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# A non-cooperative approach to the folk rule in minimum cost spanning tree problems

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ABSTRACT

This paper deals with the problem of finding a way to distribute the cost of a minimum cost spanning tree problem between the players. A rule that assigns a payoff to each player provides this distribution. An optimistic point of view is considered to devise a cooperative game. Following this optimistic approach, a sequential game provides this construction to define the action sets of the players. The main result states the existence of a unique cost allocation in subgame perfect equilibria. This cost allocation matches the one suggested by the folk rule.



## 1. Introduction

In this paper, we study the implementation of the folk solution associated with a minimum cost spanning tree problem. This research is part of a relevant agenda known as the *Nash program* for cooperative games. The Nash program arises from Nash (1953) as a tool to bridge the gap between cooperative and non-cooperative games by finding non-cooperative procedures yielding cooperative solutions as their equilibrium payoffs (Serrano, 2020).

To this end, we consider an optimistic point of view to devise a cooperative game. Following this optimistic approach, we define a sequential game that allows players, acting strategically, to construct an optimal network. The main result states the existence of a unique cost allocation in subgame perfect equilibria. This cost allocation matches the one suggested by the folk rule.

The situation of constructing a tree with the lowest possible cost known as *minimum* cost spanning tree problems is quite familiar in the literature of operations research, economics or management, among others. Let us assume that a group of players requires a service that can only be provided by a source. A network, the edges of which entail some cost to build or to use, provides access to this source. Players can either connect to the source directly or through an existing network that already provides the service to other players. No congestion nor depreciation of the service is assumed, which implies that the optimal network is always a tree. Videostream, voice-conference or software distribution applications, or an irrigation system that supplies water from a water dam, are some examples of such situations.

Assuming that players agree to build a network and decide on how to share its cost, there are two possible approaches to tackle this situation.

The first approach arises when the players leave the decision to a central planner. This planner may either be a regulator whose decision is mandatory for the players, or an adviser whose proposal is not compulsory, but all the players have incentives to follow. In this sense, a fundamental property is *core selection*, which ensures that no coalition of players can connect to the source by themselves at a lower cost than the one suggested by the central planner.[1] A relevant core-selection rule is the folk solution (Bergantiños & Vidal-Puga, 2007a; Bogomolnaia & Moulin, 2010; Feltkamp, Tijs, & Muto, 1994) which, moreover, also satisfies many other relevant properties (Bergantiños & Vidal-Puga, 2008). The second approach arises when the players achieve agreements directly among themselves, following the rules of a non-cooperative game. In this second case, the final network in equilibrium is not

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[1] Non-emptiness of the core in minimum cost spanning tree problems has been first noted by Bird (1976) and deeply studied by Granot & Huberman (1981, 1984). More recently, Dutta & Mishra (2012) and Sziklai, Fleiner, & Solymosi (2016) proved the non-emptiness of the core in two more general classes of games, and Kobayashi & Okamoto (2014) focus on concave problems, where the core has a well-known structure.

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guaranteed to be optimal nor the final payoff allocation to be efficient. Joining the two approaches, it could be suitable to find a mechanism leading to an optimal network along with a fair allocation of its cost.

In this paper, we focus on the second approach. We define a non-cooperative game in which utility-maximizing players agree on how to share the cost of an efficient graph. The non-cooperative game is as follows: first, we fix a random order of choices of the players. Then, players act sequentially according to the above order: the first player selects to whom she connects to, looking for the cheapest connection; then, the second player decides with whom she wants to connect taking into account that, in case the first player had previously connected to her, then she can choose an edge adjacent to the first player, and so on.[2] The only restriction is that no cycles are allowed. At the end of the last round, an optimal tree arises. The cost allocation that arises by charging each player with the cost of her chosen edge provides a stable share of the total cost such that the final share is fair. Consequently, players accept both the optimal tree and a cost-share given by the folk solution.

Mutuswami & Winter (2002), in a more general framework, propose a mechanism in which players move sequentially. When it is a player's turn to move, she announces a set of links that she wants to see formed and her conditional cost contribution to the spanning tree. Given the announcement, a planner selects the largest compatible coalition, and proposes a tree to be built and the allocation of each player. Unlike this mechanism, in our non-cooperative game there is no planner, and the players choose only one link and agree to pay the cost of the selected link. The results of this paper applied to minimal cost spanning tree problems, imply that the allocations to players in all subgame perfect equilibria correspond with the *Kar* rule (Kar, 2002), defined as the Shapley value of the associated cooperative game. As mentioned in Mutuswami & Winter (2002):

> Immunity to deviations by coalition is a desirable property of any mechanism. Unfortunately, our mechanisms do not possess this property [... ].

In contrast, the equilibrium payoff allocations in our non-cooperative game satisfy immunity to deviations by coalitions, i.e., they satisfy *core selection*.

Norde, Moretti, & Tijs (2004) present the Subtraction Algorithm that computes for every minimum cost spanning tree a population monotonic allocation scheme which, in turn, also recovers the folk solution. Contrary to our approach, they compute the contribution of each player, for each possible coalition of players that contain her.

Bergantiños & Vidal-Puga (2010) propose a non-cooperative game in which players always agree on an optimal tree and a cost-share given by the folk solution. In the first stage, the players offer prices to each other. These prices represent the amount that the players are willing to pay to other players if they connect to the source. Then, the player with a maximum net offer is asked to connect to the source or to propose a different network. Unlike this mechanism, in our non-cooperative game the players only propose to construct an edge, and there are no offers to other players to incentivize their connection to the source.

