{31}}{\partial x_i}m_{43} + m_{31}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{33} - m_{41}\frac{\partial m_{33}}{\partial x_i}) + \\
+\frac{\partial m_{14}}{\partial x_i}(m_{31}m_{42} - m_{41}m_{32}) + \\
+m_{14}(\frac{\partial m_{31}}{\partial x_i}m_{43} + m_{31}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{32} - m_{41}\frac{\partial m_{32}}{\partial x_i})
\end{gathered}
\tag{41}
$$

Additionally, it holds that

$$
\begin{gathered}
\frac{\partial M_{24}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{32}m_{43} - m_{42}m_{33}) + \\
+m_{11}(\frac{\partial m_{32}}{\partial x_i}m_{43} + m_{32}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{33} - m_{42}\frac{\partial m_{33}}{\partial x_i}) - \\
-\frac{\partial m_{12}}{\partial x_i}(m_{31}m_{43} - m_{41}m_{33}-) \\
-m_{12}(\frac{\partial m_{31}}{\partial x_i}m_{43} + m_{31}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{33} - m_{41}\frac{\partial m_{33}}{\partial x_i}) + \\
+\frac{\partial m_{13}}{\partial x_i}(m_{31}m_{42} - m_{41}m_{22}) + \\
+m_{13}(\frac{\partial m_{31}}{\partial x_i}m_{42} + m_{31}\frac{\partial m_{42}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{22} - m_{41}\frac{\partial m_{22}}{\partial x_i})
\end{gathered}
\tag{42}
$$

In this context one obtains

$$
\begin{gathered}
\frac{\partial M_{31}}{\partial x_i} = \frac{\partial m_{12}}{\partial x_i}(m_{23}m_{44} - m_{43}m_{24}) + \\
+m_{12}(\frac{\partial m_{23}}{\partial x_i}m_{44} + m_{23}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{43}}{\partial x_i}m_{24} - m_{43}\frac{\partial m_{24}}{\partial x_i}) - \\
-\frac{\partial m_{13}}{\partial x_i}(m_{22}m_{44} - m_{42}m_{24}-) \\
-m_{13}(\frac{\partial m_{22}}{\partial x_i}m_{44} + m_{22}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{24} - m_{42}\frac{\partial m_{24}}{\partial x_i}) + \\
+\frac{\partial m_{14}}{\partial x_i}(m_{22}m_{43} - m_{42}m_{23}) + \\
+m_{14}(\frac{\partial m_{32}}{\partial x_i}m_{43} + m_{22}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{23} - m_{42}\frac{\partial m_{23}}{\partial x_i})
\end{gathered}
\tag{43}
$$

Additionally, one has

$$\begin{array}{c}\frac{\partial M_{32}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{23}m_{44} - m_{42}m_{24}) + \\ +m_{11}(\frac{\partial m_{23}}{\partial x_i}m_{44} + m_{23}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{24} - m_{42}\frac{\partial m_{24}}{\partial x_i}) - \\ -\frac{\partial m_{13}}{\partial x_i}(m_{12}m_{44} - m_{41}m_{24} -) \\ -m_{13}(\frac{\partial m_{12}}{\partial x_i}m_{44} + m_{12}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{24} - m_{41}\frac{\partial m_{24}}{\partial x_i}) + \\ +\frac{\partial m_{14}}{\partial x_i}(m_{12}m_{43} - m_{41}m_{23}) + \\ +m_{14}(\frac{\partial m_{12}}{\partial x_i}m_{43} + m_{12}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{23} - m_{41}\frac{\partial m_{23}}{\partial x_i})\end{array} \tag{44}$$

Furthermore, one has

$$\begin{array}{c}\frac{\partial M_{33}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{22}m_{44} - m_{42}m_{24}) + \\ +m_{11}(\frac{\partial m_{22}}{\partial x_i}m_{44} + m_{22}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{24} - m_{42}\frac{\partial m_{24}}{\partial x_i}) - \\ -\frac{\partial m_{12}}{\partial x_i}(m_{21}m_{44} - m_{41}m_{24} -) \\ -m_{12}(\frac{\partial m_{21}}{\partial x_i}m_{44} + m_{21}\frac{\partial m_{44}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{24} - m_{41}\frac{\partial m_{24}}{\partial x_i}) + \\ +\frac{\partial m_{14}}{\partial x_i}(m_{41}m_{22} - m_{21}m_{42}) + \\ +m_{14}(\frac{\partial m_{41}}{\partial x_i}m_{22} + m_{41}\frac{\partial m_{22}}{\partial x_i} - \frac{\partial m_{21}}{\partial x_i}m_{42} - m_{21}\frac{\partial m_{42}}{\partial x_i})\end{array} \tag{45}$$

Continuing in this manner one gets

$$\begin{array}{c}\frac{\partial M_{34}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{22}m_{43} - m_{42}m_{23}) + \\ +m_{11}(\frac{\partial m_{22}}{\partial x_i}m_{43} + m_{22}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{42}}{\partial x_i}m_{23} - m_{42}\frac{\partial m_{23}}{\partial x_i}) - \\ -\frac{\partial m_{12}}{\partial x_i}(m_{21}m_{43} - m_{41}m_{23} -) \\ -m_{12}(\frac{\partial m_{21}}{\partial x_i}m_{43} + m_{21}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{23} - m_{41}\frac{\partial m_{23}}{\partial x_i}) + \\ +\frac{\partial m_{13}}{\partial x_i}(m_{21}m_{42} - m_{41}m_{22}) + \\ +m_{13}(\frac{\partial m_{22}}{\partial x_i}m_{42} + m_{22}\frac{\partial m_{42}}{\partial x_i} - \frac{\partial m_{41}}{\partial x_i}m_{22} - m_{41}\frac{\partial m_{22}}{\partial x_i})\end{array} \tag{46}$$

Besides, one has

$$\begin{array}{c}\frac{\partial M_{41}}{\partial x_i} = \frac{\partial m_{12}}{\partial x_i}(m_{22}m_{34} - m_{33}m_{24}) + \\ +m_{12}(\frac{\partial m_{22}}{\partial x_i}m_{34} + m_{22}\frac{\partial m_{34}}{\partial x_i} - \frac{\partial m_{33}}{\partial x_i}m_{24} - m_{33}\frac{\partial m_{24}}{\partial x_i}) - \\ -\frac{\partial m_{13}}{\partial x_i}(m_{23}m_{34} - m_{32}m_{23} -) \\ -m_{13}(\frac{\partial m_{22}}{\partial x_i}m_{34} + m_{22}\frac{\partial m_{34}}{\partial x_i} - \frac{\partial m_{32}}{\partial x_i}m_{23} - m_{32}\frac{\partial m_{23}}{\partial x_i}) + \\ +\frac{\partial m_{14}}{\partial x_i}(m_{22}m_{33} - m_{32}m_{23}) + \\ +m_{14}(\frac{\partial m_{22}}{\partial x_i}m_{33} + m_{22}\frac{\partial m_{33}}{\partial x_i} - \frac{\partial m_{32}}{\partial x_i}m_{23} - m_{32}\frac{\partial m_{23}}{\partial x_i})\end{array} \tag{47}$$

Equivalently, one obtains

$$\begin{array}{c}\frac{\partial M_{42}}{\partial x_i} = \frac{\partial m_{11}}{\partial x_i}(m_{32}m_{43} - m_{33}m_{42}) + \\ +m_{11}(\frac{\partial m_{32}}{\partial x_i}m_{43} + m_{32}\frac{\partial m_{42}}{\partial x_i} - \frac{\partial m_{33}}{\partial x_i}m_{42} - m_{33}\frac{\partial m_{42}}{\partial x_i}) - \\ -\frac{\partial m_{31}}{\partial x_i}(m_{12}m_{43} - m_{13}m_{42} -) \\ -m_{31}(\frac{\partial m_{12}}{\partial x_i}m_{43} + m_{12}\frac{\partial m_{43}}{\partial x_i} - \frac{\partial m_{13}}{\partial x_i}m_{42} - m_{13}\frac{\partial m_{42}}{\partial x_i}) + \\ +\frac{\partial m_{41}}{\partial x_i}(m_{12}m_{33} - m_{13}m_{32}) + \\ +m_{41}(\frac{\partial m_{12}}{\partial x_i}m_{33} + m_{12}\frac{\partial m_{33}}{\partial x_i} - \frac{\partial m_{13}}{\partial x_i}m_{32} - m_{13}\frac{\partial m_{32}}{\partial x_i})\end{array} \tag{48}$$

In a similar manner one gets

$$\begin{gathered}
\frac{\partial M_{43}}{\partial x_i}=\frac{\partial m_{11}}{\partial x_i}(m_{22}m_{34}-m_{32}m_{24})+ \\
+m_{11}(\frac{\partial m_{22}}{\partial x_i}m_{34}+m_{22}\frac{\partial m_{34}}{\partial x_i}-\frac{\partial m_{32}}{\partial x_i}m_{24}-m_{32}\frac{\partial m_{24}}{\partial x_i})- \\
-\frac{\partial m_{12}}{\partial x_i}(m_{21}m_{34}-m_{31}m_{24}-) \\
-m_{12}(\frac{\partial m_{21}}{\partial x_i}m_{34}+m_{21}\frac{\partial m_{34}}{\partial x_i}-\frac{\partial m_{31}}{\partial x_i}m_{24}-m_{31}\frac{\partial m_{24}}{\partial x_i})+ \\
+\frac{\partial m_{14}}{\partial x_i}(m_{21}m_{32}-m_{31}m_{22})+ \\
+m_{14}(\frac{\partial m_{21}}{\partial x_i}m_{32}+m_{21}\frac{\partial m_{32}}{\partial x_i}-\frac{\partial m_{31}}{\partial x_i}m_{22}-m_{31}\frac{\partial m_{22}}{\partial x_i})
\end{gathered} \tag{49}$$

