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Rational stability of choice functions

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Abstract

Two independent approaches have been used to analyze choices. A prominent notion is rationalizability: individuals choose maximizing binary relations. An alternative is to analyze choices in terms of standards of behavior with the notion of von Neumann–Morgenstern (vNM)-stability. We introduce a new concept (r-stability) that in turn extends the notion of stability and rationality. Our main result establishes that every rationalizable choice function is r-stable and every vNM-stable choice has an r-stable selection. An appealing property of r-stability is that well-known solution concepts (top cycle, uncovered set, ...) are r-stable, while they are neither rationalizable nor vNM-stable.

KEYWORDS

binary choice, rationalizable choice, stable set

JEL CLASSIFICATION

D11, D71

1 | INTRODUCTION

In social choice, two key concepts when analyzing decisions are stability and rationality. However, neither of these two different concepts is compatible with important choice methods, such as the top cycle, the uncovered set, minimal covering, and so forth. In this paper we introduce the notion of r-stability which, in some sense, generalizes both stability and rationality and besides, it is compatible with the main choice functions.

A basic modeling of individual or collective decisions is given by a *choice function*, which associates to every feasible set of alternatives X a nonempty subset that, in some sense, is intended as the *best alternatives in* X. This is the case, for instance, of an individual maximizing a utility function, or a collective decision made by aggregation of preferences, majority voting, and so forth.

When analyzing choice functions a natural question is whether these choices can be considered *rational*, which requires that every choice has been made as if the individuals were maximizing some preference relation. As pointed out by several authors, this is a very demanding condition: apart from Arrow's general impossibility result (Arrow, 1950) that rationalizability induces, it is highly questionable whether collective choices meet this requirement. Well-known choice functions as the *top cycle*, the *uncovered set* (Fishburn, 1977; Miller, 1980), the *minimal covering set* (Dutta, 1988), or the *bipartisan set* (Laffond et al., 1993) do not satisfy this property.

Several weakenings of the rationalizability concept have appeared in the literature, as the notions of *subrationalizability* (Deb, 1983) or *basically subrationalizability* (Subiza & Peris, 2000). Alternative notions, although no weaker, are those of *set-rationalizable* and *self-stable* choice functions (Brandt & Harrenstein, 2011).

A different approach used to analyze the behavior of choice functions is that of *stability* (Von Neumann & Morgenstern, 1944). They interpret this notion in terms of accepted standards of behavior based on internal stability ("no inner contradictions") and external stability ("used to discredit nonconforming procedures"). The notion of stability may also be interpreted as a way of selecting the best alternatives. Nevertheless, usual choice functions do not satisfy this stability notion: again this is the case for the *top cycle*, *uncovered set*, *minimal covering set*, and so forth.

As mentioned, our aim is to introduce a new stability concept that choice functions should satisfy. At a first sight, r-stability is directly linked to the notion of von Neumann–Morgenstern stable sets (*vNM*-stable, in what follows), or its generalizations: the generalized stable set (Harsanyi, 1974; Van Deemen, 1991), m-stable set (Peris & Subiza, 2013), or w-stable set (Han & Van Deemen, 2016). These notions, as the classical stability, are based on internal/external conditions. But the way in which the internal/external conditions are formulated is crucial when analyzing the behavior of a choice function. As we will see, our new concept uses the internal condition in m-stability and the external condition in generalized stability. By mixing these conditions we obtain a new concept with interesting properties:

(i) It generalizes both the notion of vNM-stability and rationalizability: Theorem 3 shows that every rationalizable choice function is r-stable and that every vNM-stable choice function has an r-stable selection.
(ii) The top cycle, uncovered set, minimal covering, or bipartisan set satisfy r-stability, whereas they do not satisfy the other stability notions.
(iii) The concept of r-stability is intermediate between rationalizable and basically subrationalizable choice functions (Theorem 4), a very weak condition when analyzing rationality.

As m-stability, the r-stability condition is related to the *generalized optimal choice set* (Schwartz, 1972) and we argue that this is the kind of stability/rationalizability that should be fulfilled by any choice function.

The paper is organized as follows. Section 2 presents the preliminary concepts and Section 3 introduces our proposal: r-stability. Section 4 contains our main results and in Section 5 we study the case in which the binary choice is univalued (*complete binary discrimination*). Some final remarks close the paper.

2 | PRELIMINARIES

In this introductory section we survey some basic definitions and properties of binary relations. Then we analyze choice functions from the two different approaches: rationalizability and stability concepts. Apart from the basic notions of rationalizable or vNM-stable choice function, we provide some known generalizations of these concepts.

2.1 | Binary relations

A binary relation $\succ$ defined on a set (of alternatives) X is any subset $A(\succ) \subseteq X \times X$. If a pair $(x, y) \in A(\succ)$, we write $x \succ y$, which is interpreted to mean that alternative x is *better than* alternative y. The binary relation $\succ$ is said to be:

- *Irreflexive* if for all x, *not* $(x \succ x)$.
- *Asymmetric* if $x \succ y$ implies *not* $(y \succ x)$.
- *Complete* if for all $x \neq y, x \succ y$, or $y \succ x$. Asymmetric, not necessarily complete binary relations are known as *weak tournaments*. Asymmetric and complete binary relations are called *tournaments*.
- *Acyclic* if $x_1 \succ x_2 \succ \cdots \succ x_k$ implies *not* $(x_k \succ x_1)$.
- *Transitive* if for all $x, y, z, x \succ y$ and $y \succ z$ implies $x \succ z$. A *linear order* is a complete, transitive, and irreflexive binary relation.

