Threefold way to black hole entropy

Pedro Bargueño*

Departamento de Física Aplicada, Universidad de Alicante,
Campus de San Vicente del Raspeig, E-03690 Alicante, Spain

Ernesto Contreras†

Departamento de Física, Colegio de Ciencias e Ingeniería, Universidad San Francisco de Quito USFQ,
Quito 170901, Ecuador

(Received 27 August 2022; accepted 24 February 2023; published 15 March 2023)

In this work we propose a correspondence between black hole entropy and a topological quantity defined for projective spaces based on the real, complex, and quaternion numbers. After interpreting Weinstein's integer as the normalized volume of the quantum phase space, whose logarithm gives place to the area law (in the real case) and to logarithmic corrections with $-\frac{1}{2}$ and $-\frac{3}{2}$ coefficients (in the complex and quaternionic cases, respectively), the exact Bekenstein-Hawking entropy is obtained when certain equally spaced spectrum for the event horizon area is imposed. Even more, the minimal area(s) which emerge from our model, are of the form $4\log k$, $k \in \{2, 4, 16\}$, in complete agreement with previous works. Finally, the role played by global (complex and quaternionic) phases in different descriptions of black hole entropy is clarified.

DOI: 10.1103/PhysRevD.107.066013

I. INTRODUCTION

The entropy of a black hole, which equals one quarter of the area of its event horizon in units of the Planck area to leading order [1,2], is usually considered one of the key pieces of gravity where looking for questions trying to open the way toward a complete theory of quantum gravity. Thermodynamically, it behaves like the entropy of non-gravitational physics and, therefore, we must have a statistical interpretation in terms of underlying microstates. Providing a statistical mechanical interpretation of black hole entropy has been a longstanding goal for all candidates for a complete quantum gravitational theory. Nowadays, the problem is not that we do not have an (incomplete) theory for quantum gravity, but we have several of them. Some of these, although different (in principle and in practice), predict the same qualitative logarithmic correction to the aforementioned entropy by adopting a particular definition of black hole microstates together with their counting. This is the case, for example, of loop quantum gravity (LQG) [3,4], string theory [5,6] and the AdS/CFT correspondence [7]. Essentially, what is generally accepted is that, when quantum corrections are switched on, black hole entropy is given by $S = \frac{A}{4} - k\log\frac{A}{4} + \cdots$, where A is the area of the event horizon.

In this work, we take the very same existence of the aforementioned coincidences between calculations performed for black hole entropies in different theories, as an indicative of some underlying structure to all of them. More precisely, we propose a duality between black hole microstates and a topological quantity for projective spaces entering into the description of the quantum sector of gravity, such that it is capable of easily encoding both the area law and the $k = 1/2, 3/2$ corrections, which are of fundamental importance within LQG and two dimensional CFT and the Cardy formula (calculations within string theory show (see, for example, [6]), that the aforementioned corrections deviate from the previously mentioned values for k).

Specifically, we will appeal to a very basic fact: quantum theory may be formulated using Hilbert spaces over the real numbers ($\mathbb{R}$), the complex numbers ($\mathbb{C}$) and the quaternions ($\mathbb{H}$), which form the three associative normed division algebras. This "three-fold way", as Dyson called it [8], will be revealed as a powerful tool to analyze the quantum phase space, whose volume is encoded in the Weinstein integer. We will show that it allows for a simple counting of black hole microstates.

II. MOTIVATION: THE SPACE OF PURE STATES

If we adopt a conventional point of view on quantum theory, pure states are given by vectors in a Hilbert space, $\mathcal{H}$. If a finite dimensional Hilbert space is considered, then