Finally, one has that

$$\begin{gathered}
\frac{\partial M_{44}}{\partial x_i}=\frac{\partial m_{22}}{\partial x_i}(m_{22}m_{33}-m_{32}m_{23})+ \\
+m_{11}(\frac{\partial m_{22}}{\partial x_i}m_{33}+m_{22}\frac{\partial m_{33}}{\partial x_i}-\frac{\partial m_{32}}{\partial x_i}m_{23}-m_{32}\frac{\partial m_{23}}{\partial x_i})- \\
-\frac{\partial m_{12}}{\partial x_i}(m_{22}m_{33}-m_{31}m_{23}-) \\
-m_{12}(\frac{\partial m_{22}}{\partial x_i}m_{33}+m_{23}\frac{\partial m_{33}}{\partial x_i}-\frac{\partial m_{31}}{\partial x_i}m_{23}-m_{31}\frac{\partial m_{23}}{\partial x_i})+ \\
+\frac{\partial m_{13}}{\partial x_i}(m_{21}m_{32}-m_{31}m_{22})+ \\
+m_{13}(\frac{\partial m_{21}}{\partial x_i}m_{32}+m_{21}\frac{\partial m_{32}}{\partial x_i}-\frac{\partial m_{31}}{\partial x_i}m_{22}-m_{31}\frac{\partial m_{22}}{\partial x_i})
\end{gathered} \tag{50}$$

About the partial derivatives of the determinant $detM$ one has for $i=1,2,\cdots,8$

$$\begin{gathered}
\frac{\partial detM}{\partial x_i}=\frac{\partial m_{11}}{\partial x_i}M_{11}+m_{11}\frac{\partial M_{11}}{\partial x_i}-\frac{\partial m_{12}}{\partial x_i}M_{12}-m_{12}\frac{\partial M_{12}}{\partial x_i}+ \\
+\frac{\partial m_{13}}{\partial x_i}M_{13}+m_{13}\frac{\partial M_{13}}{\partial x_i}-\frac{\partial m_{14}}{\partial x_i}M_{14}-m_{14}\frac{\partial M_{14}}{\partial x_i}
\end{gathered} \tag{51}$$

Next, the derivatives of the elements of the inertia matrix M are computed.

It holds that $m_{11}=p_1+p_2cos(x_3)$. Thus one has: $\frac{\partial m_{11}}{\partial x_1}=0$, $\frac{\partial m_{11}}{\partial x_2}=0$, $\frac{\partial m_{11}}{\partial x_3}=-p_2sin(x_3)$, $\frac{\partial m_{11}}{\partial x_4}=0$, $\frac{\partial m_{11}}{\partial x_5}=0$, $\frac{\partial m_{11}}{\partial x_6}=0$, $\frac{\partial m_{11}}{\partial x_7}=0$, $\frac{\partial m_{11}}{\partial x_8}=0$.

Besides, it holds that $m_{12}=m_{21}=p_3+0.5p_2sin(x_3)x_4$, thus $\frac{\partial m_{12}}{\partial x_1}=\frac{\partial m_{21}}{\partial x_1}=0$, $\frac{\partial m_{12}}{\partial x_2}=\frac{\partial m_{21}}{\partial x_2}=0$, $\frac{\partial m_{12}}{\partial x_3}=\frac{\partial m_{21}}{\partial x_3}=0.5p_2cos(x_3)x_4$, $\frac{\partial m_{12}}{\partial x_4}=\frac{\partial m_{21}}{\partial x_4}=0.5p_2sin(x_3)$, $\frac{\partial m_{12}}{\partial x_5}=\frac{\partial m_{21}}{\partial x_5}=0$, $\frac{\partial m_{12}}{\partial x_6}=\frac{\partial m_{21}}{\partial x_6}=0$, $\frac{\partial m_{12}}{\partial x_7}=\frac{\partial m_{21}}{\partial x_7}=0$, $\frac{\partial m_{12}}{\partial x_8}=\frac{\partial m_{21}}{\partial x_8}=0$.

Moreover, it holds that $m_{13}=m_{31}=0$, thus $\frac{\partial m_{13}}{\partial x_1}=\frac{\partial m_{31}}{\partial x_1}=0$, $\frac{\partial m_{13}}{\partial x_2}=\frac{\partial m_{31}}{\partial x_2}=0$, $\frac{\partial m_{13}}{\partial x_3}=\frac{\partial m_{31}}{\partial x_3}=0$, $\frac{\partial m_{13}}{\partial x_4}=\frac{\partial m_{31}}{\partial x_4}=0$, $\frac{\partial m_{13}}{\partial x_5}=\frac{\partial m_{31}}{\partial x_5}=0$, $\frac{\partial m_{13}}{\partial x_6}=\frac{\partial m_{31}}{\partial x_6}=0$, $\frac{\partial m_{13}}{\partial x_7}=\frac{\partial m_{31}}{\partial x_7}=0$, $\frac{\partial m_{13}}{\partial x_8}=\frac{\partial m_{31}}{\partial x_8}=0$.

Additionally, it holds that $m_{14}=m_{41}=-p_5$, thus $\frac{\partial m_{14}}{\partial x_1}=\frac{\partial m_{41}}{\partial x_i}=0$ $\frac{\partial m_{14}}{\partial x_2}=\frac{\partial m_{41}}{\partial x_2}=0$, $\frac{\partial m_{14}}{\partial x_3}=\frac{\partial m_{41}}{\partial x_3}=0$, $\frac{\partial m_{14}}{\partial x_5}=\frac{\partial m_{41}}{\partial x_5}=0$, $\frac{\partial m_{14}}{\partial x_6}=\frac{\partial m_{41}}{\partial x_6}=0$, $\frac{\partial m_{14}}{\partial x_7}=\frac{\partial m_{41}}{\partial x_7}=0$, $\frac{\partial m_{14}}{\partial x_8}=\frac{\partial m_{41}}{\partial x_8}=0$

Moreover, it holds that $m_{22}=p_2$, thus $\frac{\partial m_{22}}{\partial x_1}=0$, $\frac{\partial m_{22}}{\partial x_2}=0$, $\frac{\partial m_{22}}{\partial x_3}=0$, $\frac{\partial m_{22}}{\partial x_4}=0$, $\frac{\partial m_{22}}{\partial x_5}=0$, $\frac{\partial m_{22}}{\partial x_6}=0$, $\frac{\partial m_{22}}{\partial x_7}=0$, $\frac{\partial m_{22}}{\partial x_8}=0$

Furthermore, it holds that $m_{23}=m_{32}=0$, thus $\frac{\partial m_{23}}{\partial x_1}=\frac{\partial m_{32}}{\partial x_1}=\frac{\partial m_{23}}{\partial x_2}=\frac{\partial m_{32}}{\partial x_2}=0$, $\frac{\partial m_{23}}{\partial x_3}=\frac{\partial m_{32}}{\partial x_3}=0$, $\frac{\partial m_{23}}{\partial x_4}=\frac{\partial m_{32}}{\partial x_4}=0$, $\frac{\partial m_{23}}{\partial x_5}=\frac{\partial m_{32}}{\partial x_5}=0$, $\frac{\partial m_{23}}{\partial x_6}=\frac{\partial m_{32}}{\partial x_6}=0$, $\frac{\partial m_{23}}{\partial x_7}=\frac{\partial m_{32}}{\partial x_7}=0$, $\frac{\partial m_{23}}{\partial x_8}=\frac{\partial m_{32}}{\partial x_8}=0$.

Besides, it holds that $m_{24}=m_{42}=-p_5$, thus $\frac{\partial m_{24}}{\partial x_1}=\frac{\partial m_{42}}{\partial x_1}=\frac{\partial m_{24}}{\partial x_2}=\frac{\partial m_{42}}{\partial x_2}=0$, $\frac{\partial m_{24}}{\partial x_3}=\frac{\partial m_{42}}{\partial x_3}=0$, $\frac{\partial m_{24}}{\partial x_4}=\frac{\partial m_{42}}{\partial x_4}=0$, $\frac{\partial m_{24}}{\partial x_5}=\frac{\partial m_{42}}{\partial x_5}=0$, $\frac{\partial m_{24}}{\partial x_6}=\frac{\partial m_{42}}{\partial x_6}=0$, $\frac{\partial m_{24}}{\partial x_7}=\frac{\partial m_{42}}{\partial x_7}=0$, $\frac{\partial m_{24}}{\partial x_8}=\frac{\partial m_{42}}{\partial x_8}=0$.

Moreover, it holds that $m_{33}=p_4$, thus $\frac{\partial m_{33}}{\partial x_1}=0$, $\frac{\partial m_{33}}{\partial x_2}=0$, $\frac{\partial m_{33}}{\partial x_3}=0$, $\frac{\partial m_{33}}{\partial x_4}=0$, $\frac{\partial m_{33}}{\partial x_5}=0$, $\frac{\partial m_{33}}{\partial x_6}=0$, $\frac{\partial m_{33}}{\partial x_7}=0$, $\frac{\partial m_{33}}{\partial x_8}=0$

Additionally, it holds that $m_{34} = m_{43} =$, thus $\frac{\partial m_{34}}{\partial x_1} = \frac{\partial m_{43}}{\partial x_1} =$, $\frac{\partial m_{34}}{\partial x_2} = \frac{\partial m_{43}}{\partial x_2} =$, $\frac{\partial m_{34}}{\partial x_3} = \frac{\partial m_{43}}{\partial x_3} =$, $\frac{\partial m_{34}}{\partial x_4} = \frac{\partial m_{43}}{\partial x_4} =$, $\frac{\partial m_{34}}{\partial x_5} = \frac{\partial m_{43}}{\partial x_5} =$, $\frac{\partial m_{34}}{\partial x_6} = \frac{\partial m_{43}}{\partial x_6} =$, $\frac{\partial m_{34}}{\partial x_7} = \frac{\partial m_{43}}{\partial x_7} =$,. $\frac{\partial m_{34}}{\partial x_8} = \frac{\partial m_{43}}{\partial x_8} =$.