The *transitive closure* $\ggcurly$ of an asymmetric relation $\succ$ on X is defined as $x \ggcurly y$ if and only if there exist $x_1, x_2, ..., x_k \in X$, $k \geq 2$, such that

$$x = x_1 \succ x_2 \succ \cdots \succ x_k = y.$$

When $x \ggcurly y$, we say that the alternative y is *indirectly dominated* by the alternative x. The *congruence* (equivalence) relation is defined as

$$x \cong y \text{ if and only if } x = y \text{ or } \left[x \ggcurly y \text{ and } y \ggcurly x\right].$$

We denote the quotient set by $\mathbb{X}$, and $[x]$ will denote the equivalence class in $\mathbb{X}$ containing the element $x \in X$. The extended preference relation $\triangleright$ on $\mathbb{X}$ is defined, for all $[x], [y] \in \mathbb{X}$, by $[x] \triangleright [y]$ if and only if

$$[x] \neq [y] \text{ and there exists } a \in [x], b \in [y] \text{ such that } a \ggcurly b.$$

2.2 | Choice functions and rationalizability

Throughout the paper, we denote by $\mathcal{S}$ the (finite) set of all possible alternatives and by $\mathcal{P}(\mathcal{S})$ the family of nonempty subsets of $\mathcal{S}$. A *choice function* associates a nonempty subset to each feasible set $X \in \mathcal{P}(\mathcal{S})$:

$$F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S}) \text{ such that for all } X \in \mathcal{P}(\mathcal{S}), \varnothing \neq F(X) \subseteq X.$$

A choice function G is a selection of F if for all $X \in \mathcal{P}(\mathcal{S})$, $G(X) \subseteq F(X)$.
Associated with any choice function F, the *base relation* $\succ^b$ is defined by

$$\text{for all } x, y \in \mathcal{S} \text{ with } x \neq y, x \succ^b y \quad \text{if and only if} \quad F(\{x, y\}) = \{x\}.$$

When there is no confusion, we will omit the subindex and simply denote the base relation by $\succ^b$. This relation is asymmetric, although it may not be complete.

Given an asymmetric binary relation $\succ$, the set of *undominated* elements, or *maximal set*, in a subset $X \in \mathcal{P}(\mathcal{S})$ is

$$M(X, \succ) = \{x \in X : \text{for all } y \in X, \textit{not}\,(y \succ x)\}.$$

It should be noticed that the maximal set $M(X, \succ)$ may be empty for general asymmetric binary relations. A necessary and sufficient condition to ensure the nonemptyness of the maximal set for every $X \in \mathcal{P}(\mathcal{S})$ is the acyclicity of the asymmetric relation $\succ$ (see, e.g., Peris & Subiza, 1994). A choice function is rationalizable if the choice is based on the maximization of some binary relation.

Definition 1. A choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is *rationalizable* if there exists an asymmetric binary relation $\succ$ such that

$$F(X) = M(X, \succ) \quad \text{for all } X \in \mathcal{P}(\mathcal{S}).$$

We say that the binary relation $\succ$ *rationalizes* F.

Remark 1. If a choice function F is rationalizable by means of the asymmetric binary relation $\succ$, it is immediate to show that the base relation $\succ^b$ is acyclic and rationalizes the choice function; that is, $F(X) = M(X, \succ^b)$, since

$$x \succ^b y \Leftrightarrow F(\{x, y\}) = \{x\} = M(\{x, y\}, \succ) \Leftrightarrow x \succ y,$$

If the choice function represents the election made by a group of agents (e.g., by using a majority voting procedure), the absence of cycles is not a realistic assumption. Therefore, the set of maximal elements could be empty in some feasible subsets $X \in \mathcal{P}(\mathcal{S})$. In these contexts, as pointed out by Schwartz (1986), *"how reasonable is rationalizability?"* Schwartz (1972, 1986) introduced the *generalized optimal choice set* as an extension of the set of maximal elements. When the binary relation is a tournament, this set coincides with the well-known notion of *top cycle*.

Definition 2. Given an asymmetric binary relation $\succ$, the *generalized optimal choice set*, $Goc(X, \succ)$, is defined by the union of the maximal elements (classes) in the problem $(\mathbb{X}, \rhd)$,

$$Goc(X, \succ) = \bigcup_{[x] \in M(\mathbb{X}, \rhd)} [x].$$

Remark 2. The *Goc* set is always nonempty and it selects alternatives as if individuals (or society) were maximizing some preference relation (in a quotient set). The main drawback of this solution concept is that it frequently selects too many alternatives, even the entire set of alternatives.

In Subiza and Peris (2000) a weaker notion of rationalizability is presented (extending the notion of subrationalizable choice functions; Deb, 1983).

Definition 3. A choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is *basically subrationalizable* if $F(X) \supseteq M(X, \succ^b)$ for all $X \in \mathcal{P}(\mathcal{S})$.

The relevant fact in this notion is that the acyclicity of the base relation is not required, and the *top cycle* or the *uncovered set* define basically subrationalizable choice functions (for additional properties and a more detailed discussion of basically subrationalizable choice functions, see Subiza & Peris, 2000, where this condition was introduced).

Brandt and Harrenstein (2011) introduce the notion of *self-stable* choice function as an alternative to the concept of rationalizability.

Definition 4. A choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is *self-stable* if for all $X, Y \in \mathcal{P}(\mathcal{S})$

1. $F(X) \subseteq Y \subseteq X$ implies $F(X) = F(Y)$,[1] and
2. $F(X) = F(Y)$ implies $F(X \cup Y) = F(X) = F(Y)$.

They show that the top cycle, the minimal covering, or the bipartisan set are also self-stable choice functions. Self-stable choice functions are basically subrationalizable, but there is no inclusion relationship between rationalizable and self-stable choice functions (Brandt & Harrenstein, 2011).

2.3 | Standards of behavior: Stability

The relevance of the maximal (*undominated* alternatives) may be interpreted in terms of stability. Nonmaximal (dominated) alternatives can be upset by other alternatives showing to be *unstable choices*. The notion of *stability* (Von Neumann & Morgenstern, 1944) captures this idea. Stable sets were originally introduced in the context of (cooperative) game theory rather than social choice. Nevertheless, this notion is usually applied to choice situations: marriage problems, school allocation, and so forth.