Moulin & Velez (2013) and Hougaard & Tvede (2012) consider two mixed approaches, respectively. In Moulin & Velez (2013), vertices are sellers who bid to supply individual edges, so that a single buyer purchases a minimum cost spanning tree. They show that an optimal tree arises in equilibrium. In Hougaard & Tvede (2012), a planner asks for the costs of the edges to the adjacent players, who have a priori private information about their actual costs. With this information, the planner builds the optimal network (under the assumption of truth-telling), so that costs become common knowledge for the edges that belong to this optimal network. They show that the folk rule causes truthful announcements to be a Nash equilibrium for every allocation problem.

As opposed to these previous results, the non-cooperative game we propose in this paper does not require neither the presence of a planner to implement the cost sharing nor the players to offer prices nor bids to make proposals, making the strategies significantly simpler. Moreover, we have two relevant properties. Firstly, the equilibrium is strong, i.e., no coalition of players can improve their aggregate payoff by coordinating their strategies. Secondly, players use undominated strategies in equilibrium. In particular, their strategies in equilibrium are optimal independently of the strategies of other players, which make them immune to irrational deviations by other players.

Finally, in Hernández, Peris, & Silva-Reus (2016), a different strategic game is defined associated to a minimum cost spanning tree problem. This game is based on the existence of a social transfer structure that establishes side-payments to ensure that a particular tree is obtained. Under this approach, the minimum cost spanning tree appears as a subgame perfect equilibrium. The allocation associated with this subgame perfect equilibrium depends on the initial social transfer structure, and may coincide or not with the folk rule. Moreover, in the game defined in Hernández et al. (2016) subgame perfect equilibria may appear, such that the provided spanning tree is not efficient. This inefficiency cannot occur under our approach.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we introduce the non-cooperative game. In Section 4, we discuss the results. We close with the acknowledgements.

## 2. The model

Let $N_0 = N \cup \{0\}$ be a set of *vertices* where $N = \{1, 2, \ldots, n\}$ is a finite set of *players* and 0 is the *source* they need to connect to.

Let $C = (c_{ij})_{i,j \in N_0}$ be the *cost matrix*, where $c_{ij} \in \mathbb{R}_+$ represents the connection cost between vertices $i$ and $j$. We assume, as usual, that $c_{ii} = 0$ and $c_{ij} = c_{ji}$ for all $i, j \in N_0$. We denote the set of all cost matrices on $N$ as $\mathcal{C}^N$. A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair $(N_0, C)$.

A *network* $g$ over $N_0$ is a subset of $\{(i, j) : i, j \in N_0\}$. The elements of $g$ are called *edges*. We assume that the edges are undirected, i.e. $(i, j)$ and $(j, i)$ represent the same edge.

Given a network $g$ and a pair of vertices $i$ and $j$, a *path* from $i$ to $j$ in $g$ is a sequence of distinct vertices $\{i_0, \ldots, i_l\}$ satisfying $i = i_0$, $j = i_l$ and $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, \ldots, l\}$.

A *spanning tree* over $N_0$ is a network $t$ such that for all $i, j \in N_0$ there exists a unique path in $t$ from $i$ to $j$. Let $\mathcal{T}_0^N$ denote the set of all spanning trees over $N_0$. Given $t \in \mathcal{T}_0^N$, we define the *cost* associated with $t$ in $(N_0, C)$ as

$$c(N_0, C, t) = \sum_{(i,j) \in t} c_{ij}.$$

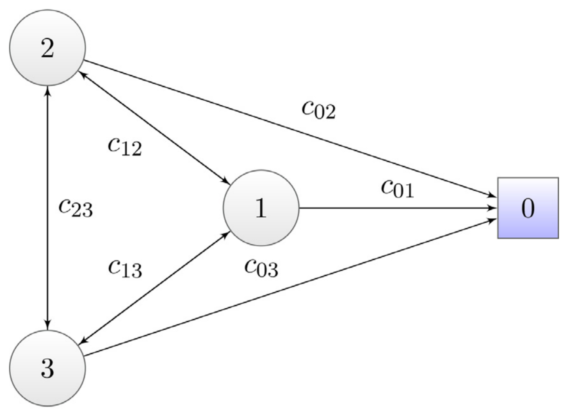
When there is no ambiguity, we write $c(t)$ instead of $c(N_0, C, t)$.

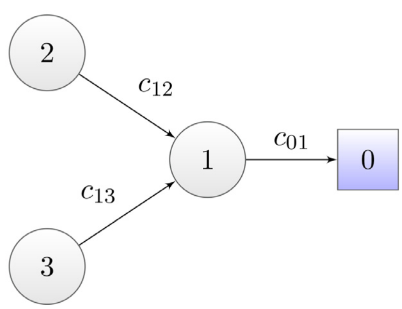
A *minimum cost spanning tree* for $(N_0, C)$, briefly an *mt*, is a spanning tree $t^* \in \mathcal{T}_0^N$ such that $c(t^*) = \min_{t \in \mathcal{T}_0^N}\{c(t)\}$. Given a *mcstp* $(N_0, C)$, an *mt* always exists, but it may not be unique. We denote the cost associated with any *mt* on $(N_0, C)$ as $c(N_0, C)$.

There are several algorithms in the literature to construct an *mt*. Prim (1957) provides one. Sequentially, the players connect, either directly or indirectly to the source. At each stage, we add one of the cheapest edges between the connected and the unconnected vertices.