*pedro.bargueno@ua.es
†econtreras@usfq.edu.ec

 © 2023 American Physical Society

$\mathcal{H} = \mathbb{C}^n$ equipped with a scalar product. However, given that two states are related by multiplication by any complex number, the true quantum phase space is the space of rays in the Hilbert space. For the special case in which $\mathcal{H}$ is $\mathbb{C}^{n+1}$, the space of rays is $\mathbb{C}P^n$ (the so-called projective Hilbert space, $\mathcal{H}^{\mathcal{P}}$) [9]. The dimension n is $n = k - 1$, where k labels the number of relevant (energy, spin,…) levels which are needed to describe the physics under study. At this point, the term *quantum phase space* refers to the space of physical states which we will assume as a truly phase space in the statistical mechanical sense. If this is the case, then, what is the basic unit of phase space? In standard statistical mechanics, it is true that Planck's constant can be thought as the quantum of phase space volume [10]. For example, for an ideal gas, it enters in the ratio between the volume, V, and λ_{dB}^3, where $\lambda_{\mathrm{dB}} \sim \hbar$ is the thermal de Broglie wavelength. As we are promoting quantum phase spaces to truly phase spaces, a natural characterization of the aforementioned granularity is mandatory. The problem is that quantum phase spaces, as we have defined them, are pure mathematical objects. For instance, in the space of qubits, it is $\mathbb{C}P^1 = S^2$. Then, a term like $\mathrm{Vol}(\mathbb{C}P^1)/\mathrm{Vol}(X)$, where X is to be determined, should play an equivalent role to the standard statistical-mechanical terms like $V^2/\lambda_{\mathrm{dB}}^2$ [11].

It is customary by geometers and topologists to take the sphere as, let us say, the canonical and paradigmatic object. Roughly speaking, the study of "deviations from the sphere" has always been an intense field of study in geometry and topology. Here we will rely on mathematics to define quantum phase spaces. For example, for complex projective spaces we will promote a term like $\mathrm{Vol}(\mathbb{C}P^n)/\mathrm{Vol}(S^n)$ to the measure of the volume of truly quantum phase spaces of pure n-level systems. In this sense, the choice of these "normalized volumes" as quantum phase spaces can be considered, at this point, the first hypothesis of the present work.

As commented before, we are mainly interested in projective spaces. Specifically, we will work with $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$. For brevity, we will collect these spaces by writing $\mathbb{K}P^n$, specifying the field if necessary. The spheres S^n will be also considered in order to introduce "deviations from classicality," as we will show along the manuscript.

III. WEINSTEIN'S INTEGER

A Riemannian n-dimensional manifold (M^n, g) will be called a C_L-manifold if all the geodesics on M^n are closed and have length $2\pi L$, i.e., if all the orbits of the geodesic flow on the unit tangent bundle $U(M^n, g)$ are periodic with least period $2\pi L$. The following important results, which we cite without proof, were reported by A. Weinstein during the 1970s [12].

Theorem 1.—If (M^n, g) is C_L, then

$$i(M^n, g) = \frac{\mathrm{Vol}(M^n, g)}{L^n \mathrm{Vol}(S^n, \mathrm{can})} \tag{1}$$

is an integer.
Here, *can* stands for the canonical metric.

Theorem 2.—The number $j(M^n, g) = 2i(M^n, g)$ is a topological invariant of the fibration of the unit tangent bundle $U(M^n, g)$ by the orbits of the geodesic flow.
For example, all compact symmetric spaces of rank one, which are S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and the octonionic projective plane, are C_L-manifolds [13]. Note that the octonionic case is the only one which is not n-dimensional. For that reason we will not consider it along the manuscript. Even more, note that, apart from a numerical factor, Weinstein's integer is exactly what we were looking for to describe quantum phase spaces following the first hypothesis previously introduced.

With these tools at hand, we can compute the corresponding volumes of $\mathbb{K}P^n$ and S^n, following [14]. From these, the Weinstein integer [12] for our cases of interest are [12]:

$$i(S^n, \mathrm{can}) = 1 \tag{2}$$

$$i(\mathbb{R}P^n, \mathrm{can}) = 2^{n-1} \tag{3}$$

$$i(\mathbb{C}P^n, \mathrm{can}) = \binom{2n-1}{n-1} \tag{4}$$

$$i(\mathbb{H}P^n, \mathrm{can}) = \frac{1}{2n+1}\binom{4n-1}{2n-1}, \tag{5}$$

[15], whose meaning will be easily clarified in what follows.