Given a feasible set $X \in \mathcal{P}(\mathcal{S})$, a nonempty subset $V \subseteq X$ is *vNM-stable* in X with respect to some asymmetric relation $\succ$ if:

(a) for all $x, y \in V$, *not* $(x \succ y)$; and
(b) for all $z \in X \backslash V$ there is $x \in V$ such that $x \succ z$.

The first condition (*internal stability*) makes the elements in V not comparable, so none of them can be considered better off. The second condition (*external stability*) is used to discredit

[1]This condition is known in the social choice literature as the *strong superset* property (Chernoff, 1954).

alternatives outside V. The elements in a vNM-stable set may be, in this sense, considered as the *best elements*. Then, choice functions should return vNM-stable sets.

Definition 5. A choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is *vNM-stable* if there is an asymmetric binary relation $\succ$ such that for all $X \in \mathcal{P}(\mathcal{S}), F(X)$ is vNM-stable in X with respect to $\succ$.

Remark 3. As in the case of rationalizable choice functions, it is immediate that, if a choice function is stable, then the base relation $\succ^b$ provides the vNM-stability since

$$x \succ^b y \Leftrightarrow F(\{x, y\}) = \{x\} \Leftrightarrow x \succ y,$$

where the second equivalence is due to external stability of the set $F(\{x, y\})$. Then, the vNM-stability conditions can be written in the following way: a choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is vNM-stable with respect to $\succ^b$ if for all $X \in \mathcal{P}(\mathcal{S})$,

(a) $x, y \in F(X)$ implies $F(\{x, y\}) = \{x, y\}$; and
(b) $z \in X \backslash F(X)$ implies $F(\{x, z\}) = \{x\}$ for some $x \in F(X)$.

Although it seems that vNM-stability is a weakening of rationalizable choice function, this only happens whenever the binary relation $\succ^b$ is complete (*complete binary discrimination*, see Section 5). Example 1 proves that these properties are independent, showing a rationalizable choice function that is not vNM-stable, and a vNM-stable choice function that is not rationalizable.

Example 1. Let $\mathcal{S} = \{a, b, c\}$ be the set of alternatives and the choice function $F(X) = X$ if $|X| = 1, F(\{a, b\}) = \{a\}, F(\{a, c\}) = \{a, c\}, F(\{b, c\}) = \{b\}$, and $F(\mathcal{S}) = \{a\}$. Then F is rationalizable, although $F(\mathcal{S}) = \{a\}$ is not a vNM-stable set.

On the other hand, let $\mathcal{S} = \{a, b, c\}$ be the set of alternatives and the choice function $G(X) = X$ if $|X| = 1, G(\{a, b\}) = \{a\}, G(\{a, c\}) = \{a, c\}, G(\{b, c\}) = \{b\}$, and $G(\mathcal{S}) = \{a, c\}$. Then G is vNM-stable, although it is not rationalizable, since

$$G(\mathcal{S}) = \{a, c\} \neq M(\mathcal{S}, \succ^b) = \{a\}.$$

Remark 4. As far as we know, Knoblauch (2020) is the first reference analyzing if a choice function is vNM-stable (vNM-rationalizable). Knoblauch (2020) shows that when transitivity of the asymmetric binary relation is required, then the notions of vNM-stable and rationalizable choice functions coincide. We show in Section 5 that this is also true for acyclic binary relations, when complete binary discrimination is fulfilled (this is not the case of the choice functions in Example 1). A significant difference between our model and the one in Knoblauch (2020) is that we consider every *agenda* (every $X \subseteq S$) as a possible feasible set for the choice function, whereas Knoblauch (2020) allows for a domain containing only some of the possible agendas.

Several generalizations of the Von Neumann and Morgenstern stability have appeared in the literature. These extensions are based on the use of the transitive closure (indirect dominance) and are referred to as *farsighted stability*. This concept was first used by Harsanyi

(1974) in the context of cooperative game theory. For the sake of completeness we define these stability notions, although we are not going to use them.

Let $V \subseteq X$, $V \neq \emptyset$, let $\succ$ be an asymmetric binary relation defined on X, and let $\gg$ be the transitive closure of $\succ$. Then,

- V is a *generalized stable set* (Harsanyi, 1974; Van Deemen, 1991) if
  (a) for all $x, y \in V$, *not* $(x \gg y)$; and
  (b) for all $z \in X \setminus V$ there is $x \in V$ such that $x \gg z$.
- V is an *m-stable set* (Peris & Subiza, 2013) if
  (a) for all $x, y \in V$, $x \gg y$ implies $y \gg x$; and
  (b) for all $z \in X \setminus V$, $z \gg x$ for no $x \in V$.
- V is a *w-stable set* (Han & Van Deemen, 2016) if
  (a) for all $x, y \in V$, *not* $(x \gg y)$; and
  (b) for all $x \in V, z \in X \setminus V$, if $z \gg x$ then $x \gg z$.

We could now replicate Definition 5 to introduce the notions of generalized stable, m-stable, or w-stable choice function. However, as Example 2 shows, the usual choice functions (top cycle, uncovered set, ...) do not satisfy any of these notions of stability. Since our goal is to reconcile stability with rationality, a new notion of stability is required. As mentioned in Schwartz (1972) and Moulin (1986) "a minimal rationality requirement of any choice function is that it produces a subset of the top cycle."

Example 2. Let $\mathcal{S} = \{a, b, c, d, x, y, z\}$ be the universal set of alternatives and the tournament T defined by Figure 1.

If we compute some usual choice functions (*top cycle, uncovered set, minimal covering*), we obtain

$$TC(\mathcal{S}, T) = \mathcal{S}, \quad UC(\mathcal{S}, T) = \{a, b, c, d\}, \quad MC(\mathcal{S}, T) = \{a, b, c\}.$$

None of them is a vNM-stable, generalized stable, or w-stable set, so these choice functions do not satisfy the above-mentioned stability conditions. The uncovered set and the minimal covering are not m-stable either.