[2] Such a mechanism resembles Kruskal's and Prim's algorithms.

**Example 2.1.** Consider the *mcstp*$(N_0, C)$ with $N = \{1, 2, 3\}$ and a cost matrix $C \in \mathcal{C}^N$ satisfying $c_{12} < c_{13} < c_{23} < c_{01} < c_{02} < c_{03}$. The Prim's algorithm proceeds as follows: At stage 1, the edge formed is (0,1), because this is the cheapest one between a connected vertex (the source), and a non-connected one (players in $N$). At stage 2, the edge formed is (1,2) because this is the cheapest one between a connected vertex (the source and player 1) and a non-connected one (players 2 and 3). At stage 3, the edge formed is (1,3) because this is the cheapest one between a connected vertex (the source and players 1 and 2) and a non-connected one (player 3). The *mt* formed is then $\{(0,1),(1,2),(1,3)\}$, which in this example is unique.




Given $S \subset N$, we denote the restriction to $S$ of the *mcstp* $(N_0, C)$ as $(S_0, C)$, and the cost associated with any *mt* on $(S_0, C)$ as $c(S_0, C)$; that is, $c(S_0, C)$ is the cost of connection of the players in $S$ to the source.

Given $N$ a finite set of players, a *cooperative cost game* for $N$ is given by a *characteristic function* $v : 2^N \to \mathbb{R}$ where $v(S) \in \mathbb{R}$ for each $S \subseteq N$ represents the cost of providing service to players in $S$. Moreover, $v(\emptyset) = 0$, i.e., it is costless to provide no service.

For each minimum cost spanning tree problem $(N_0, C)$, we construct an associated cooperative cost game $v_C$ given by $v_C(S) = c(S_0, C)$ for each $S \subseteq N$, where the worth of a coalition $S$ depends on vertices only in $S$, *i.e.*, those vertices outside $S$ are unavailable. This approach is pessimistic because each coalition $S$ should build their network without counting with players in $N \setminus S$.

**Example 2.2.** With the data in Example 2.1, the cost game $v_C$ is given by $v_C(\{i\}) = c_{0i}$ for all $i \in N$, $v_C(\{1,2\}) = c_{01} + c_{12}$, $v_C(\{1,3\}) = c_{01} + c_{13}$, $v_C(\{2,3\}) = c_{02} + c_{23}$, and $v_C(N) = c_{01} + c_{12} + c_{13}$.

Nevertheless, we may consider an optimistic approach by defining for each $S$, the cost matrix $C^S$ given by $c^S_{ij} = c_{ij}$ for all $i, j \in S$ and $c^S_{i0} = \min\left\{c_{ij} : j \in N_0 \setminus S\right\}$ for all $i \in S$. This formulation means that each coalition $S$ can build a network assuming that players in $N \setminus S$ are already connected. The cooperative cost game $v^+_C$ is then defined where $v^+_C(S) = c\left(S_0, C^S\right)$ for all $S \subseteq N$. Bergantiños & Vidal-Puga (2007b) are the first to propose this alternative associated cooperative cost game $v^+_C$. See Christian Trudeauand Vidal-Puga (2020) for other possible associated cost games for $(N_0, C)$.

**Example 2.3.** With the data in Example 2.1, the optimistic cost game $v^+_C$ is given by $v^+_C(\{1\}) = v^+_C(\{2\}) = c_{12}$, $v^+_C(\{3\}) = c_{13}$, $v^+_C(\{1,2\}) = c_{12} + c_{13}$, $v^+_C(\{1,3\}) = c_{13} + c_{12}$, $v^+_C(\{2,3\}) = c_{12} + c_{13}$, $v^+_C(N) = c_{01} + c_{12} + c_{13}$.

Let $\Pi^N$ be the set of orders $\pi : \{1, \ldots, n\} \to N$. For simplicity, we denote $\pi(k)$ as $\pi_k$ for all $k \in \{1, \ldots, n\}$. Then, given some $\pi \in \Pi^N$, the *marginal contributions payoff allocation* of the optimistic game $v^+_C$ with order $\pi$ is $m^\pi$ given by $m^\pi_{\pi_1} = v^+_C(\{\pi_1\})$ and, for $k = 2, \ldots, n$,

$$m^\pi_{\pi_k} = v^+_C(\{\pi_1, \pi_1, \ldots, \pi_k\}) - v^+_C(\{\pi_1, \pi_1, \ldots, \pi_{k-1}\}).$$

A rule is a function that assigns to each *mcstp* a payoff allocation. Notice that a payoff corresponds to each player whereas a payoff allocation is a vector whose coordinates are the respective players' payoffs.

The *folk rule* (Bergantiños & Vidal-Puga, 2007a), provides a criterion for sharing the cost of an *mt* between the players. The definition of the folk rule is made by applying the Prim's algorithm to an irreducible[3] cost matrix $C^*$. Remarkably, the folk rule can also be defined as the Shapley value of the optimistic game $v^+_C$ or as the Shapley value of the pessimistic cost game $v^*_C$ obtained from the irreducible cost matrix.[4]

**Example 2.4.** Since the Shapley value is the average of marginal contributions payoff allocations, we can obtain the folk rule by computing these payoff allocations in the optimistic game $v^+_C$ for each possible order. Table 1 represents these vectors with the data in Example 2.1 and the average of these contributions that corresponds with the folk rule.