IV. WEINSTEIN ENTROPY

The fundamental postulate of statistical mechanics expresses the entropy, S, of a physical system, P, composed of N particles, as a function of the accessible volume in phase space, $\Omega[P(N)]$, following Boltzmann's formula:

$$S = k_B \log \Omega[P(N)], \tag{6}$$

where k_B is Boltzmann's constant.

Let us make the identification

$$P \to (M^n, g) \tag{7}$$

$$N \to n \tag{8}$$

$$\Omega[P(N)] \to i[(M^n, g)]. \tag{9}$$

Even more, let us go one step further and define the *Weinstein entropy* of a C_L-manifold, $S_g[(M^n, g)]$ as

$$S_i[(M^n, g)] = \log i[(M^n, g)]. \tag{10}$$

In a further analogy with statistical mechanics, let us take the limit of large n (not necessary for the spheres, which have $i(S^n, \mathrm{can}) = 1$), which corresponds to a large number of particles, N. We get, for our cases of interest

$$i(S^n, \mathrm{can}) = 1 \tag{11}$$

$$i(\mathbb{R}P^n, \mathrm{can}) \sim 2^n \tag{12}$$

$$i(\mathbb{C}P^n, \mathrm{can}) \sim \frac{4^n}{2\sqrt{\pi} n^{1/2}} \tag{13}$$

$$i(\mathbb{H}P^n, \mathrm{can}) \sim \frac{16^n}{4\sqrt{2\pi} n^{3/2}}. \tag{14}$$

Therefore, following

$$\boxed{S_i = \log[i(\mathbb{K}P^n)]\ n \text{ large},} \tag{15}$$

we get

$$S_i(S^n) = \log[i(S^n, \mathrm{can})] = 0 \tag{16}$$

$$S(\mathbb{R}P^n) = \log[i(\mathbb{R}P^n, \mathrm{can})] \sim n \log 2 \tag{17}$$

$$S_i(\mathbb{C}P^n) = \log[i(\mathbb{C}P^n, \mathrm{can})] \sim n \log 4 - \frac{1}{2} \log n \tag{18}$$

$$S_i(\mathbb{H}P^n) = \log[i(\mathbb{H}P^n, \mathrm{can})] \sim n \log 16 - \frac{3}{2} \log n. \tag{19}$$

[16].

V. BLACK HOLE ENTROPY

Let us assume an equal spacing of black hole area eigenvalues. Although there is not complete consensus on the spacing on the area eigenvalues, let us remark that LQG initially predicted a not equally spaced spectrum (see, for example, the review [18]). However, there are alternative choices within LQG where a new area operator with equidistant eigenvalues exists (see [20] and references therein) (in addition, the standard area spectrum of LQG is equally spaced in the large spin limit). We will define the number of black hole microstates by the asymptotic growth of Weinstein's integer. Thus, black hole entropy is given by the asymptotic expansion for large n, which we will assume is proportional to the area of the event horizon, of Weinsten's integer, $i(\mathbb{K}P^n)$ [19].

Specifically, we will assume that the area spectrum is given by

$$A = nA_{\mathrm{min}} = 4n \log k, \tag{20}$$

where k is an integer. The argument for our choice, following Mukhanov and Bekenstein [21–23], is the following. They considered that $g_n = e^S$ is the degeneracy of the nth area eigenvalue and, therefore, the accepted thermodynamic relation between black hole surface area and entropy can be met with the requirement that g_n has to be an integer for every n only when $A_{\mathrm{min}} = 4 \log k$, $k \in \mathbb{N}$. In this sense, statistical physics arguments force the dimensionless constant g_n to be of the form of A_{min} given by Eq. (20).