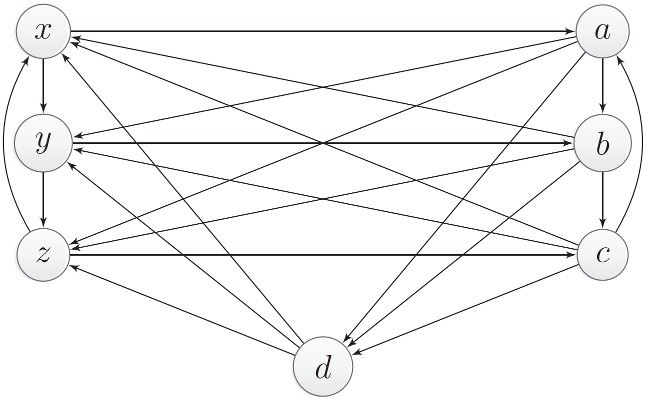

FIGURE 1 Tournament in Example 2

3 | A NEW NOTION: r –STABILITY

Our stability proposal tries to *combine* the concepts of stability and rationalizability (then, we name it r(ational)-stability).

Definition 6. Let $X \in \mathcal{P}(\mathcal{S})$ and $\succ$ an asymmetric binary relation defined on X. A subset $V \subseteq X$ is *r-stable* in X with respect to $\succ$ if:

(a) for all $x, y \in V, x \gg y$ implies $y \gg x$; and
(b) for all $z \in X \setminus V$, there is $x \in V$ such that $x \gg z$.

Note that the first condition (*internal stability*) coincides with the corresponding condition in m-stability. This condition says that if some element $x \in V$ indirectly dominates some other element $y \in V, x \gg y$, then y also indirectly dominates x and both relations could be interpreted as if they cancel each other out, so both elements x, y could be jointly selected.

The second condition in Definition 6 (*external stability*) coincides with the corresponding one in generalized stability. So, we use the farsighted dominance relation to discredit unselected alternatives. Although we combine conditions of two previous stability notions, the new notion (and its properties) is very different, as the results and examples in Section 4 will show.

Remark 5. The *generalized optimal choice set*, $Goc(X, \succ)$ can be alternatively defined as

$$Goc(X, \succ) = \left\{x \in X : \text{for any } y \gg x, \text{ then } x \gg y\right\}.$$

Then, r-stable sets are subsets $V \subseteq Goc(X, \succ)$ such that every alternative $z \in X \setminus V$ is farsighted dominated by some alternative $x \in V$. In particular, we will show that $Goc(X, \succ)$ is r-stable.

Definition 7. A choice function $F : \mathcal{P}(\mathcal{S}) \rightarrow \mathcal{P}(\mathcal{S})$ is *r-stable* if there is an asymmetric binary relation $\succ$ such that for all $X \in \mathcal{P}(\mathcal{S}), F(X)$ is an r-stable set in X with respect to $\succ$.

As in the case of rationalizable or vNM-stable choice functions, the binary relation providing r-stability coincides with the base relation. To check this property, note that if F is r-stable, for all $x, y \in \mathcal{S}, F(\{x, y\})$ is an r-stable set with respect to some asymmetric relation $\succ$. Then, one of the following cases must be true:

1. $F(\{x, y\}) = \{x, y\}$: then, as $\succ$ is asymmetric, r-stability implies that none of the possibilities $x \succ y$, or $y \succ x$ holds.
2. $F(\{x, y\}) = \{x\}$ if and only if $x \succ y$.
3. $F(\{x, y\}) = \{y\}$ if and only if $y \succ x$.

In any case, relation $\succ$ coincides with the base relation.

4 | RESULTS

This section deals with the general results of r-stability and compares it with rationalizable, basically subrationalizable, vNM-stable, or self-stable choice functions. First, we show the characteristics of r-stable sets, that only select alternatives in all maximal classes in the quotient set. Our main result shows that every rationalizable choice function is r-stable, and that any vNM-stable choice function has an r-stable selection. Therefore, in this sense, r-stability performs better than other extensions of the classical stability. Another property we find for r-stability is that this concept can be placed between the strongest rationalizability concept and the very weak notion of basically subrational choice function. Finally, we also compare r-stability with self-stability. Figure 2 summarizes the results.

Our first result completely characterizes the form of r-stable sets. It consists of alternatives selected from, and exclusively from, each *maximal class* of the extended preference relation $\rhd$ in the quotient set $\mathbb{X}$.

Theorem 1. *Let $X \in \mathcal{P}(\mathcal{S})$ and let $\succ$ be an asymmetric binary relation defined on X. Then, V is an r-stable set if and only if it intersects every maximal class in the quotient set:*

$$\text{for all } [x] \in M(\mathbb{X}, \rhd), V \cap [x] \neq \varnothing.$$

Furthermore, any r-stable set is included in $Goc(X, \succ)$, and a vNM-stable set always includes an r-stable set.

Proof. If V is an r-stable set, it is clear that V must intersect every maximal class, since if $V \cap [x] = \varnothing$, for some maximal class $[x]$, then $x \notin V$ and there is no element $y \in V \setminus [x]$ such that $y \gg x$, so the external stability condition is violated. Moreover, it is clear that if V contains some element y outside the maximal classes, there is some maximal class $[x]$, and some alternative $x \in V \cap [x]$, such that $x \gg y$ and *not* $(y \gg x)$. So, the internal stability condition would be violated. Therefore, the set V takes the required form.