## 3. The non-cooperative extensive game

We define the non-cooperative game inductively as follows:

- At the first stage ($k = 0$), nature chooses some order $\pi \in \Pi^N$, being each $\pi$ chosen with the same probability $\frac{1}{n!}$. We define $\Omega^0_i = \{i\}$ for all $i \in N_0$.
- At stage $k = 1$, player $\pi_1$ chooses an action from the following set:

  $$S_{\pi_1} = \left\{(i, j) : i \in \Omega^0_{\pi_1}, j \in N_0 \setminus \Omega^0_{\pi_1}\right\}.$$

  That is, player $\pi_1$ selects edge $s_{\pi_1} = (i_1 = \pi_1, j_1) \in S_{\pi_1}$ to be built. Once done, vertices $i_1$ and $j_1$ become connected, and we set $\Omega^1_{i_1} = \Omega^1_{j_1} = \{i_1, j_1\}$. We also define $\Omega^1_i = \Omega^0_i$ for any other $i \in N_0 \setminus \{i_1, j_1\}$.
- In general, at stage $k \geq 1$, player $\pi_k$ chooses an action from the set:

  $$S_{\pi_k} = \left\{(i, j) : i \in \Omega^{k-1}_{\pi_k}, j \in N_0 \setminus \Omega^{k-1}_{\pi_k}\right\}.$$

  That is, player $\pi_k$ selects some edge $s_{\pi_k} = (i_k, j_k) \in S_{\pi_k}$ to be built. Once this action is done, vertices $i_k$ and $j_k$ become connected and we set $\Omega^k_{i_k} = \Omega^k_{j_k} = \Omega^{k-1}_{i_k} \cup \Omega^{k-1}_{j_k}$. We also define $\Omega^k_l = \Omega^k_{i_k}$ for all $l \in \Omega^{k-1}_{i_k} \cup \Omega^{k-1}_{j_k}$, and $\Omega^k_l = \Omega^{k-1}_l$ in another case.

[3] The irreducible cost matrix $C^*$ is the corresponding matrix such that no edge cost can be reduced without reducing the cost of the grand coalition to connect to the source.

[4] See Bergantiños & Vidal-Puga (2007a,b) for details and additional properties.

**Table 1**
Marginal contributions of Example 2.1.

| order | player 1 | player 2 | player 3 |
|---|---|---|---|
| [123] | $v_C^+(\{1\}) = c_{12}$ | $v_C^+(\{1,2\}) - v_C^+(\{1\}) = c_{13}$ | $v_C^+(N) - v_C^+(\{1,2\}) = c_{01}$ |
| [132] | $v_C^+(\{1\}) = c_{12}$ | $v_C^+(N) - v_C^+(\{1,3\}) = c_{01}$ | $v_C^+(\{1,3\}) - v_C^+(\{1\}) = c_{13}$ |
| [213] | $v_C^+(\{1,2\}) - v_C^+(\{2\}) = c_{13}$ | $v_C^+(\{2\}) = c_{12}$ | $v_C^+(N) - v_C^+(\{1,2\}) = c_{01}$ |
| [231] | $v_C^+(N) - v_C^+(\{2,3\}) = c_{01}$ | $v_C^+(\{2\}) = c_{12}$ | $v_C^+(\{2,3\}) - v_C^+(\{2\}) = c_{13}$ |
| [312] | $v_C^+(\{1,3\}) - v_C^+(\{3\}) = c_{12}$ | $v_C^+(N) - v_C^+(\{1,3\}) = c_{01}$ | $v_C^+(\{3\}) = c_{13}$ |
| [321] | $v_C^+(N) - v_C^+(\{2,3\}) = c_{01}$ | $v_C^+(\{2,3\}) - v_C^+(\{3\}) = c_{12}$ | $v_C^+(\{3\}) = c_{13}$ |
| Average | $\frac{2c_{01} + 3c_{12} + c_{13}}{6}$ | $\frac{2c_{01} + 3c_{12} + c_{13}}{6}$ | $\frac{2c_{01} + 4c_{13}}{6}$ |

- At stage $k = n + 1$, the game finishes and the payoff for each player $i \in N$ is given by

  $u_i(s_i) = c_{s_i}$.

  That is, player $i$ pays the cost of the edge she selected.

Notice that, in the first stage, for each $i \in inN$, $\Omega_i^0$ is a singleton because player $i$ is not connected to anyone else.

Following Maschler, Solan, & Zamir (2013), we define the non-cooperative game in extensive form with perfect information and chance moves as:

$$\Gamma = (N, V, E, x^0, (V_i)_{i\in N_0}, (p_x)_{x\in V_0}, u)$$

where

- $N = \{1, 2, \ldots, n\}$ is the set of players.
- $V$ is the set of nodes in the game tree.[5]
  Each $v \in V$ is determined by the following triple $(k, \pi, f_k^\pi)$ :
  - stage $k \in \{0, 1, \ldots, n + 1\}$,
  - $\pi \in \Pi^N$ that determines the order (only for $k > 0$),
  - some function $f_k^\pi : \{1, \ldots, k-1\} \to N \times N_0$ such that $f_k^\pi(l) \in S_{\pi_l}$ for all $l = 1, \ldots, k-1$.