It must be emphasized that, in principle, there is not any fundamental reason to say that the dimension of the projective space must coincide with the area quantization rate. However, (20) can be thought as an heuristic constraint which makes the work. It does not mean a weakness in our approach. Instead, our connection could be thought as a bridge between abstract and unexplored mathematics and the physics realm. Interestingly, the number n in Eq. (20) can also be interpreted as the number of basics units which tessellates the event horizon, each one of these units having a minimum are proportional to l_p^2. This number of units, usually referred to as *holographic degrees of freedom*, has inspired a large number of quantum black hole models following Wheeler's "it from bit" proposal [24] (see, for example, [25,26] and references therein).

Let us assume that the event horizon of a black hole can be described by a pure state, $|\mathrm{hor}\rangle$, living on $\mathcal{H} = \mathbb{C}^n$. As we have discussed, the space of physical states is the corresponding projective space, $\mathbb{C}P^n$. Under this point of view, we note that the states

$$|\mathrm{hor}\rangle \sim |\mathrm{hor}'\rangle = e^{\imath\theta}|\mathrm{hor}\rangle \tag{21}$$

are equivalent in the sense that *a global phase makes no difference between them* (we will come to this sentence in brief).

From Eq. (18) it is clear that, after imposing the aforementioned equally spaced area spectrum as then, $S_i(\mathbb{C}P^n)$ is given by

$$S_i(\mathbb{C}P^n) = \frac{A}{4} - \frac{1}{2} \log \frac{A}{4} + \cdots, \tag{22}$$

when

$$A_{\mathrm{min}} = 4 \log 4. \tag{23}$$

Let us interpret our results from a more technical point of view. We are forming $\mathbb{C}P^n$ from S^1 fibers over S^{2n+1} spheres, i.e., we are Hopf-fibring (first fibration) to write $\mathbb{C}P^n = \frac{S^{2n+1}}{S^1} = \frac{S^{2n+1}}{U(1)}$, which physically implies ignoring the $S^1 = U(1)$ global phase. In order to see what is happening, we consider the simplest $n = 1$ case. It should be clear that the three-sphere has a different global geometry than does a circle mapped to every point of a two-sphere. Even more, in the language of quantum information we could say that "one cannot choose phase factors for all qubits that would vary continuously over the entire Bloch sphere (S^2)" [27]. Therefore, the global phase can not be universally defined. These global phases are not a mere gauge, being routinely measured in the laboratory. Then, if we do not want to eliminate the global phase, the most natural mathematical object to encode the state space of a qubit is a unit vector on a three-sphere, or a unit quaternion [28].

Following this line of thought, let us now consider that the space of physical states is $\mathbb{H}P^n = \frac{S^{4n+3}}{S^3} = \frac{S^{4n+3}}{SU(2)}$ (second Hopf fibration). In this case, a similar analysis to that performed in the previous $\mathbb{C}P^n$ case allows us to write

$$S_i(\mathbb{H}P^n) = \frac{A}{4} - \frac{3}{2}\log\frac{A}{4} + \cdots, \tag{24}$$

where, in this case, we have that

$$A_{\min} = 4\log 16. \tag{25}$$

Finally, we note that if the space of physical states is taken to be $\mathbb{R}P^n$, no logarithmic corrections are found. In this case, the same techniques allow us to write

$$S_i(\mathbb{R}P^n) = \frac{A}{4} + \cdots, \tag{26}$$

inspired a large number of quantum black hole models where the corresponding $A_{\min}$ is

$$A_{\min} = 4\log 2. \tag{27}$$

The relationship with the approached we are following in the present work is beautifully summarized by J. C. Baez, who writes [29]: "Quantum theory may be formulated using Hilbert spaces over any of the three associative normed division algebras: the real numbers, the complex numbers and the quaternions." In this sense, we interpret the Bekenstein-Hawking entropy not as a consequence of classical physics but as a consequence of quantum physics based on $\mathbb{R}P^n$ which although, "real," it is able to provide us with the desired degeneracy, as Eq. (12) shows. Even more, deviations from the area law are not exactly intrinsically quantum, but a consequence of the extra structures which appear both in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ (Hilbert spaces of any one of the three kinds, real, complex and quaternionic, can be seen as Hilbert spaces of the other kinds, equipped with extra structure [29]).