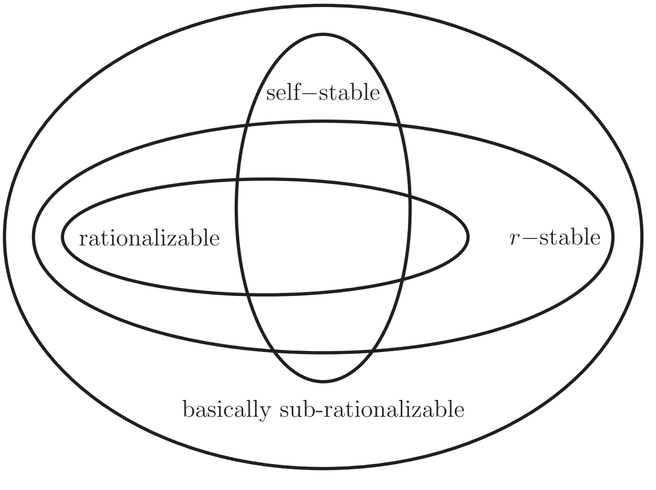

FIGURE 2 Inclusion relationship: general case

On the other hand, if V intersects all maximal classes it can be written in the following way (as the union of nonempty subsets of any maximal class):

$$V = \bigcup_{[x]\in M(\mathbb{X},\triangleright)} \{V_x : \varnothing \neq V_x \subseteq [x]\}$$

by simply setting $V_x = V \cap [x]$. Let us show that the above set is r-stable. If $x, y \in V$, we have two possibilities:

(1) Both alternatives belong to the same maximal class, in which case $x \gg y$ and $y \gg x$.
(2) They belong to different maximal classes. In this case, no dominance relation exists among these alternatives.

So the internal condition is satisfied. Now, if $z \notin V$, then:

(a) If $z \in [x]$, and $[x]$ is a maximal class, then there is some alternative $a \in [x] \cap V$, so $a \gg z$; or
(b) If alternative z lies in a nonmaximal class, there is some maximal class $[x]$ such that $[x] \triangleright [z]$. Moreover, there is some alternative $a \in [x] \cap V$, and therefore $a \gg z$, so the external condition in r-stability is also satisfied, and V is an r-stable set.

Now, as the generalized optimal choice set, *$Goc(X, \succ)$, coincides with the union of all maximal classes, any r-stable set is included in $Goc(X, \succ)$.*

Finally, Peris and Subiza (2013) show that any vNM-stable set W must intersect every maximal class, although W may contain elements in nonmaximal classes. Then, if $y \in W$ does not belong to a maximal class, there is some maximal class $[x]$, and some alternative $x \in W \cap [x]$, such that $x \gg y$ and *not* $(y \gg x)$. So W does not satisfy the internal condition in r-stability. Nevertheless, we can find a subset of W (e.g., eliminating the elements which do not belong to a maximal class), satisfying r-stability. □

As a consequence we obtain that r-stable sets always exist.

Corollary 1. *For any $X \in \mathcal{P}(\mathcal{S})$ and any asymmetric binary relation $\succ$ defined on X, there always exists an r-stable set in $(X, \succ)$.*

Remark 6. Then, as in the generalized optimal choice, *Goc*, the r-stable choice functions select subsets as if individuals (or society) were maximizing preferences, but we allow the elements in each maximal class to be chosen arbitrarily, so the result can be more discriminatory.

4.1 | Rationalizable and stable choice

As we have already shown rationalizability and vNM-stability are independent properties. However, the following result shows that r-stability is also an extension of the notion of rationalizable choice functions.

Theorem 2. *Every rationalizable choice function is r-stable, and the converse is not true.*

Proof. Example 1 shows an r-stable choice function which is not rationalizable. Let us now consider a rationalizable choice function F. We know that

$$F(X) = M(X, \succ^b) \quad \text{for all } X \in \mathcal{P}(\mathcal{S}).$$

Given $x, y \in F(X)$, since both are maximal elements, no dominance exists among them and the internal condition is fulfilled.

If $z \notin F(X)$, then it is not a maximal alternative, so there is some $y_1 \in X$ such that $y_1 \succ^b z$. If $y_1 \in F(X)$, the external condition holds. In the other case, y_1 is not maximal and some $y_2 \in X$ exists such that $y_2 \succ^b y_1$, so $y_2 \gg^b z$. By repeating this reasoning, since the feasible set X is finite, and the relation $\succ^b$ is acyclic, we will find some alternative $x \in F(X)$ such that $x \gg^b z$. Then, the external condition is fulfilled and F is an r-stable choice function. □

Then, joining the results in Theorems 1 and 2 we obtain the following result that shows how r-stability extends both vNM-stability and rationalizability notions.

Theorem 3. *Let F be a choice function. Then,*

1. *If F is stable, it has an r-stable selection.*
2. *If F is rational, it is r-stable.*

The first property is interesting because vNM-stable sets are not guaranteed to exist, whereas r-stable sets are. Thus one could, for instance, look for the smallest r-stable choice function. In Section 6 we propose some ways of defining a choice function by using r-stable sets.

4.2 | Other rationality notions

As mentioned in Section 2.2, a more general notion is that of basically subrationalizable choice function. This notion is fulfilled by usual tournament solutions as *top cycle* or *uncovered set*. The problem with this concept is that, as shown in the following example, some "not so reasonable" choice functions may fulfill this property.

Example 3. Let us consider a finite set of alternatives and a choice function F such that for every subset X with $|X| \neq 2$ selects all alternatives, that is, $F(X) = X$, and for sets with two alternatives selects exactly one of them in an arbitrary way. Then, this choice function is basically subrationalizable.

In the next result we show that r-stability is an intermediate step between rationalizable and basically subrationalizable choice functions. It must be noticed that choice functions as in Example 3 (random binary choice) are not r-stable.

Theorem 4. *Every r-stable choice function is basically subrationalizable, and the converse is not true.*

Proof. To show that r-stability is a stronger condition, let us consider an r-stable choice function F and suppose that, for some feasible subset of alternatives $X \in \mathcal{P}(\mathcal{S}), F(X) \supseteq M(X, \succ^b)$ is not fulfilled. Then, there exists some alternative $a \in M(X, \succ^b)$ such that $a \notin F(X)$. External stability implies the existence of $z \in X$, such that $z \gg^b a$, which contradicts that a is a maximal element. Example 3 shows that the converse is not true. □

Regarding self-stability (Brandt & Harrenstein, 2011), Example 4 shows an r-stable choice function that is not self-stable and a self-stable choice function that is not r-stable. In consequence, these properties are independent.