  Therefore, for $k = 1, \ldots, n$, pairs $\left(\pi, f_k^\pi\right)$ determine the history, i.e., the (feasible) choice of each predecessor of $\pi_k$ in $\pi$. Hence, the set of edges already paid, before $\pi_k$ chooses, is

  $$\left\{f_k^\pi(\pi_1), f_k^\pi(\pi_2), \ldots, f_k^\pi(\pi_{k-1})\right\}.$$

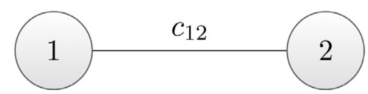
  Notice that this set is empty for $k = 1$. For $k = n + 1$, the node is a terminal one. If $k = 0$, it is nature's decision node, and $\pi_k$'s otherwise.
- $E \subset V \times V$ is the set of arcs. For a node $v$ determined by $\left(k, \pi, f_k^\pi\right)$, arc $\left(v, v'\right)$ belongs to $E$ when $v'$ is determined by $\left(k+1, \pi, f_{k+1}^\pi\right)$ such that $f_{k+1}^\pi(l) = f_k^\pi(l)$ for all $l < k$.
- $x^0$ is the node determined by $k = 0$.
- $(V_i)_{i\in N_0}$ is a partition of the set of non-terminal nodes, and it determines which player (or nature, when $i = 0$) makes the decision at that node. In particular, $V_0 = \{x^0\}$ and, given $i \in N$, we have $v \in V_i$ when $v$ is determined by $(k, \pi, f_k^\pi)$ with $k \in \{1, \ldots, n\}$ and $\pi_k = i$.
- $p_0$ is a probability distribution over the arcs emanating from $x^0$. In particular, $p_0(e) = \frac{1}{n!}$ for each such an arc $e$.
- $u$ is the function that associates each terminal node with a game outcome. In particular, if the terminal node is given by $\left(n+1, \pi, f_{n+1}^\pi\right)$, the game outcome is the payoff allocation $\left(c_{f_{n+1}^\pi(k)}\right)_{k\in\{1,\ldots,n\}}$ provided by the spanning tree $t = \left\{f_{n+1}^\pi(k)\right\}_{k\in\{1,\ldots,n\}}$.

Given $\pi \in \Pi$, we denote as $\Gamma_\pi$ the subgame that begins after nature chooses $\pi$.

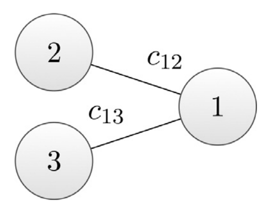
[5] To avoid ambiguities, we use the terms *nodes* and *arcs* in the game tree, as opposed to *vertices* and *edges* defined for the spanning tree.

**Example 3.1.** With the data in Example 2.1, let us now construct $\Gamma_\pi$ with $\pi_i = i$ for all $i$. Initially, $\Omega_i^0 = \{i\}$ for all $i \in N_0$.

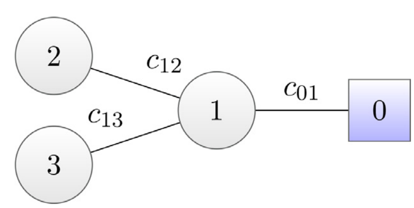
- At the first stage, player 1 decides the edge she wants to pay, $s_1 \in \{(0,1), (1,2), (1,3)\}$. Say, for example $s_1 = (1,2)$.




  Hence, $\Omega_1^1 = \Omega_2^1 = \{1,2\}$, $\Omega_3^1 = \{3\}$ and $\Omega_0^1 = \{0\}$.
- Now, player 2 decides which edge $s_2$ to pay by taking into account $s_1$. Assuming that $s_1 = (1,2)$, we have $s_2 \in \{(0,1), (0,2), (1,3), (2,3)\}$, i.e., player 2 cannot choose (1,2) (already taken) but she can choose (0,1) or (1,3) (because she is already connected to player 1). Say, for example, $s_2 = (1,3)$.




  Hence, $\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = \{1,2,3\}$ and $\Omega_0^2 = \{0\}$.
- Finally, player 3 decides which edge $s_3$ to pay by taking into account $s_1$ and $s_2$. Assuming $s_1 = (1,2)$ and $s_2 = (1,3)$, we have $s_3 \in \{(0,1), (0,2), (0,3)\}$. In either case, the three players get connected to the source simultaneously through a spanning tree. Say, for example, $s_3 = (0,1)$.




  Hence, $\Omega_1^3 = \Omega_2^3 = \Omega_3^3 = \Omega_0^3 = N_0$.

Figure 1 depicts the nodes in $V$ and the arcs in $E$ that follow this particular path.

The formed spanning tree determines the payoffs. For instance, if the players select their cheapest available options, the spanning tree is $\{(1,2), (1,3), (0,1)\}$ and the cost of each edge is distributed in the following way: Player 1 pays $c_{s_1} = c_{12}$; player 2 pays $c_{s_2} = c_{13}$; and player 3 pays $c_{s_3} = c_{01}$. Table 2 represents the payoff allocation for each $\pi$, assuming each player selects her cheapest available option.

Given the sequential structure of $\Gamma_\pi$, we will study the subgame perfect equilibria. The equilibrium strategies should specify optimal behavior from any information node up to the end of the game. That is, any player's strategy should prescribe what is optimal from that node onwards given the other players' strategies.

As Example 3.1 shows, the only equilibrium payoff in $\Gamma_\pi$ is $f^{\pi^{-1}}$, where $\pi^{-1} \in \Pi^N$ is the order defined as $\pi_k^{-1} = \pi_{n-k+1}$.