Finally, it holds that $m_{44} = p_5$, thus $\frac{\partial m_{44}}{\partial x_1} = 0$, $\frac{\partial m_{44}}{\partial x_2} = 0$, $\frac{\partial m_{44}}{\partial x_3} = 0$, $\frac{\partial m_{44}}{\partial x_4} = 0$, $\frac{\partial m_{44}}{\partial x_5} = 0$, $\frac{\partial m_{44}}{\partial x_6} = 0$, $\frac{\partial m_{44}}{\partial x_7} = 0$,.$\frac{\partial m_{44}}{\partial x_8} = 0$.

Finally, about the computation of the partial derivatives of the Coriolis forces vector one has

$$C(x,\dot{x})\dot{x} = \begin{pmatrix} c_{11}x_2 + c_{12}x_4 + c_{13}x_6 + c_{14}x_8 \\ c_{21}x_2 + c_{22}x_4 + c_{23}x_6 + c_{24}x_8 \\ c_{31}x_2 + c_{22}x_4 + c_{23}x_6 + c_{24}x_8 \\ c_{41}x_2 + c_{42}x_4 + c_{43}x_6 + c_{44}x_8 \end{pmatrix} \tag{52}$$

It holds that for $i = 1.3, 5, 7$

$$\frac{\partial c_1}{\partial x_i} = \frac{\partial c_{11}}{\partial x_i}x_2 + \frac{\partial c_{12}}{\partial x_i}x_4 + \frac{\partial c_{13}}{\partial x_i}x_6 + \frac{\partial c_{14}}{\partial x_i}x_8 \tag{53}$$

and also

$$\frac{\partial c_1}{\partial x_2} = \frac{\partial c_{11}}{\partial x_2}x_2 + c_{11} + \frac{\partial c_{12}}{\partial x_2}x_4 + \frac{\partial c_{13}}{\partial x_2}x_6 + \frac{\partial c_{14}}{\partial x_2}x_8 \tag{54}$$

$$\frac{\partial c_1}{\partial x_4} = \frac{\partial c_{11}}{\partial x_4}x_2 + \frac{\partial c_{12}}{\partial x_4}x_4 + c_{12} + \frac{\partial c_{13}}{\partial x_4}x_6 + \frac{\partial c_{14}}{\partial x_4}x_8 \tag{55}$$

$$\frac{\partial c_1}{\partial x_6} = \frac{\partial c_{11}}{\partial x_6}x_2 + \frac{\partial c_{12}}{\partial x_6}x_4 + \frac{\partial c_{13}}{\partial x_6}x_6 + c_{13} + \frac{\partial c_{14}}{\partial x_6}x_8 \tag{56}$$

$$\frac{\partial c_1}{\partial x_8} = \frac{\partial c_{11}}{\partial x_8}x_2 + \frac{\partial c_{12}}{\partial x_8}x_4 + \frac{\partial c_{13}}{\partial x_8}x_6 + \frac{\partial c_{14}}{\partial x_8}x_8 + c_{14} \tag{57}$$

Equivalently it holds that for $i = 1.3, 5, 7$

$$\frac{\partial c_2}{\partial x_i} = \frac{\partial c_{21}}{\partial x_i}x_2 + \frac{\partial c_{22}}{\partial x_i}x_4 + \frac{\partial c_{23}}{\partial x_i}x_6 + \frac{\partial c_{24}}{\partial x_i}x_8 \tag{58}$$

and also

$$\frac{\partial c_2}{\partial x_2} = \frac{\partial c_{21}}{\partial x_2}x_2 + c_{21} + \frac{\partial c_{22}}{\partial x_2}x_4 + \frac{\partial c_{23}}{\partial x_2}x_6 + \frac{\partial c_{24}}{\partial x_2}x_8 \tag{59}$$

$$\frac{\partial c_2}{\partial x_4} = \frac{\partial c_{21}}{\partial x_4}x_2 + \frac{\partial c_{22}}{\partial x_4}x_4 + c_{22} + \frac{\partial c_{23}}{\partial x_4}x_6 + \frac{\partial c_{24}}{\partial x_4}x_8 \tag{60}$$

$$\frac{\partial c_2}{\partial x_6} = \frac{\partial c_1}{\partial x_6}x_2 + \frac{\partial c_{22}}{\partial x_6}x_4 + \frac{\partial c_{23}}{\partial x_6}x_6 + c_{23} + \frac{\partial c_{24}}{\partial x_6}x_8 \tag{61}$$

$$\frac{\partial c_2}{\partial x_8} = \frac{\partial c_{21}}{\partial x_8}x_2 + \frac{\partial c_{22}}{\partial x_8}x_4 + \frac{\partial c_{23}}{\partial x_8}x_6 + \frac{\partial c_{24}}{\partial x_8}x_8 + c_{24} \tag{62}$$

Similarly, it holds that for $i = 1.3, 5, 7$

$$\frac{\partial c_3}{\partial x_i} = \frac{\partial c_{31}}{\partial x_i}x_2 + \frac{\partial c_{32}}{\partial x_i}x_4 + \frac{\partial c_{33}}{\partial x_i}x_6 + \frac{\partial c_{34}}{\partial x_i}x_8 \tag{63}$$

and also

$$\frac{\partial c_3}{\partial x_2} = \frac{\partial c_{31}}{\partial x_2}x_2 + c_{31} + \frac{\partial c_{32}}{\partial x_2}x_4 + \frac{\partial c_{33}}{\partial x_2}x_6 + \frac{\partial c_{34}}{\partial x_2}x_8 \tag{64}$$

$$\frac{\partial c_3}{\partial x_4} = \frac{\partial c_{31}}{\partial x_4}x_2 + \frac{\partial c_{32}}{\partial x_4}x_4 + c_{32} + \frac{\partial c_{33}}{\partial x_4}x_6 + \frac{\partial c_{34}}{\partial x_4}x_8 \tag{65}$$

$$\frac{\partial c_3}{\partial x_6} = \frac{\partial c_3}{\partial x_6}x_2 + \frac{\partial c_{32}}{\partial x_6}x_4 + \frac{\partial c_{33}}{\partial x_6}x_6 + c_{33} + \frac{\partial c_{34}}{\partial x_6}x_8 \tag{66}$$

$$\frac{\partial c_3}{\partial x_8} = \frac{\partial c_{31}}{\partial x_8}x_2 + \frac{\partial c_{32}}{\partial x_8}x_4 + \frac{\partial c_{33}}{\partial x_8}x_6 + \frac{\partial c_{34}}{\partial x_8}x_8 + c_{34} \tag{67}$$

Finally, it holds that for $i = 1.3, 5, 7$

$$\frac{\partial c_4}{\partial x_i} = \frac{\partial c_{41}}{\partial x_i}x_2 + \frac{\partial c_{42}}{\partial x_i}x_4 + \frac{\partial c_{43}}{\partial x_i}x_6 + \frac{\partial c_{44}}{\partial x_i}x_8 \tag{68}$$

and also

$$\frac{\partial c_4}{\partial x_2} = \frac{\partial c_{41}}{\partial x_2}x_2 + c_{41} + \frac{\partial c_{42}}{\partial x_2}x_4 + \frac{\partial c_{43}}{\partial x_2}x_6 + \frac{\partial c_{44}}{\partial x_2}x_8 \tag{69}$$

$$\frac{\partial c_4}{\partial x_4} = \frac{\partial c_{41}}{\partial x_4}x_2 + \frac{\partial c_{42}}{\partial x_4}x_4 + c_{42} + \frac{\partial c_{43}}{\partial x_4}x_6 + \frac{\partial c_{44}}{\partial x_4}x_8 \tag{70}$$

$$\frac{\partial c_4}{\partial x_6} = \frac{\partial c_4}{\partial x_6}x_2 + \frac{\partial c_{42}}{\partial x_6}x_4 + \frac{\partial c_{43}}{\partial x_6}x_6 + c_{43} + \frac{\partial c_{44}}{\partial x_6}x_8 \tag{71}$$

$$\frac{\partial c_4}{\partial x_8} = \frac{\partial c_{41}}{\partial x_8}x_2 + \frac{\partial c_{42}}{\partial x_8}x_4 + \frac{\partial c_{43}}{\partial x_8}x_6 + \frac{\partial c_{44}}{\partial x_8}x_8 + c_{44} \tag{72}$$

Next, the following partial derivatives of the elements c_{ij} $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$ of the Coriolis matrix are computed.

It holds $c_{11} = -p_2 sin(x_1)x_4$, thus one has that: $\frac{\partial c_{11}}{\partial x_1} = -p_2 cos(x_1)x_4$, $\frac{\partial c_{11}}{\partial x_2} = 0$, $\frac{\partial c_{11}}{\partial x_3} = 0$, $\frac{\partial c_{11}}{\partial x_4} = -o_2 sin(x_1)$, $\frac{\partial c_{11}}{\partial x_5} = 0$, $\frac{\partial c_{11}}{\partial x_6} = 0$, $\frac{\partial c_{11}}{\partial x_7} = 0$, $\frac{\partial c_{11}}{\partial x_8} = 0$.