VI. SOME SIMILARITIES WITH OTHER APPROACHES

Regarding the isolated horizon approach of LQG, it is well known that its classical degrees of freedom can be described dynamically by a Chern-Simons theory which was first found in its $U(1)$ gauge. The problem was that this $U(1)$ gauge fixed theory failed in the quantum theory when one tried to look for compatibility of this gauge fixing with Heisenberg's uncertainty principle. This difficulty is circumvented in the $SU(2)$ formulation which was formulated later (see [30] for a review). Leaving aside some technicalities that show differences between these two approaches, let us note that black hole entropy is given exactly by Eqs. (22) and (24) in the $U(1)$ and $SU(2)$ formulations, respectively.

Concerning the role played by global phases in our model for the horizon, let us summarize by noting that

$$\mathbb{R}P^n = \frac{S^n}{S_0} = \frac{S^n}{Z_2} \tag{28}$$

$$\mathbb{C}P^n = \frac{S^{2n+1}}{S^1} = \frac{S^{2n+1}}{U(1)} \tag{29}$$

$$\mathbb{H}P^n = \frac{S^{4n+3}}{S^3} = \frac{S^{4n+3}}{SU(2)}. \tag{30}$$

In the complex case, the $U(1)$ fibers correspond to a global phase of the corresponding $n+1$-level system, $|n+1\rangle \in \mathbb{C}P^n$. Therefore, $U(1)$ remember us that $|n+1\rangle$ and $e^{\imath\theta}|n+1\rangle$ belong to the same class of equivalence, which gives place to the $-\frac{1}{2}\log\frac{A}{4}$ correction. Correspondingly, in the $SU(2)$ case, invariance under quaternionic rotations leads to the $-\frac{3}{2}\log\frac{A}{4}$ correction. In this sense, it is very appealing to show how invariance under $U(1)$ or $SU(2)$, which is of paramount importance in LQG, also emerges in the first and second Hopf fibrations, which are intimately linked to our proposal based on Weinstein's entropy. In addition, we note that if no equivalence classes are introduced in the spheres, S^n, all of its points are trivially equivalent (there are no privileged points on S^n). Therefore, from a statistical point of view, S^n consists only of one microstate. We conjecture that this the reason why $S_g(S^n) = 0$.

We close this section by noting a coincidence between our approach and a CFT-based counting. Carlip shown [7] that the density of states for a CFT with central charge c and eigenvalues Δ grows as

$$\rho(\Delta) \sim \left(\frac{c}{96\Delta^3}\right)^{1/4} e^{2\pi\sqrt{\frac{c\Delta}{6}}}, \tag{31}$$

giving place, for a generic black hole in arbitrary dimensions [31], to a generic density of states given by:

$$\rho(\Delta) \sim \frac{c}{12}\left(\frac{A}{8\pi}\right)^{-3/2} e^{A/4}. \tag{32}$$

$$
\begin{tikzcd}
& \boxed{S_i = \log[i(\mathbb{K}P^n)]} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{R}P^n \arrow[r, "\text{complex}"'] \arrow[d, "S^0\sim Z_2"'] & \mathbb{C}P^n \arrow[r, "\text{quaternionic}"'] \arrow[d, "S^1\sim U(1)"'] & \mathbb{H}P^n \arrow[d, "S^3\sim SU(2)"'] \\
S_i = n\log 2 \arrow[r] \arrow[d] & S_i = n\log 4 - \frac{1}{2}\log\frac{n}{4} \arrow[r] \arrow[d] & S_i = n\log 16 - \frac{3}{2}\log\frac{n}{4} \arrow[d] \\
A = n\,4\log 2 \arrow[r] \arrow[d] & A = n\,4\log 4 \arrow[r] \arrow[d] & A = n\,4\log 16 \arrow[d] \\
S_i = \frac{A}{4} \arrow[r] \arrow[dr] & S_i = \frac{A}{4} - \frac{1}{2}\log\frac{A}{4} \arrow[r] \arrow[d] & S_i = \frac{A}{4} - \frac{3}{2}\log\frac{A}{4} \arrow[dl] \\
& \boxed{\text{Black hole entropy}} &
\end{tikzcd}
$$

FIG. 1. The three-fold way to black hole entropy. See text for details.