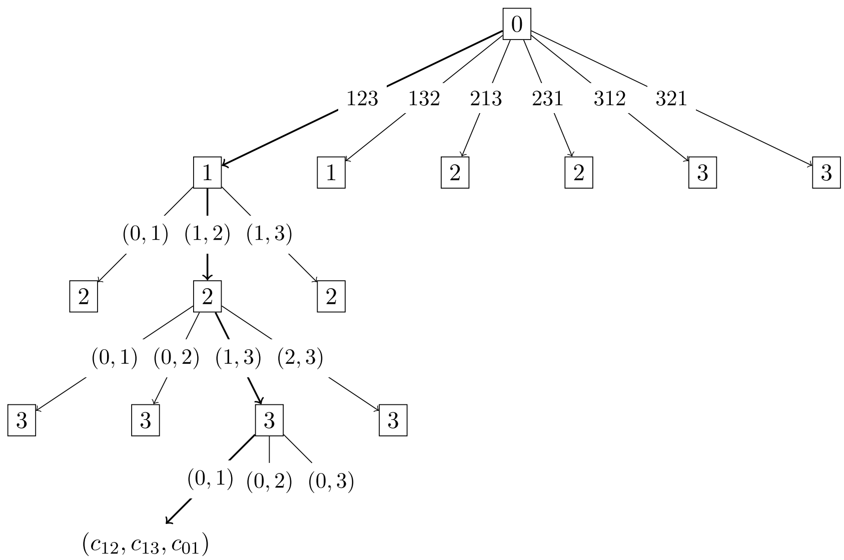



**Fig. 1.** Game tree in Example 3.1. The digit at each non-terminal node (squared) represents the player (or nature) that makes the decision at that particular node.

**Table 2**
Payoff allocation with the cheapest available option.

| order | $mt$ in $C$ | player 1 | player 2 | player 3 |
|---|---|---|---|---|
| [123] | $\{(1,2),(1,3),(0,1)\}$ | $c_{12}$ | $c_{13}$ | $c_{01}$ |
| [132] | $\{(1,2),(1,3),(0,1)\}$ | $c_{12}$ | $c_{01}$ | $c_{13}$ |
| [213] | $\{(1,2),(1,3),(0,1)\}$ | $c_{13}$ | $c_{12}$ | $c_{01}$ |
| [231] | $\{(1,2),(1,3),(0,1)\}$ | $c_{01}$ | $c_{12}$ | $c_{13}$ |
| [312] | $\{(1,3),(1,2),(0,1)\}$ | $c_{12}$ | $c_{01}$ | $c_{13}$ |
| [321] | $\{(1,3),(1,2),(0,1)\}$ | $c_{01}$ | $c_{12}$ | $c_{13}$ |
| Average | | $\frac{2c_{01}+3c_{12}+c_{13}}{6}$ | $\frac{2c_{01}+3c_{12}+c_{13}}{6}$ | $\frac{2c_{01}+4c_{13}}{6}$ |

Tables 1 and 2 show that the marginal contributions allocation of the optimistic game $v_C^+$ coincides with the payoff allocation when players select their cheapest available edge. Hence, in this example, the expected equilibrium payoff allocation in $\Gamma$ is the one provided by the folk rule.

Our main result establishes that this happens in general.

**Theorem 3.1.** *Given $\pi \in \Pi^N$, there exists a unique subgame perfect equilibrium payoff allocation for $\Gamma_\pi$, given by the marginal contributions payoff allocation of the optimistic game $v_C^+$ with order $\pi$. Moreover, this equilibrium is strong and uses undominated strategies.*

**Proof.** We will prove that for all $\Gamma_\pi$, each player $\pi_k$ has a strategy that assigns her a cost so that she pays at most $m^\pi_{\pi_k} = v_C^+(\{\pi_1, \dots, \pi_k\}) - v_C^+(\{\pi_1, \dots, \pi_{k-1}\})$, independently of the strategies of the other players. In other words, the payoff for each player is bounded from above independently of the strategies of the other players. Also, this strategy is protected from any coordinated actions by the other players, who cannot extract a higher payment from her. Thus, this strategy profile constitutes a strong subgame perfect equilibrium and the strategies are undominated.

By a standard backwards argument, it is clear that there exists a subgame perfect equilibrium for each $\Gamma_\pi$ and, moreover, each player will select one of her cheapest available edges. Hence, even though the subgame perfect equilibrium may not be unique, the subgame perfect equilibrium payoff is.

Assume w.l.o.g. $\pi_i = i$ for all $i \in N$. Hence, at the first stage, player 1 would choose one of her cheapest adjacent edges $f_1^\pi(1) = (1, i)$ for some $i \in N_0 \setminus \{1\}$, the cost of which is precisely $v_C^+(\{1\}) = c_{01}^{\{1\}}$.