Additionally, it holds that $c_{12} = -0.5p_2 sin(x_3)x_4$, thus $\frac{\partial c_{12}}{\partial x_1} = 0$, $\frac{\partial c_{11}}{\partial x_2} = 0$, $\frac{\partial c_{12}}{\partial x_3} = -0.5p_2 cos(x_3)x_4$, $\frac{\partial c_{12}}{\partial x_4} = -0.5p_2 sin(x_3)$, $\frac{\partial c_{12}}{\partial x_5} = 0$, $\frac{\partial c_{12}}{\partial x_6} = 0$, $\frac{\partial c_{12}}{\partial x_7} = 0$, $\frac{\partial c_{11}}{\partial x_8} = 0$.

Moreover, it holds that $c_{13} = 0$, thus one obtains: $\frac{\partial c_{13}}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$

Additionally, it holds that $c_{14} = 0$, thus one obtains: $\frac{\partial c_{14}}{\partial x_i} =$, for $i = 1, \cdots, 8$

Additionally, it holds that $c_{21} = 0.5p_2 sin(x_3)x_2$, thus, one obtains: $\frac{\partial c_{21}}{\partial x_1} = 0$, $\frac{\partial c_{21}}{\partial x_2} = 0.5p_2 sin(x_3)$, $\frac{\partial c_{21}}{\partial x_3} = 0.5p_2 cos(x_3)x_2$, $\frac{\partial c_{21}}{\partial x_4} = 0$, $\frac{\partial c_{21}}{\partial x_5} = 0$, $\frac{\partial c_{21}}{\partial x_6} = 0$, $\frac{\partial c_{21}}{\partial x_7} = 0$, $\frac{\partial c_{21}}{\partial x_8} = 0$.

Additionally, it holds that $c_{22} = 0$ thus one obtains: $\frac{\partial c_{22}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$

Moreover, it holds that $c_{23} = 0$, thus, one obtains: $\frac{\partial c_{23}}{\partial x_i} = 0$ for $i = 1, 2, \cdots, 8$.

Furthermore, it holds that $c_{24} = 0$ thus, one obtains $\frac{\partial c_{24}}{\partial x_i} = 0$ for $i = 1, 2, \cdots, 8$

Additionally, one has that $c_{31} = 0$ thus, one obtains: $\frac{\partial c_{31}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$

Furthermore, it holds $c_{32} = 0$, thus, one obtains: $\frac{\partial c_{32}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$.

Moreover, it holds that $c_{33} = 0$ thus, one obtains: $\frac{\partial c_{33}}{\partial x_1} = 0$, for $i = 1, 2, \cdots, 8$

Additionally, it holds that $c_{34} = 0$, thus, one obtains: $\frac{\partial c_{34}}{\partial x_1} = 0$, for $i = 1, 2, \cdots, 8$

Furthermore, one has that $c_{41} = 0$ thus, one obtains: $\frac{\partial c_{41}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$

Moreover, it holds that $c_{42} = 0$ thus, one obtains: $\frac{\partial c_{42}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$

Furthermore, it holds that $c_{43} = 0$ thus, one obtains: $\frac{\partial c_{43}}{\partial x_i} =$, for $i = 1, 2, \cdots, 8$.

Finally, it holds that $c_{44} = 0$. Thus, one obtains $\frac{\partial c_{44}}{\partial x_i} = 0$, for $i = 1, \cdots, 8$.

In a similar manner one computes the partial derivatives of the elements of the gravitational forces matrix. It holds that $g_1 = 0$ that one obtains $\frac{\partial g_1}{\partial x_i} = 0$, for $i = 1, \cdots, 8$.

Additionally, it holds that $g_2 = 0$ that one obtains $\frac{\partial g_2}{\partial x_i} = 0$, for $i = 1, \cdots, 8$.

Moreover it holds that $g_3 = p_4 \bar{g}$, thus, one obtains $\frac{\partial g_3}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$.

Finally it holds that $g_4 = 0$ thus, one obtains $\frac{\partial g_4}{\partial x_i} = 0$, for $i = 1, 2, \cdots, 8$

In a similar manner one computes the partial derivatives of the elements of the disturbances vector $\tilde{d}$. It holds that $d_1 = b_1 x_2$ that one obtains $\frac{\partial d_1}{\partial x_i} = 0$, for $i = 1, \cdots, 8$ and $i \neq 2$, while $\frac{\partial d_1}{\partial x_2} = b_1$.

Additionally, it holds that $d_2 = b_2 x_4$ that one obtains $\frac{\partial d_2}{\partial x_i} = 0$, for $i = 1, \cdots, 8$ and $i \neq 4$, while $\frac{\partial d_2}{\partial x_4} = b_2$.

Moreover it holds that $d_3 = b_3 x_6$ that one obtains $\frac{\partial d_3}{\partial x_i} = 0$, for $i = 1, \cdots, 8$ and $i \neq 6$, while $\frac{\partial d_3}{\partial x_6} = b_3$.

Finally it holds that $d_4 = b_4 x_8$ that one obtains $\frac{\partial d_4}{\partial x_i} = 0$, for $i = 1, \cdots, 8$ and $i \neq 8$, while $\frac{\partial d_4}{\partial x_8} = b_4$.

4 Design of an H-infinity nonlinear feedback controller

4.1 Equivalent linearized dynamics of the 4-DOF SCARA robot

After linearization around its current operating point, the dynamic model for the 4-DOF SCARA robot is written as

$$\dot{x} = Ax + Bu + d_1 \tag{73}$$

Parameter d_1 stands for the linearization error in the 4-DOF SCARA robot's model that was given previously in Eq. (17). The reference setpoints for the state vector of the aforementioned dynamic model are denoted by $\mathbf{x_d} = [x_1^d, \cdots, x_8^d]$. Tracking of this trajectory is achieved after applying the control input u^*. At every time instant the control input u^* is assumed to differ from the control input u appearing in Eq. (73) by an amount equal to Δu, that is $u^* = u + \Delta u$

$$\dot{x}_d = Ax_d + Bu^* + d_2 \tag{74}$$

The dynamics of the controlled system described in Eq. (73) can be also written as

$$\dot{x} = Ax + Bu + Bu^* - Bu^* + d_1 \tag{75}$$

and by denoting $d_3 = -Bu^* + d_1$ as an aggregate disturbance term one obtains

$$\dot{x} = Ax + Bu + Bu^* + d_3 \tag{76}$$

By subtracting Eq. (74) from Eq. (76) one has

$$\dot{x} - \dot{x}_d = A(x - x_d) + Bu + d_3 - d_2 \tag{77}$$

By denoting the tracking error as $e = x - x_d$ and the aggregate disturbance term as $\tilde{d} = d_3 - d_2$, the tracking error dynamics becomes

$$\dot{e} = Ae + Bu + \tilde{d} \tag{78}$$

The above linearized form of the 4-DOF SCARA robot's model can be efficiently controlled after applying an H-infinity feedback control scheme.

4.2 The nonlinear H-infinity control

The initial nonlinear model of the 4-DOF SCARA robot is in the form

$$\dot{x} = f(x, u) \quad x \in R^n, \ u \in R^m \tag{79}$$

Linearization of the model of the 4-DOF SCARA robot is performed at each iteration of the control algorithm round its present operating point $(x^*, u^*) = (x(t), u(t - T_s))$. The linearized equivalent of the system is described by

$$\dot{x} = Ax + Bu + L\tilde{d} \quad x \in R^n, \ u \in R^m, \ \tilde{d} \in R^q \tag{80}$$

where matrices A and B are obtained from the computation of the previously defined Jacobians and vector $\tilde{d}$ denotes disturbance terms due to linearization errors. The problem of disturbance rejection for the linearized model that is described by

$$\begin{array}{c} \dot{x} = Ax + Bu + L\tilde{d} \\ y = Cx \end{array} \tag{81}$$

where $x \in R^n$, $u \in R^m$, $\tilde{d} \in R^q$ and $y \in R^p$, cannot be handled efficiently if the classical LQR control scheme is applied. This is because of the existence of the perturbation term $\tilde{d}$. The disturbance term $\tilde{d}$ apart from modeling (parametric) uncertainty and external perturbation terms can also represent noise terms of any distribution.

In the H_∞ control approach, a feedback control scheme is designed for trajectory tracking by the system's state vector and simultaneous disturbance rejection, considering that the disturbance affects the system in the worst possible manner. The disturbances' effects are incorporated in the following quadratic cost function:

$$J(t) = \frac{1}{2}\int_0^T [y^T(t)y(t) + ru^T(t)u(t) - \rho^2\tilde{d}^T(t)\tilde{d}(t)]dt, \quad r, \rho > 0 \tag{82}$$

The significance of the negative sign in the cost function's term that is associated with the perturbation variable $\tilde{d}(t)$ is that the disturbance tries to maximize the cost function $J(t)$ while the control signal $u(t)$ tries to minimize it. The physical meaning of the relation given above is that the control signal and the disturbances compete to each other within a min-max differential game. This problem of min-max optimization can be written as $min_u max_{\tilde{d}} J(u, \tilde{d})$.

The objective of the optimization procedure is to compute a control signal $u(t)$ which can compensate for the worst possible disturbance, that is externally imposed to the 4-DOF SCARA robot. However, the solution to the min-max optimization problem is directly related to the value of the parameter ρ. This means that there is an upper bound in the disturbances magnitude that can be annihilated by the control signal.