Example 4. Consider the set of alternatives $\mathcal{S} = \{a, b, c\}$ and the choice function $F(X) = X$ if $|X| = 1, F(\{a, b\}) = \{a\}, F(\{a, c\}) = \{c\}, F(\{b, c\}) = \{b\}$, and $F(\mathcal{S}) = \{a, b\}$. Then F is not self-stable, although it is r-stable.

Let $\mathcal{S} = \{a, b, c, y\}$ be the universal set of alternatives and the choice function defined by $F(X) = X$ if $|X| = 1$, and the choices:

X	$\{a,b\}$	$\{a,c\}$	$\{a,y\}$	$\{b,c\}$	$\{b,y\}$	$\{c,y\}$
$F(X)$	$\{a\}$	$\{c\}$	$\{a\}$	$\{b\}$	$\{b\}$	$\{c\}$
X	$\{a,b,c\}$	$\{a,b,y\}$	$\{a,c,y\}$	$\{b,c,y\}$	$\{a,b,c,y\}$	
$F(X)$	$\{a,b,c\}$	$\{a\}$	$\{c\}$	$\{b\}$	$\{a,b,c,y\}$	

It is clear that F is self-stable, but $F(\mathcal{S}) = \mathcal{S}$ is not an r-stable set. If we observe the choices, alternative y is rejected in pairwise comparison with any other alternative. However, this alternative is selected when the whole set is considered.

Figure 2 summarizes the obtained general relationships between rationalizable, basically subrationalizable, self-stable, and r-stable choice functions.

Remark 7. Brandt and Harrenstein (2011) show that many social choice functions may fail to be self-stable. They show that is the case, for instance, of "all scoring rules (e.g., plurality rule or Borda's rule), all scoring runoff rules (e.g., Hare's rule or Coombs' rule), all weak Condorcet extensions as well as a number of other common SCFs." Their argument also shows that the choice functions mentioned above are not r-stable. In general, social choice functions that can select a non-Condorcet winner (e.g., Borda's rule) are not r-stable, due to the external stability condition (the same occurs with other stability notions).

Knoblauch (2020) analyzes acyclic vNM-stable choice functions; that is, choice functions F such that for all $X \in \mathcal{P}(\mathcal{S}), F(X)$ is a vNM-stable set and, additionally, the base relation $\succ^b$ is acyclic. The following example shows that although we require acyclicity, vNM-stability and r-stability do not coincide.

Example 5. Let $\mathcal{S} = \{a, b, c\}$ and the choice function $F(X) = X$ if $|X| = 1$,

$$F(\{a, b\}) = \{a\}, \quad F(\{a, c\}) = \{a, c\}, \quad F(\{b, c\}) = \{b\}, \quad F(\mathcal{S}) = \{a, c\}.$$

This is a vNM-stable choice function and the base relation $\succ^b$ is acyclic. Nevertheless, F is not r-stable, since $F(\mathcal{S})$ is not an r-stable set.

The following result shows that acyclic r-stability is equivalent to rationalizability.

Theorem 5. *Let F be a choice function such that $\succ^b$ acyclic. Then F is r-stable if and only if it is rationalizable.*

Proof. Let us suppose that F is r-stable and that $\succ^b$ is acyclic. We show that for all $X \subseteq \mathcal{S}, F(X) = M(X, \succ^b)$ and therefore F is rationalizable.

If $x \in F(X)$, but $x \notin M(X, \succ^b)$, then there is $z \in X$ such that $z \succ^b x$. If $z \in F(X)$ from the internal condition of r-stability, $x \gg^b z$ that implies the existence of a cycle, a contradiction. If $z \notin F(X)$ the external condition of r-stability implies the existence of $w \in F(X)$ such that $w \gg^b z$, so $w \gg^b x$ and internal r-stability implies $x \gg^b w$ and the existence of a cycle, contradicting the hypothesis. Therefore,

$$F(X) \subseteq M(X, \succ^b).$$

If $x \in M(X, \succ^b)$, but $x \notin F(X)$, external r-stability implies the existence of $z \in F(X)$ such that $z \gg^b x$, contradicting that x is a maximal element. Therefore,

$$M(X, \succ^b) \subseteq F(X).$$

Conversely, if F is rationalizable the base relation $\succ^b$ is acyclic and Theorem 2 implies that F is r-stable. □

Knoblauch (2020) proves that if the binary relation that provides rationalizability or vNM-stability is transitive, then both conditions coincide. As a consequence, the following result is obtained.

Corollary 2. *Let F be a choice function $F : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ such that $\succ^b$ is transitive. Then, the following conditions are equivalent:*

(a) F is rationalizable.
(b) F is vNM-stable.
(c) F is r-stable.

5 | COMPLETE BINARY DISCRIMINATION

This section deals with a particular but very relevant case: We assume that the choice between two alternatives is always decisive; that is, the *selector* (the agent, the society, ...) can always select one of any two alternatives. Although decisiveness is controversial when many alternatives are presented to choose from, it is less restrictive when applied to binary options. In the case of sports competitions, this means that ties are not allowed. We obtain that many important solution concepts in this case are r-stable choice functions. As an important result we will see that in this particular case the concepts of rationalizable and vNM-stable choice function coincide. Figure 3 summarizes the results in this particular context.

If the choice function completely discriminates whenever the set presented for choice has just two alternatives, which we call *complete binary discrimination* (CBD), that is

$$\text{for all } x, y \in \mathcal{S}, |F(\{x, y\})| = 1, \qquad \text{(CBD)}$$

the base relation $\succ_F^b$ is complete, so it defines a *tournament*. The particular class of decision problems in which the asymmetric binary relation is a *tournament* has been widely analyzed (see, e.g., Brandt, 2011; Laslier, 1997; or Moulin, 1986). In this context, the quotient set contains just one maximal class, the *top cycle*, and every nonempty subset of the *top cycle* is an r-stable set. In particular, if the maximal class contains just one element, this alternative dominates every other in the feasible set (the *Condorcet winner* of the tournament).