For clarification purposes, we analyse stage 2. At this stage, player 1 has selected some edge $(1, j_1)$ and player 2 would choose her cheapest adjacent edge $(2, j_2)$, whose cost is $c_{02}^{\{2\}}$, unless $2 = f_1^\pi(1)$ and $j_2 = 1$. In this latter case, player 2 cannot choose edge $(2,1)$, but other edges (those adjacent to player 1) would be available, and in particular the chosen edge would cost $\min\left\{c_{01}^{\{1,2\}}, c_{02}^{\{1,2\}}\right\}$. We show that, in either case, player 2 pays at most $v_C^+(\{1,2\}) - v_C^+(\{1\})$. We distinguish two cases:

(a) If $f_1^\pi(1) \neq 2$, or $f_1^\pi(1) = 2$ and $j_2 \neq 1$, then player 2 chooses her cheapest adjacent edge $(2, j_2)$ and pays $c_{2j_2} = c_{02}^{\{2\}}$. In this case,

$$v_C^+(\{1,2\}) = \min\left\{c_{12} + c_{01}^{\{1\}}, c_{12} + c_{02}^{\{2\}}, c_{01}^{\{1\}} + c_{02}^{\{2\}}\right\}$$

$$v_C^+(\{1\}) = c_{01}^{\{1\}}$$

and

$$v_C^+(\{1,2\}) - v_C^+(\{1\}) = \min\left\{c_{12}, c_{12} + c_{02}^{\{2\}} - c_{01}^{\{1\}}, c_{02}^{\{2\}}\right\} = c_{02}^{\{2\}}$$

so player 2 pays $c_{2j_2} = v_C^+(\{1,2\}) - v_C^+(\{1\})$.

(b) If $f_1^\pi(1) = 2$, and $j_2 = 1$, then player 2 selects the edge that minimizes $c_{ij}$, $i \in \{1,2\}$, $j \in N_0 \setminus \{1,2\}$. We have two subcases:
   - $c_{01}^{\{1\}} = c_{12}$, then player 2 pays $\min\left\{c_{01}^{\{1,2\}}, c_{02}^{\{1,2\}}\right\} = v_C^+(\{1,2\}) - v_C^+(\{1\})$.
   - $c_{01}^{\{1\}} < c_{12}$, then player 2 pays $c_{01}^{\{1\}} < v_C^+(\{1,2\}) - v_C^+(\{1\})$.

We now prove the result in general. Assume that we are in stage $k$, so that player $\pi_k = k$ chooses an edge to be built. Notice that we do not assume that the previous players, denoted as $S=\{1,\ldots,k-1\}$, have followed any particular strategy profile. Player $k$ would choose one of her cheapest adjacent edges, that may connect her to a *previous* player (some $j\in S$) or not (some $j\notin S\cup\{k\}$). The cost of this edge is $c_{kj_k}=\min_{i\in N_0,i\neq k}\{c_{ki}\}$. However, as in the case of stage 2, this edge might be available or not. We distinguish the following possibilities:

(a) If $k\notin\bigcup_{i\in S}\Omega_i^{k-1}$, then we have three subcases:
- If $j_k\in S$ and $(k,j_k)$ is not one of the cheapest edges that connects a vertex in $S$ with a vertex in $N_0\setminus S$, then $v_C^+(S\cup\{k\})=v_C^+(S)+c_{kj_k}$, so player $k$ would pay $c_{kj_k}=v_C^+(S\cup\{k\})-v_C^+(S)$.
- If $j_k\in S$ and $(k,j_k)$ is one of the cheapest edges that connects a vertex in $N_0\setminus S$ with a vertex in $S$; that is, then there is some *mt* $t_S$ in $S_0$ such that $k$ is connected with players $S^k\subseteq S$ throughout $t_S$. In this case,

  $$v_C^+(S\cup\{k\})=v_C^+(S)+\min_{i\in S^k\cup\{k\},l\notin S^k\cup\{k\}}\{c_{il}\}$$

  and $\min_{i\in S^k\cup\{k\},l\notin S^k\cup\{k\}}\{c_{il}\}\geq c_{kj}$. So player $k$ would pay

  $$c_{kj_k}\leq\min_{i\in S^k\cup\{k\},l\notin S^k\cup\{k\}}\{c_{il}\}=v_C^+(S\cup\{k\})-v_C^+(S).$$

- If $j_k\notin S$, then $v_C^+(S\cup\{k\})=v_C^+(S)+c_{kj_k}$, so player $k$ would pay $c_{kj_k}=v_C^+(S\cup\{k\})-v_C^+(S)$.

(b) If $k\in\bigcup_{i\in S}\Omega_i^{k-1}$, this means that edge $(r,k)$ has been built for some $r\in S$, so $k\in\Omega_r^{k-1}$. If there is $j_k\in N_0\setminus\Omega_r^{k-1}$ such that $c_{kj_k}=\min_{i\in N_0,i\neq k}\{c_{ki}\}$, edge $(k,j_k)$ is available for player $k$, and the same reasoning as in the previous case applies.

(c) Finally, it remains the case in which $k\in\bigcup_{i\in S}\Omega_i^{k-1}$ and for each $j_k$ such that $c_{kj_k}=\min_{i\in N_0,i\neq k}\{c_{ki}\}$, the edge $(k,j_k)$ is not available for player $k$; that is, $k,j_k\in\Omega_r^{k-1}$, for some $r\in S$. Then, player $k$ would choose one of the cheapest available edges $(j,l)$ with $j\in\Omega_r^{k-1}$ and $l\notin\Omega_r^{k-1}$, so that

$$c_{jl}=\min_{i\in\Omega_r^{k-1},i^*\notin\Omega_r^{k-1}}\{c_{ii^*}\}.\tag{1}$$

The cost of this edge and the final payoff for player $k$ is $c_{jl}$. Let $t^*$ be an *mt*, and let $t_S^*=\{(i,i^*)\in t^*: i,i^*\in S\}$ be the restriction of $t^*$ to edges whose both vertices are in $S$. Clearly, $t_S^*$ induces a partition $\{S_1,\ldots,S_\lambda\}$ of $S$ into $\lambda\geq1$ connected components. For each $\alpha=1,\ldots,\lambda$, let $(i_\alpha,i_\alpha^*)\in t^*$ such that $i_\alpha\in S_\alpha$, $i_\alpha^*\notin S_\alpha$, and