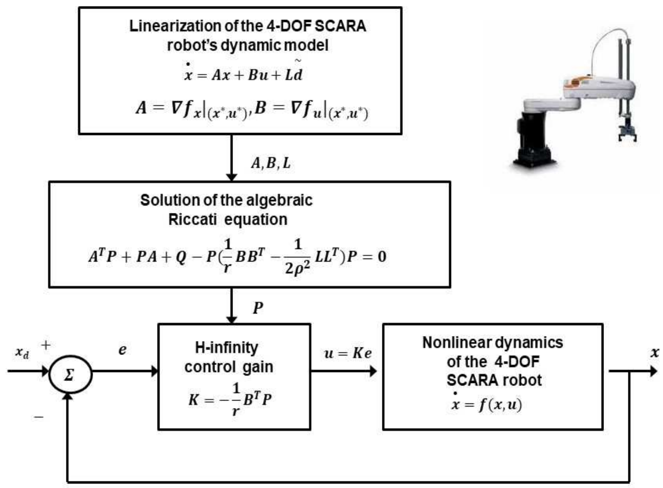

Figure 2: Diagram of the control scheme for the 4-DOF SCARA robotic manipulator

4.3 Computation of the feedback control gains

For the linearized system given by Eq. (81) the cost function of Eq. (82) is defined, where the coefficient r determines the penalization of the control input and the weight coefficient ρ determines the reward of the disturbances' effects. It is assumed that (i) The energy that is transferred from the disturbances signal $\tilde{d}(t)$ is bounded, that is $\int_0^\infty \tilde{d}^T(t)\tilde{d}(t)dt < \infty$, (ii) matrices $[A, B]$ and $[A, L]$ are stabilizable, (iii) matrix $[A, C]$ is detectable. In the case of a tracking problem the optimal feedback control law is given by

$$u(t) = -Ke(t) \tag{83}$$

with $e = x - x_d$ to be the tracking error, and $K = \frac{1}{r}B^T P$ where P is a positive definite symmetric matrix. As it will be proven in Section 5, matrix P is obtained from the solution of the Riccati equation

$$A^T P + PA + Q - P(\tfrac{2}{r}BB^T - \tfrac{1}{\rho^2}LL^T)P = 0 \tag{84}$$

where Q is a positive semi-definite symmetric matrix. The worst case disturbance is given by

$$\tilde{d}(t) = \tfrac{1}{\rho^2}L^T Pe(t) \tag{85}$$

The solution of the H-infinity feedback control problem for the 4-DOF SCARA robot and the computation of the worst case disturbance that the related controller can sustain, comes from superposition of Bellman's optimality principle when considering that the robot is affected by two separate inputs (i) the control input u (ii) the cumulative disturbance input $\tilde{d}(t)$. Solving the optimal control problem for u, that is for the minimum variation (optimal) control input that achieves elimination of the state vector's tracking error, gives $u = -\frac{1}{r}B^T Pe$. Equivalently, solving the optimal control problem for $\tilde{d}$, that is for the worst case disturbance that the control loop can sustain gives $\tilde{d} = \frac{1}{\rho^2}L^T Pe$.

The diagram of the considered control loop for the 4-DOF SCARA robot is depicted in Fig. 2.

5 Lyapunov stability analysis

5.1 Stability proof

Through Lyapunov stability analysis it will be shown that the proposed nonlinear control scheme assures H_∞ tracking performance for the 4-DOF SCARA robot, and that in case of bounded disturbance terms asymptotic convergence to the reference setpoints is achieved. The tracking error dynamics for the 4-DOF SCARA robot is written in the form

$$\dot{e} = Ae + Bu + L\tilde{d} \tag{86}$$

where in the 4-DOF SCARA robot's case $L = \in R^{8\times 8}$ to be the disturbance inputs gain matrix. Variable $\tilde{d}$ denotes model uncertainties and external disturbances of the 4-DOF SCARA robot's model. The following Lyapunov equation is considered

$$V = \frac{1}{2}e^T P e \tag{87}$$

where $e = x - x_d$ is the tracking error. By differentiating with respect to time one obtains

$$\dot{V} = \frac{1}{2}\dot{e}^T P e + \frac{1}{2}eP\dot{e} \Rightarrow \dot{V} = \frac{1}{2}[Ae + Bu + L\tilde{d}]^T P e + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow \tag{88}$$

$$\dot{V} = \frac{1}{2}[e^T A^T + u^T B^T + \tilde{d}^T L^T]Pe + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow \tag{89}$$

$$\dot{V} = \frac{1}{2}e^T A^T P e + \frac{1}{2}u^T B^T P e + \frac{1}{2}\tilde{d}^T L^T P e + \frac{1}{2}e^T P A e + \frac{1}{2}e^T P B u + \frac{1}{2}e^T P L\tilde{d} \tag{90}$$

The previous equation is rewritten as

$$\dot{V} = \frac{1}{2}e^T(A^T P + PA)e + \left(\frac{1}{2}u^T B^T P e + \frac{1}{2}e^T P B u\right) + \left(\frac{1}{2}\tilde{d}^T L^T P e + \frac{1}{2}e^T P L\tilde{d}\right) \tag{91}$$

Assumption: For given positive definite matrix Q and coefficients r and ρ there exists a positive definite matrix P, which is the solution of the following matrix equation

$$A^T P + PA = -Q + P\left(\frac{2}{r}BB^T - \frac{1}{\rho^2}LL^T\right)P \tag{92}$$

Moreover, the following feedback control law is applied to the system

$$u = -\frac{1}{r}B^T P e \tag{93}$$

By substituting Eq. (92) and Eq. (93) one obtains

$$\dot{V} = \frac{1}{2}e^T\left[-Q + P\left(\frac{2}{r}BB^T - \frac{1}{\rho^2}LL^T\right)P\right]e + e^T PB\left(-\frac{1}{r}B^T P e\right) + e^T P L\tilde{d} \Rightarrow \tag{94}$$

$$\begin{aligned}\dot{V} = &-\frac{1}{2}e^T Q e + \frac{1}{r}e^T P B B^T P e - \frac{1}{2\rho^2}e^T P L L^T P e \\ &-\frac{1}{r}e^T P B B^T P e + e^T P L\tilde{d}\end{aligned} \tag{95}$$

which after intermediate operations gives

$$\dot{V} = -\frac{1}{2}e^T Q e - \frac{1}{2\rho^2}e^T P L L^T P e + e^T P L\tilde{d} \tag{96}$$

or, equivalently

$$\dot{V} = -\frac{1}{2}e^T Q e - \frac{1}{2\rho^2}e^T P L L^T P e + \frac{1}{2}e^T P L\tilde{d} + \frac{1}{2}\tilde{d}^T L^T P e \tag{97}$$

Lemma: The following inequality holds

$$\tfrac{1}{2}e^TL\tilde{d}+\tfrac{1}{2}\tilde{d}L^TPe-\tfrac{1}{2\rho^2}e^TPLL^TPe\leq\tfrac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{98}$$

Proof: The binomial $(\rho\alpha-\frac{1}{\rho}b)^2$ is considered. Expanding the left part of the above inequality one gets

$$\begin{array}{c}\rho^2a^2+\frac{1}{\rho^2}b^2-2ab\geq 0\Rightarrow\frac{1}{2}\rho^2a^2+\frac{1}{2\rho^2}b^2-ab\geq 0\Rightarrow\\ ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2\Rightarrow\frac{1}{2}ab+\frac{1}{2}ab-\frac{1}{2\rho^2}b^2\leq\frac{1}{2}\rho^2a^2\end{array} \tag{99}$$

The following substitutions are carried out: $a=\tilde{d}$ and $b=e^TPL$ and the previous relation becomes

$$\tfrac{1}{2}\tilde{d}^TL^TPe+\tfrac{1}{2}e^TPL\tilde{d}-\tfrac{1}{2\rho^2}e^TPLL^TPe\leq\tfrac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{100}$$

Eq. (100) is substituted in Eq. (97) and the inequality is enforced, thus giving

$$\dot{V}\leq-\tfrac{1}{2}e^TQe+\tfrac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{101}$$

Eq. (101) shows that the H_∞ tracking performance criterion is satisfied. The integration of $\dot{V}$ from 0 to T gives

$$\begin{array}{c}\int_0^T\dot{V}(t)dt\leq-\frac{1}{2}\int_0^T||e||_Q^2dt+\frac{1}{2}\rho^2\int_0^T||\tilde{d}||^2dt\Rightarrow\\ 2V(T)+\int_0^T||e||_Q^2dt\leq 2V(0)+\rho^2\int_0^T||\tilde{d}||^2dt\end{array} \tag{102}$$

Moreover, if there exists a positive constant $M_d>0$ such that

$$\int_0^\infty||\tilde{d}||^2dt\leq M_d \tag{103}$$

then one gets

$$\int_0^\infty||e||_Q^2dt\leq 2V(0)+\rho^2M_d \tag{104}$$

Thus, the integral $\int_0^\infty||e||_Q^2dt$ is bounded. Moreover, $V(T)$ is bounded and from the definition of the Lyapunov function V in Eq. (87) it becomes clear that $e(t)$ will be also bounded since $e(t)\in\Omega_e=\{e|e^TPe\leq 2V(0)+\rho^2M_d\}$. According to the above and with the use of Barbalat's Lemma one obtains $\lim_{t\to\infty}e(t)=0$.