Surprisingly, we have that

$$\rho(\Delta) \sim i(\mathbb{H}P^n, \mathrm{can}). \tag{33}$$

Therefore, although the CFT argument also does not in general involve equal spacing and, furthermore, both the LQG and CFT derivations involve an explicit counting of a finite number of states, with no regularization, the Weinstein integer makes its appearance at some level within both approaches.

VII. SOME FINAL COMMENTS

To close our work we would like to discuss on some possible consequences of our model. First, we note that imposing $A = nA_{\mathrm{min}}$, as we previously said, we have that for

$$A_{\mathrm{min}} = (4\log 2, 4\log 4, 4\log 16), \tag{34}$$

then Eqs. (17), (18) and (19) give exactly the Bekenstein-Hawking entropy together with logarithmic corrections.

In this sense, there is a correspondence between the Weinstein entropy and black hole entropy. This is somehow reminiscent of the so-called black hole-qubit correspondence [32], which connects two previously disparate areas of theoretical physics: the Bekenstein-Hawking entropy of certain black hole solutions in string theory with certain multipartite entanglement measures in quantum information. In our case, the correspondence we show here connects very general features of black hole entropy with geometric and topological properties of projective spaces.

Clearly, we can write $A_{\mathrm{min}} = \alpha$ where the value taken by α will depend on the integer we are considering to define Weinstein's entropy. Interestingly, α can be associated to gravitational waves echoes [33]. As commented by Agullo *et al.* [33], the ability of measuring α is of interest for fundamental theories of gravity. For example, in the framework of LQG, the area spectrum for macroscopic black holes is almost continuum and, therefore, no echoes are expected. However, an alternative definition of the area operator defined by Barbero and coworkers in Ref. [20] allows an equally spaced spectrum of the area operator and it predicts $\alpha = 4\log 3$. In our approach, the entropy with the correct logarithmic correction of $-3/2$ yields $\alpha = 4\log 16$. We expect that forthcoming observations will allow to discriminate between this and other values for α.

VIII. CONCLUSIONS

In this work we have implemented a three-fold way in order to calculate black hole entropy by defining the Weinstein entropy of projective spaces based on three normed associative division algebras, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Using statistical mechanics as a guide, we have interpreted Weinstein's integer as the normalized volume of the quantum phase space (the space of physical states), whose logarithm gives place to an arealike law $(\mathbb{R})$ in addition to logarithmic corrections with $-\frac{1}{2}$ $(\mathbb{C})$ and $-\frac{3}{2}$ $(\mathbb{H})$ coefficients. The exact Bekenstein-Hawking entropy (including the aforementioned logarithmic corrections) is obtained when an equally spaced spectrum for the event horizon area is imposed. Even more, the minimal area(s) which emerge from our model (depending on $\mathbb{K}$), are of the form

$4\log k$, $k \in \mathbb{N}$, in complete agreement with previous works. Comparison of our findings with both loop quantum gravity and CFT techniques reveals surprising similarities. Specifically, we have identified the key $U(1)$ and $SU(2)$ symmetries as the fibres of the first and second Hopf fibrations which have allowed us to uncover the role played by global (complex and quaternionic) phases in the description of black hole entropy, as Fig. 1 summarizes.

ACKNOWLEDGMENTS

P. B. is funded by the Beatriz Galindo Program under the Contract No. BEAGAL 18/00207 (Ministerio de Universidades, Spain). P. B. acknowledges Anaís, Lucía, Inés and Ana for continuous support. P. B. acknowledges Prof. Alan Weinstein for his interest in our work and for sharing with him some illuminating ideas regarding Weinstein's integer.