As most tournament solutions are included in the *top cycle* (Brandt, 2011), they are r-stable choice functions, although none of them is vNM-stable.

Corollary 3. *When applied to tournaments, the top cycle* (*TC*), *the uncovered set* (*UC*), *the minimal covering set* (*MC*), *and the bipartisan set* (*BP*) *are r-stable choice functions.*

Proof. Let $X \in \mathcal{P}(\mathcal{S})$ and $\succ$ a tournament defined on X. We know (see, e.g., Brandt, 2011) the chain inclusion

$$BP(X, \succ) \subseteq MC(X, \succ) \subseteq UC(X, \succ) \subseteq TC(X, \succ)$$

and Theorem 1 implies r-stability. □

The following result shows that, although generally the intersection of r-stable choice functions is not r-stable, r-stability is preserved under intersections when the considered choice functions fulfill condition CBD.

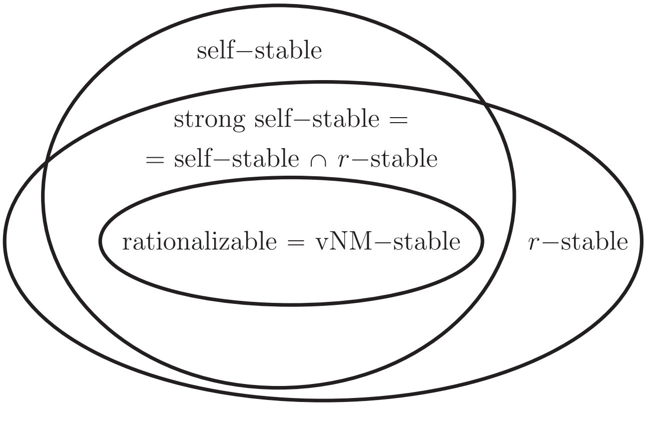

FIGURE 3 Inclusion relationships for choice functions fulfilling CBD

Theorem 6. *Let F, G be r-stable choice functions fulfilling CBD such that they have nonempty intersection for any set $X \in \mathcal{P}(\mathcal{S})$. Then $F \cap G$ also satisfies r-stability.*

Proof. Since both choice functions fulfill CBD and they do intersect for each feasible set X, we easily obtain that the base relations coincide. As the relation is a tournament, there is just one maximal class, the *generalized optimal choice set*, and r-stable choice functions just select subsets of $Goc(X, \succ^b)$, whose intersection, if nonempty, is also a subset of $Goc(X, \succ^b)$, so it is r-stable. □

Remark 8. A consequence of Theorem 6 is that intersections of tournament solutions, if nonempty, are also r-stable sets. This property allows us to obtain more discriminating choice functions. This is the case, for instance, of Banks set, BA (Banks, 1985), and minimal covering set, MC. Their intersection is always nonempty (see, e.g., Brandt, 2011), so $BA \cap MC$ is a more discriminating choice function and, as they always do intersect, it is an r-stable choice function.

Our next result shows that rationalizability and vNM-stability are equivalent conditions on choice functions fulfilling complete discrimination on binary choices.

Theorem 7. *Let F be a choice function fulfilling condition CBD. Then, F is vNM-stable if and only if it is rationalizable.*

Proof. If F is rationalizable, $F(X) = M(X, \succ^b)$ for all $X \in \mathcal{P}(\mathcal{S})$ and, under condition CBD, the base relation is a linear order. Then $F(X)$ contains just one alternative and it is a vNM-stable set, so F fulfills vNM-stability.

On the other hand, if F is a vNM-stable choice function fulfilling CBD then the base relation $\succ^b$ is acyclic. To prove this property, let us consider alternatives $x_1, x_2, \ldots, x_k \in \mathcal{S}$ such that

$$x_1 \succ^b x_2 \succ^b \cdots \succ^b x_k \quad \text{and} \quad x_k \succ^b x_1.$$

If $x_2 \succ^b x_k$, then there is no vNM-stable subset in $\{x_1, x_2, x_k\}$, a contradiction. Then, by completeness, $x_k \succ^b x_2$. If $x_3 \succ^b x_k$, then there is no vNM-stable subset in $\{x_2, x_3, x_k\}$, a contradiction. Then, $x_k \succ^b x_3$. By repeating this argument, we always obtain a contradiction. So, cycles are not possible. Then, for all $X \in \mathcal{P}(\mathcal{S})$ the only vNM-stable subset coincides with the maximal set and F is rationalizable. □

The following result shows that, contrary to what happens in the general case, under complete binary discrimination self-stability extends the notion of rationalizability (and, from Theorem 7, the notion of vNM-stable choice function).

Theorem 8. *Let F be a choice function fulfilling condition CBD. If F is rationalizable, then it is self-stable.*

Proof. We know that if F is rationalizable, under condition CBD the base relation is a linear order. Then $F(X) = M(X, \succ^b)$ contains just one element. Then, it is immediate to observe that both conditions in self-stability are fulfilled. □

The following example shows a self-stable choice function that is not rationalizable, so self-stability defines a wider class in the case of choice functions fulfilling complete binary discrimination.

Example 6. Let $\mathcal{S} = \{a, b, c\}$ be the universal set of alternatives and the choice function defined by $F(X) = X$ if $|X| = 1$, and

X	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$F(X)$	$\{a\}$	$\{c\}$	$\{b\}$	$\{a,b,c\}$

It is easy to observe that F is self-stable (both conditions 1 and 2 in Definition 4 are fulfilled). Nevertheless, F is not rationalizable since

$$F(\mathcal{S}) = \{a, b, c\} \neq M(\mathcal{S}, \succ^b) = \varnothing.$$

Even in the case of choice functions fulfilling complete binary discrimination, self-stability does not imply r-stability, as shown in Example 4. Nevertheless, we identify a natural subclass of self-stable choice functions that is included in the r-stable class. Note that self-stable choice functions fulfill the *direct* generalized Condorcet condition (Brandt & Harrenstein, 2011):

$$\text{if } y = F(\{x, y\}) \quad \text{for all } x \in X, \text{ then } F(X \cup \{y\}) = \{y\}.$$

We introduce a new notion of set-rationalizability by adding a rejecting Condorcet property: *if an alternative is rejected in pairwise choice in front of any alternative of a set X, then it is rejected in front of the whole set X.*

Definition 8. A choice function $F: \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$ is said to be *strong self-stable* if it is self-stable and fulfills

$$\textit{if } y \notin F(\{x, y\}) \quad \text{for all } x \in X, \text{then } y \notin F(X \cup \{y\}).$$

Finally, we prove that, in case of choice functions fulfilling complete binary discrimination, every strong self-stable choice function is an r-stable choice function.