After following the stages of the stability proof one arrives at Eq. (101) which shows that the H-infinity tracking performance criterion holds. By selecting the attenuation coefficient ρ to be sufficiently small and in particular to satisfy $\rho^2<||e||_Q^2/||\tilde{d}||^2$ one has that the first derivative of the Lyapunov function is upper bounded by 0. This condition holds at each sampling instance and consequently global stability for the control loop can be concluded.

5.2 Robust state estimation with the use of the H_∞ Kalman Filter

The control loop has to be implemented with the use of information provided by a small number of sensors and by processing only a small number of state variables. To reconstruct the missing information about the state vector of the 4-DOF SCARA robot it is proposed to use a filtering scheme and based on it to apply state estimation-based control [1], [32]. By denoting as $A(k)$, $B(k)$, $C(k)$ the discrete-time equivalents of matrices A, B, C which constitute the linearized state-space model of Eq. (17), the recursion of the H_∞ Kalman Filter, for the model of the boom SCARA robot, can be formulated in terms of a *measurement update* and a *time update* part

Measurement update:

$$
\begin{aligned}
&D(k) = [I - \theta W(k)P^{-}(k) + C^T(k)R(k)^{-1}C(k)P^{-}(k)]^{-1} \\
&K(k) = P^{-}(k)D(k)C^T(k)R(k)^{-1} \\
&\hat{x}(k) = \hat{x}^{-}(k) + K(k)[y(k) - C\hat{x}^{-}(k)]
\end{aligned} \tag{105}
$$

Time update:

$$
\begin{aligned}
&\hat{x}^{-}(k+1) = A(k)x(k) + B(k)u(k) \\
&P^{-}(k+1) = A(k)P^{-}(k)D(k)A^T(k) + Q(k)
\end{aligned} \tag{106}
$$

where it is assumed that parameter θ is sufficiently small to assure that the covariance matrix $P^{-}(k)^{-1} - \theta W(k) + C^T(k)R(k)^{-1}C(k)$ will be positive definite. When $\theta = 0$ the H_∞ Kalman Filter becomes equivalent to the standard Kalman Filter. One can measure only a part of the state vector of the SCARA robot, for instance state variables x_1, x_3, x_5, and x_7 and can estimate through filtering the rest of the state vector elements (x_2, x_4, x_6 and x_8). Moreover, the proposed Kalman filtering method can be used for sensor fusion purposes.

6 Simulation tests

The global stability properties of the control method and the elimination of the state vector's tracking error which were previously proven through Lyapunov analysis are further confirmed through simulation experiments. The parameters of the model of the 4-DOF SCARA robot which have been used in the simulation tests have been according to [2]. To compute the stabilizing feedback gains of the controller, the algebraic Riccati equation of Eq. (92) had to be repetitively solved at each iteration of the control algorithm. The obtained results are depicted in Fig. 3 to Fig. 18. The real values of the state variables of the 4-DOF SCARA robot are printed in blue, their estimates which are provided by the H-infinity Kalman Filter are printed in green colour while the associated setpoints are printed in red. The performance of the nonlinear optimal control method was very satisfactory Actually, through all test cases it has been confirmed that the control method can achieve fast and accurate tracking of reference trajectories (setpoints) under moderate variations of the control inputs. The simulation tests come to confirm that the control method has global (and not local) stability properties. Under the nonlinear optimal control method the state variables of the SCARA robot can track precisely setpoints with fast and abrupt changes. Moreover, the convergence to these setpoints is independent from initial conditions.

Regarding the selection of values for the controller gains it can be noted that parameters r, ρ and Q which appear in the method's algebraic Riccati equations are assigned offline constant values, where the gains vector K is updated at each sampling instance, based on the positive definite and symmetric matrix P which is the solution of the method's algebraic Riccati equation. The tracking accuracy and the transient performance of the control scheme depends on the values of coefficients r, ρ and on the values of the elements of the diagonal matrix Q. Actually, for relatively small values of r one achieves elimination of the state vectors' tracking error one. Moreover, for relatively high values of the diagonal elements of matrix Q one achieves fast convergence the state variables' reference trajectories, Finally, the smallest value of the attenuation coefficient ρ that results into a valid solution of the method's Riccati equation in the form of the positive definite and symmetric matrix P, it the one that provides the control loop with maximum robustness.

Comparing to past attempts for solving the H-infinity control problem for nonlinear dynamical systems, the article's control approach is substantially different [28]. Preceding results on the use of H-infinity control to nonlinear dynamical systems were limited to the case of affine-in-the-input systems with drift-only dynamics and considered that the control inputs gain matrix is not dependent on the values of the system's state vector. Moreover, in these approaches the linearization was performed around points of the desirable trajectory whereas in the present article's control method the linearization points are related with the value of the state vector at each sampling instant as well as with the last sampled value of the control inputs vector. The Riccati equation which has been proposed for computing the feedback gains of the controller is novel, so is the presented global stability proof through Lyapunov analysis.

The proposed H-infinity (optimal) control method for the 4-DOF SCARA robot exhibits several advantages when compared against other linear or nonlinear control schemes [28]. For instance: (i) In contrast to global linearization-based control schemes (Lie algebra-based control and differential flatness theory-based control) it does not need complicated changes of state-variables (diffeomorphisms) and does not come against singularity problems in the computation of the control inputs, (ii) In contrast to sliding-mode control or to back-stepping control the proposed nonlinear optimal control scheme does not require the state-space model of the system to be in a specific form (e.g. triangular, canonical, etc.) (iii) In contrast to PID control the proposed nonlinear optimal control method is globally stable and functions well at changes of operating points, (iv) In contrast to multi-models based control and linearization around multiple operating points, the nonlinear optimal control scheme requires linearization around one single operating point and thus it avoids the computational burden for solving multiple Riccati equations or LMIs, (v) Moreover, unlike the popular computed torque method for robotic manipulators, the new control approach is characterized by optimality and is also applicable when the number of control inputs is not equal to the robot's number of DOFs.

7 Conclusions

SCARA-type robots (Selective Compliance Articulated Robot Arms) are widely used in several industrial tasks. To improve their accuracy, and speed in tasks' execution as well as to reduce the cost of their functioning, elaborated control algorithms have to be used about them. In the present article a novel nonlinear optimal control approach has been used for the dynamic model of the 4-DOF SCARA robot with three revolute joints and one prismatic joint. At a first-stage the dynamic model of the SCARA robot undergoes approximate linearization with the use of first-order Taylor series expansion and through the computation of the associated Jacobian matrices. The linearization point is updated at each sampling instance and is defined by the present value of the system's state vector and by the last sampled value of the control inputs vector.

At a second stage a stabilizing H-infinity feedback controller is designed. The H-infinity controller achieves solution of the optimal control problem for the model of the SCARA robot under model uncertainty and external perturbations. The H-infinity controller represents a min-max differential game which takes place between (i) the control inputs which try to minimize a quadratic cost function of the state vector's tracking error, ii) the model imprecision and the external perturbation terms which try oo maximize this cost function. To compute the stabilizing feedback gains of the H-infinity controller an algebraic Riccati equation had to be repetitively solved at each time-step of the control algorithm. The global stability properties of the control scheme have been proven through Lyapunov analysis. First, it has been demonstrated that the control method satisfies the H-infinity tracking performance criterion, while under moderate conditions it has been proven that the control loop is globally asymptotically stable. Finally, to implement state est9mation-based control, the H-infinity Kalman Filter has been used as a robust state estimator. The nonlinear optimal control approach retains the advantages of the standard linear optimal control, that is fast and accurate tracking of reference setpoints under moderate variations of the control inputs.

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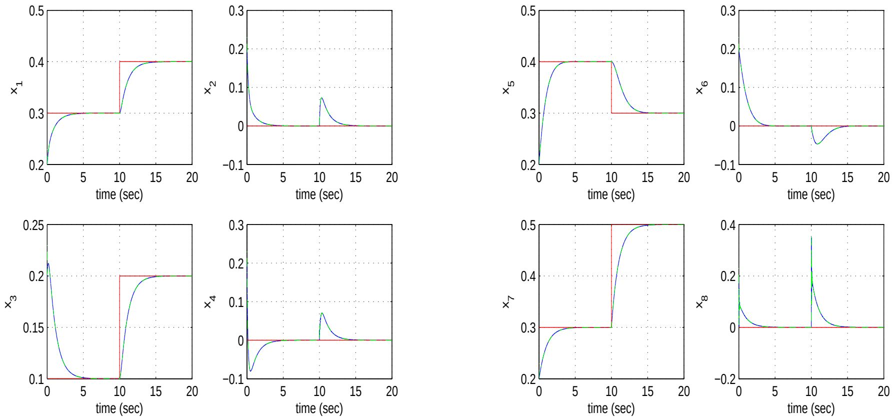
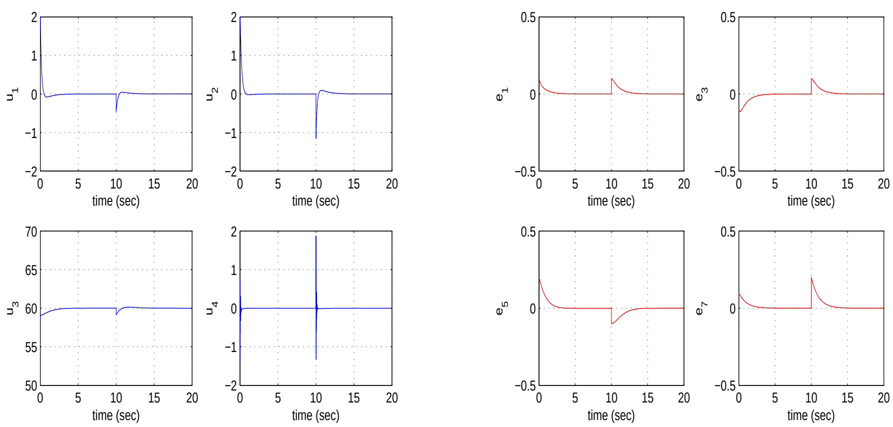