$$c_{i_\alpha i_\alpha^*}=\min_{i\in S_\alpha,i^*\notin S_\alpha}\{c_{ii^*}\}.$$

Clearly, $i_\alpha^*\notin S$ for all $\alpha$ (however, $i_\alpha^*=i_{\alpha'}^*$ is possible for some $\alpha\neq\alpha'$). Let $t=t_S^*\cup\{(i_\alpha,i_\alpha^*)\}_{\alpha=1}^{\lambda}$. It is not difficult to check that

$$v^+(S)=\sum_{(i,i^*)\in t}c_{ii^*}.\tag{2}$$

We have two subcases:
- If $k=i_\alpha^*$ for some $\alpha$, let $\hat{S}=\bigcup_{\alpha:k=i_\alpha^*}S_\alpha$. Then,

  $$v^+(S\cup\{k\})=v^+(S)+c_{hh^*}$$

  where $(h,h^*)\in t^*$, $h\in\hat{S}\cup\{k\}$, $h^*\notin\hat{S}\cup\{k\}$, and

  $$c_{hh^*}=\min_{i\in\hat{S}\cup\{k\},i^*\notin\hat{S}\cup\{k\}}\{c_{ii^*}\}.$$

  So, $m_k^\pi=c_{hh^*}$.
  Let $(i,i^*)$ be the first edge in the (unique) path in $t$ from $k$ to $l$ such that $i\in\Omega_r^{k-1}$ and $i^*\notin\Omega_r^{k-1}$. Under (1), $c_{jl}\leq c_{ii^*}$. Under (2), $c_{ii^*}\leq c_{hh^*}$. Hence, $c_{jl}\leq c_{hh^*}=v_C^+(S\cup\{k\})-v_C^+(S)$.
- If $k\neq i_\alpha^*$ for all $\alpha$,

  $$v^+(S\cup\{k\})=v^+(S)+c_{kk^*}$$

  where $(k,k^*)\in t^*$ and

  $$c_{kk^*}=\min_{i\neq k}\{c_{ki}\}=v^+(\{k\}).$$

  So, $m_k^\pi=c_{kk^*}$. In case $k^*\notin\Omega_r^{k-1}$, under (1) we deduce $c_{jl}\leq c_{kk^*}=v_C^+(S\cup\{k\})-v_C^+(S)$. In case $k^*\in\Omega_r^{k-1}$, let $(i,i^*)$ be the first edge in the (unique) path in $t^*$ from $k$ to $l$ such that $i\in\Omega_r^{k-1}$ and $i^*\notin\Omega_r^{k-1}$. Under (1), $c_{jl}\leq c_{ii^*}$. Under (2), $c_{ii^*}\leq c_{kk^*}$. Hence, $c_{jl}\leq c_{kk^*}=v_C^+(S\cup\{k\})-v_C^+(S)$.

Finally, observe that given an *mt* $t^*$ in $N_0$, with cost $c(t^*)$, the following relations are fulfilled in equilibrium, where $f(k)=f_{k+1}^\pi(k)$ denotes the edge selected by player $k$

$$c(t^*)\leq\sum_{k=1}^n c_{f(k)}\leq\sum_{k=1}^n v_C^+(\{1,\ldots,k\})-v_C^+(\{1,\ldots,k-1\})=c(t^*)$$

and the equality in the above relationships is obtained, $c_{f(k)}=v_C^+(\{1,\ldots,k\})-v_C^+(\{1,\ldots,k-1\})$, for all $k\in N$. □

The next two corollaries present properties derived from our main result.

**Corollary 3.1.** *The folk rule arises as a unique expected subgame perfect equilibrium payoff allocation for* $\Gamma$.

**Corollary 3.2.** *A minimum cost spanning tree always arises in any subgame perfect equilibrium for* $\Gamma$.

## 4. Concluding remarks

The operations research literature has explored the design of efficient algorithms to build optimal trees, as well as their computational complexity. More recently, the cost-sharing aspect has received increasing attention, from both the operational research and the economics literature. The idea is that the players involved are responsible for paying the total cost of the implementation of an optimal tree. This idea leads to taking into account the players' incentives to guarantee the construction of such an optimal network. Within this context, the problem of finding an optimal network structure does not rely only on its total cost but also on the amount that should be charged to each player.

Our non-cooperative game gets the folk rule in expected terms. Following Bag & Winter (1999) and Mutuswami & Winter (2002), we can achieve a complete implementation by adding a previous stage in which one of the players, chosen at random, proposes a spanning tree and a cost-sharing allocation. If all the other players accept this proposal (they vote sequentially in any order), both the tree and the cost-sharing allocation are imposed, and the game finishes. In case any of them rejects the proposal, they play game $\Gamma$ in the known terms. Assuming either that: a) players are risk-averse, or b) they are risk-neutral but prefer to finish as soon as possible, then the only final cost allocation is the one given by the folk rule.

Another relevant characteristic of our approach is that the equilibrium strategy profiles do not need to anticipate the moves of the following players in the order. Hence, we can define the non-cooperative game by choosing only the first player at random; after this player chooses her available edge, another player is chosen at random, and so on. Moreover, the optimal strategy is to choose the cheapest available edge. Hence, the subgame perfect equilibrium is also a strong perfect equilibrium and an equilibrium with dominant strategies.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.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