Theorem 9. *Let F be a choice function fulfilling condition CBD. If F is strong self-stable, then it is r-stable.*

Proof. We will show that, for any set $X \in \mathcal{P}(\mathcal{S})$, the choice set $F(X)$ is included in the top cycle $TC(X, \succ^b)$. Then, from Theorem 1, $F(X)$ is an r-stable set.

Let us consider $y \in X, y \notin TC(X, \succ^b)$. Then, for all $x \in TC(X, \succ^b)$ it must be $F(\{x, y\}) = \{x\}$ and the rejecting Condorcet property implies that $y \notin F(TC(X, \succ^b) \cup \{y\})$. Moreover, condition (1) in self-stability implies that

$$F(TC(X, \succ^b) \cup \{y\}) = F(TC(X, \succ^b)).$$

If we apply condition (2) in self-stability to sets $B_1 = TC(X, \succ^b) \cup \{y_1\}$ and $B_2 = TC(X, \succ^b) \cup \{y_2\}$, for $y_1, y_2 \in X \setminus TC(X, \succ^b)$, we get

$$F(TC(X, \succ^b) \cup \{y_1, y_2\}) = F(TC(X, \succ^b)).$$

And, by repeating this argument, $F(X) = F(TC(X, \succ^b)) \subseteq TC(X, \succ^b)$. □

Example 4 shows an r-stable choice function fulfilling complete binary discrimination that is not strong self-stable, since it is not self-stable, so the inclusion is strict.

Remark 9. It is easy to prove that for choice functions fulfilling complete binary discrimination r-stability implies the rejecting Condorcet property, so the intersection of r-stable and self-stable choice functions coincides with the class of strongly self-stable choice functions. Then, we have the following result.

Corollary 4. *Let F be a choice function fulfilling condition CBD. Then F is strong self-stable if and only if it is self-stable and r-stable.*

Remark 10. In social choice parlance, Definition 8 strengthens self-stability by the requirement that Condorcet losers must not be chosen. Then, Corollary 5 also follows straightforwardly from the fact that the top cycle can be characterized by self-stability and an axiom that prevents the selection of Condorcet losers (Brandt, 2011).

Figure 3 illustrates the obtained results under complete binary discrimination.

6 | CONCLUDING REMARKS

We have introduced a new stability notion that can be considered an extension of both rationalizability and classical stability properties:

> Every rationalizable choice function is r-stable and every stable choice function has an r-stable selection.

Moreover, it is an intermediate step between rationalizable and basically subrationalizable choice functions. If the base relation $\succ^b$ turns out to be acyclic, then for each $X \in \mathcal{P}(\mathcal{S})$ there is only one r-stable set that coincides with the *maximal set*. In more general cases, as tournaments, we have shown that the usual choice functions fulfill r-stability.

On the other hand, we can consider the use of r-stability itself as a *solution concept* for decision problems. As we have shown, this solution always exists. Nevertheless, in general, there are several r-stable subsets of alternatives. As each one may consist in several alternatives (at least one in each maximal class, Theorem 1), we would be interested in selecting *minimal* (with respect to the set inclusion) r-stable sets; that is, to select just one element in each maximal class of the quotient problem $(\mathbb{X}, \rhd)$.

For instance, let $X \in \mathcal{P}(\mathcal{S})$ and let $\succ$ be an asymmetric binary relation defined on X. Then, we can select in each maximal class $[x]$:

- the element(s) x^* maximizing the number of directly dominated alternatives in $[x]$, $\arg\max_{[x]}\{|(z \in [x] : x^* \succ z)|\}$; or

- the element(s) x^* maximizing the number of directly dominated alternatives in X, $\arg\max_{[x]}\{|(z \in X : x^* \succ z)|\}$.

Using the last possibility, we can define the following choice function:

$$F_r(X, \succ) = \bigcup_{[x]\in M(\mathbb{X},\triangleright)} \{\arg\max_{x^*\in[x]} \{|(z \in X : x^* \succ z)|\}\}.$$

This is obviously an r-stable choice function.

Finally, and connecting with the classical literature on choice functions, we would like to make some axiomatic considerations regarding r-stability. Although we are defining a family of choice functions (those that satisfy r-stability) rather than a single choice function (or tournament solution), we can test whether all the functions in the family satisfy some basic properties. To this end, it is easy to check that r-stable choice functions fulfill *Condorcet's consistency* (they select the Condorcet winner, whenever it exists), or *Smith's consistency*, a condition linked to the top cycle.

A related axiom is *composition consistency* (Laffond et al., 1996). This property may be interpreted as if the set of alternatives was composed of different projects and the composition consistent solution will select the best outcomes of the best projects. Like composition consistency, r-stable choice functions also choose elements in the "best projects" (the maximal classes), but since the elements in each maximal class are *equally* good this selection can be made randomly, or the entire class may be selected. Laffond et al. (1996) show that the top cycle does not fulfill this property, so r-stable choice functions may fail to fulfill composition consistency. They also prove that Condorcet's and composition consistencies imply that the selected set is included in the top cycle. As a consequence of Theorem 1, the following result is obtained.

Corollary 5. *Let F be a CBD choice function. If F fulfills composition consistency and Condorcet's consistency, then it is an r-stable choice function.*

So, r-stability is a necessary, thought not sufficient, condition for composition consistent CBD choice functions.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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