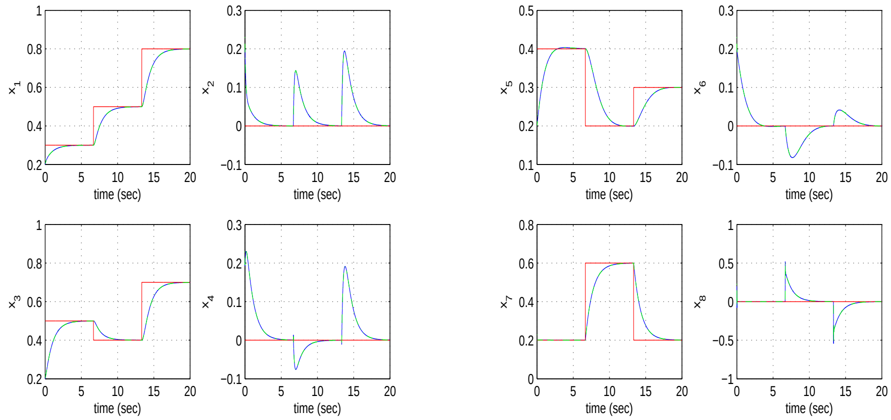
(a) **(b)**

Figure 3: Tracking of setpoint 1 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



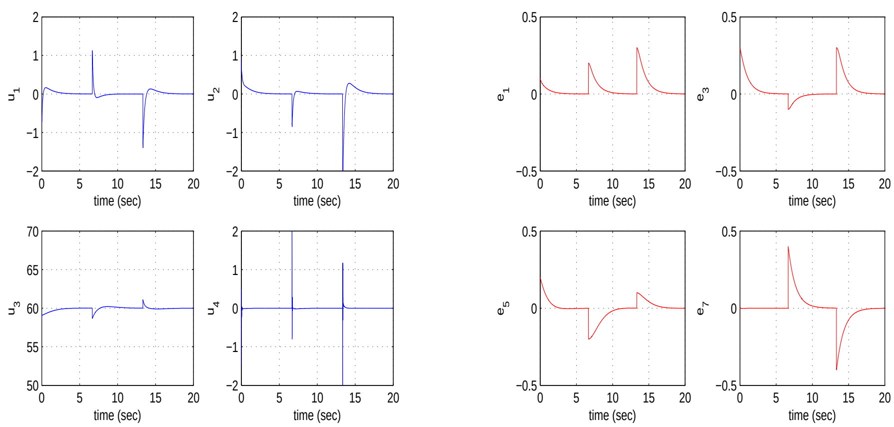
(a) **(b)**

Figure 4: Tracking of setpoint 1 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



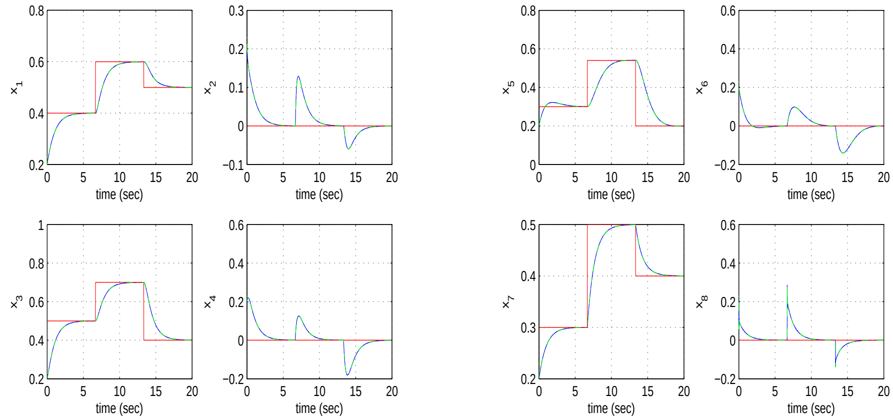
(a) **(b)**

Figure 5: Tracking of setpoint 2 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



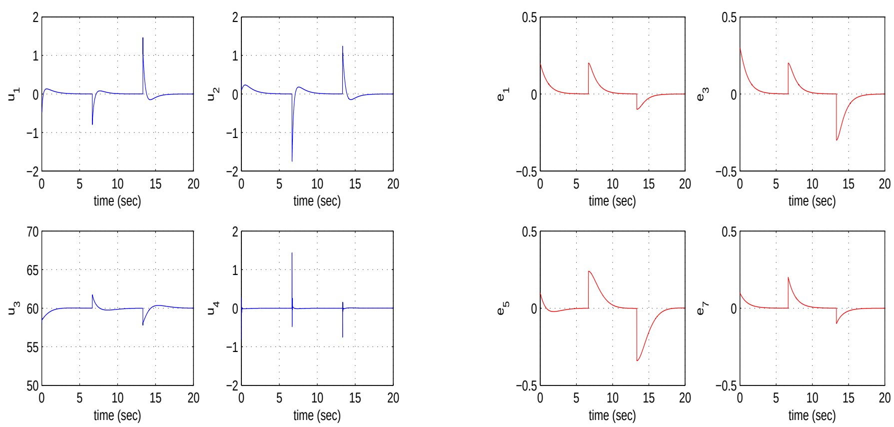
(a) **(b)**

Figure 6: Tracking of setpoint 2 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



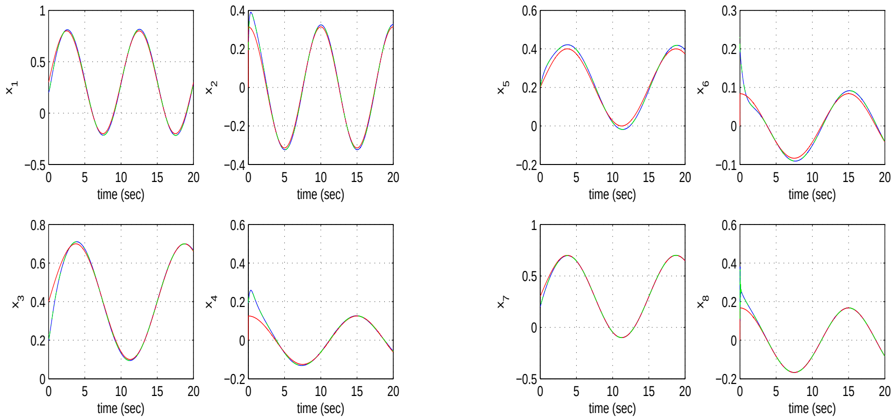
(a) **(b)**

Figure 7: Tracking of setpoint 3 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



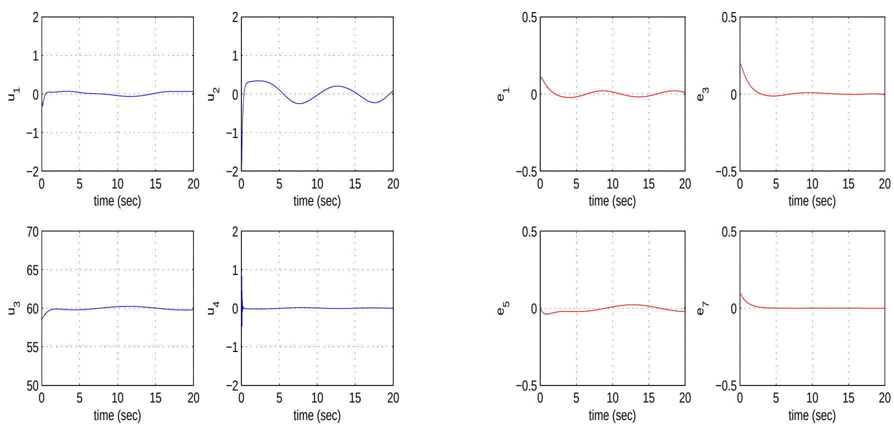
(a) **(b)**

Figure 8: Tracking of setpoint 3 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



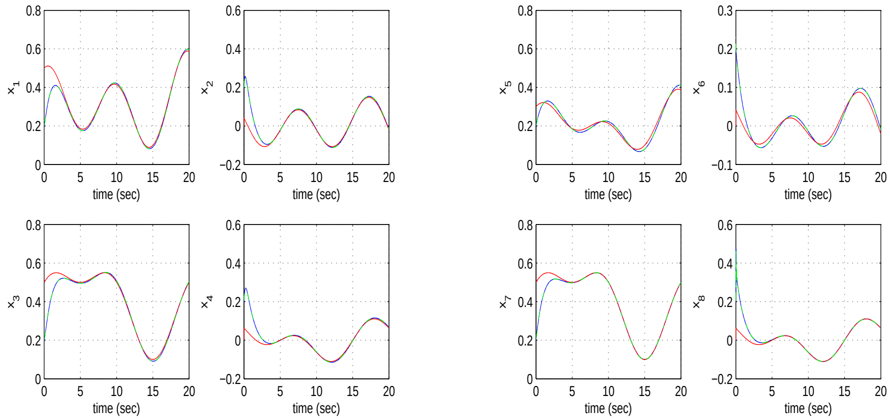
(a) **(b)**

Figure 9: Tracking of setpoint 4 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