[1] J. D. Bekenstein, Lett. Nuovo Cimento **4**, 737 (1972); Phys. Rev. D **7**, 2333 (1973).
[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[3] R. Kaul and P. Majumdar, Phys. Rev. Lett. **84**, 5255 (2000).
[4] K. A. Meissner, Classical Quantum Gravity **21**, 5245 (2004).
[5] A. Sen, J. High Energy Phys. 04 (2013) 156.
[6] S. Banerjee, R. K. Gupta, I. Mandal, and A. Sen, J. High Energy Phys. 11 (2011) 143.
[7] S. Carlip, Classical Quantum Gravity **17**, 4175 (2000).
[8] F. Dyson, J. Math. Phys. (N.Y.) **3**, 1199 (1962).
[9] I. Bengtsson and P. Zyczkowski, *Geometry of Quantum States, An Introduction to Quantum Entanglement*, 2nd ed. (Cambridge University Press, Cambridge, England, 2017).
[10] Indeed, it is the minimum volume that can occur in the phase space, and it sets the scale for its granularity or discreteness.
[11] Following standard conventions, along this work we will write Vol^n in general, although some particular cases such as, for example, $n = 2$, can be named differently (*area*, in this case).
[12] A. Weinstein, J. Diff. Geom. **9**, 513 (1974).
[13] A. L. Besse, *Manifolds All of Whose Geodesics are Closed* (Springer-Verlag, New York, 1978).
[14] M. Berger, *Lectures on Geodesics Riemannian Geometry* (Tata Institute of Fundamental Research, Bombay, 1965).
[15] As an aside note, the following nice identity can be derived:

$$j(\mathbb{C}P^n, \mathrm{can}) = (n+1)C_n \tag{35}$$

where C_n is the nth Catalan number.
[16] We would like to remark that the concept of *entropy* is also employed by mathematicians. Although there are several kinds of entropies, here we would like to emphasize some similarities of the Weinstein entropy we have introduced with the *volume entropy* [17] of a Riemannian manifold, which roughly measures the exponential growth rate of the volume of metric balls in the universal cover of the considered manifold. Despite this similarity, we have not found either in the physical nor in the mathematical literature a definition like ours. In this sense, it would be interesting to consider it also from a pure mathematical point of view.
[17] A. Manning, Ann. Math. **110**, 567 (1979).
[18] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).
[19] Note that black hole microstates can be alternatively defined in terms of $j(\mathbb{K}P^n)$. If this is the case, they are not only invariant under diffeomorphisms but, in addition, they are topological invariants in the sense of Theorem 2.
[20] J. Barbero, L. Lewandowski, and E. J. S. Villasenor, Phys. Rev. D **80**, 044016 (2009).
[21] J. D. Bekenstein, in *XVII Brazilian National Meeting on Particles and Fields*, edited by A. J. da Silva *et al.* (Brazilian Physical Society, Sao Paulo, 1996).
[22] J. D. Bekenstein, in *Proceedings of the VIII Marcel Grossmann Meeting on General Relativity*, edited by T. Piran and R. Ruffini (World Scientific, Singapore, 1998).
[23] J. D. Bekenstein and V. F. Mukhanov, Phys. Lett. B, **360**, 7 (1995).
[24] J. Wheeler, *Proceedings of the 3rd International Symposium on Foundations of Quantum Mechanics* (Physical Society of Japan (1990), Tokyo, 1989), p. 354.
[25] P. Bargueno and E. Contreras, Phys. Rev. D **105**, 046003 (2022).
[26] P. Bargueno and E. Contreras, Phys. Rev. D **106**, 066001 (2022).
[27] J. Urbantke, Am. J. Phys., **59**, 503 (1991).
[28] K. B. Wharton and D. Koch, J. Phys. A **48**, 235302 (2015).
[29] J. C. Baez, Found. Phys. **42**, 819 (2012).
[30] A. Perez, Rep. Prog. Phys. **80**, 126901 (2017).
[31] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999).
[32] L. Borsten, M. J. Duff, and P. Lévay, Classical Quantum Gravity **29**, 224008 (2012).
[33] I. Agullo, V. Cardoso, A. del Rio, M. Maggiore, and J. Pullin, Phys. Rev. Lett. **126**, 041302 (2021).