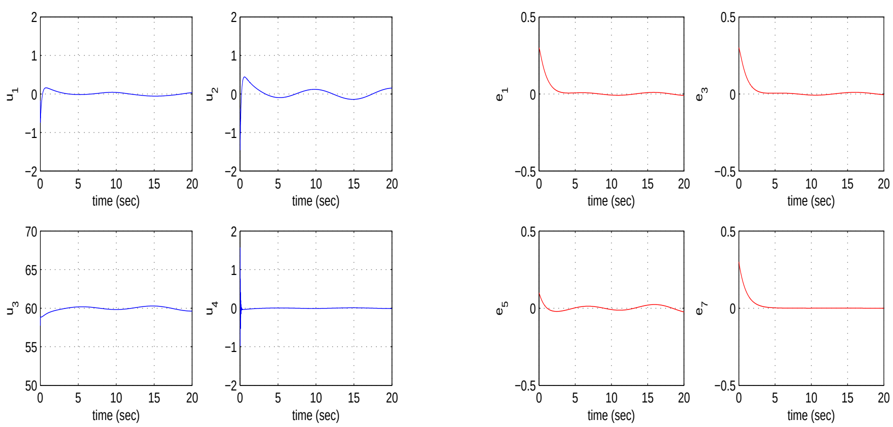
(a) **(b)**

Figure 10: Tracking of setpoint 4 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



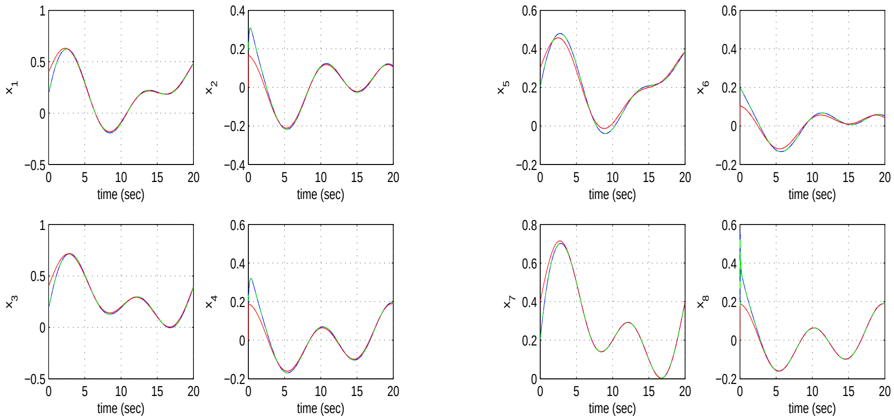
(a) **(b)**

Figure 11: Tracking of setpoint 5 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



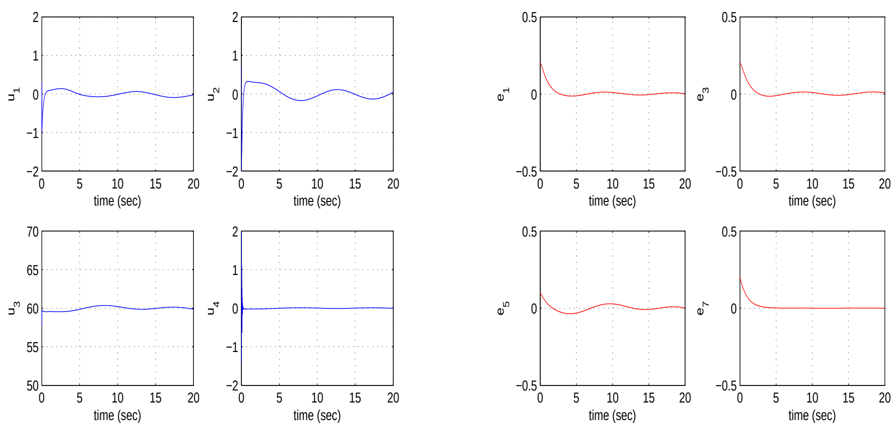
(a) **(b)**

Figure 12: Tracking of setpoint 5 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



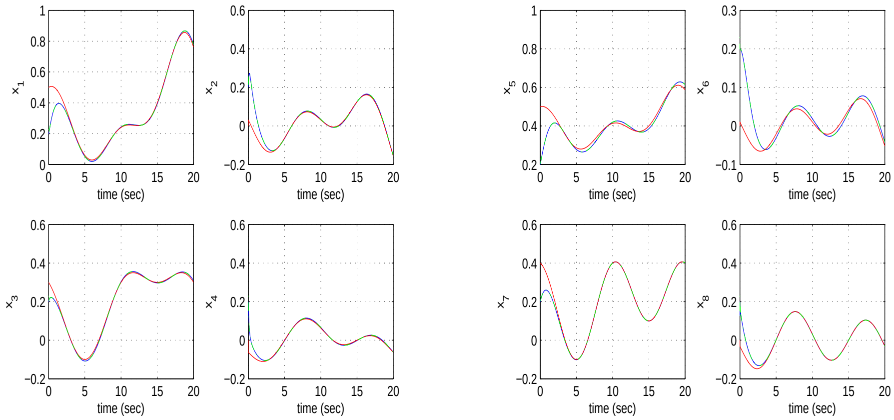
(a) **(b)**

Figure 13: Tracking of setpoint 6 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



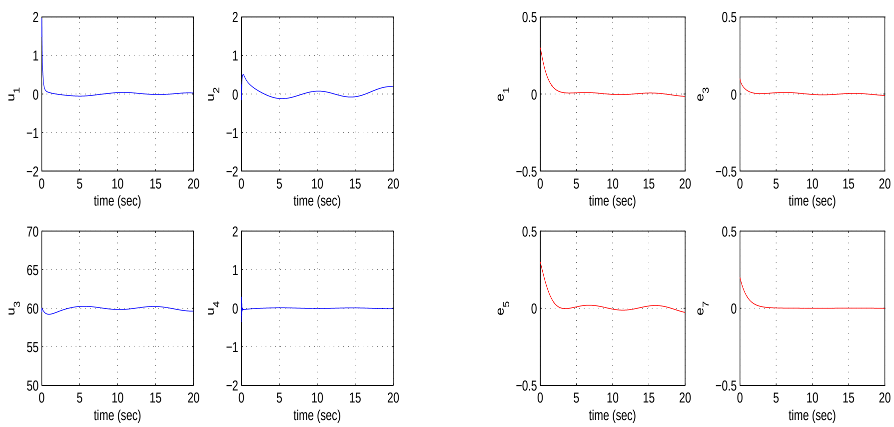
(a) **(b)**

Figure 14: Tracking of setpoint 6 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



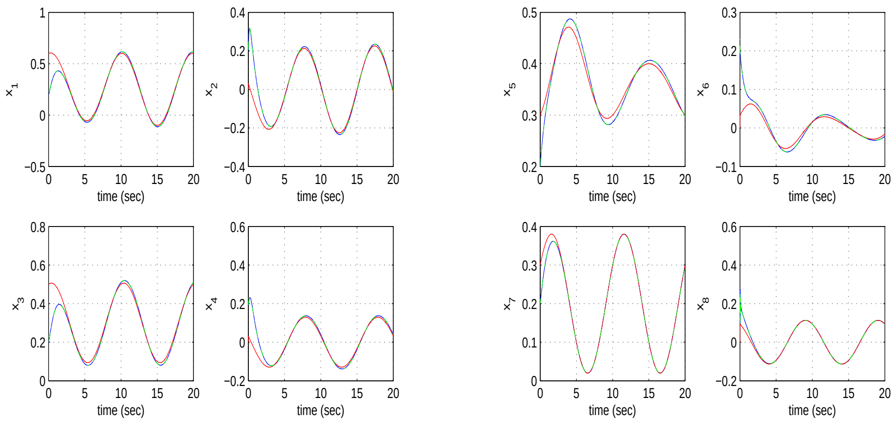
(a) **(b)**

Figure 15: Tracking of setpoint 7 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



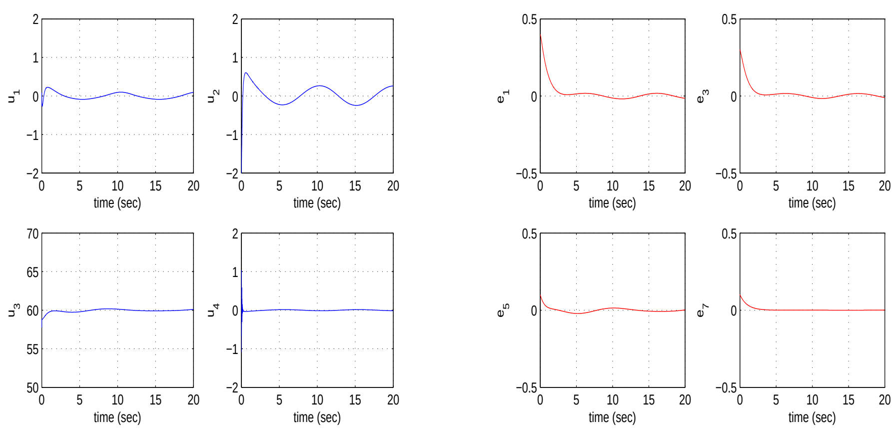
(a) **(b)**

Figure 16: Tracking of setpoint 7 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot



(a) **(b)**

Figure 17: Tracking of setpoint 8 for the SCARA robot (a) convergence of state variables x_1 to x_4 to their reference setpoints (red line: setpoint, blue line: real value, green line: estimated value), (b) convergence of state variables x_5 to x_8 to their reference setpoints



(a) **(b)**

Figure 18: Tracking of setpoint 8 for the SCARA robot (a) control inputs u_1, u_2 applied to the robot, (b) tracking error variables e_1, e_3, e_5 and e_7 of the SCARA robot

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