DEFINITIONAL AMBIGUITY: A CASE OF CONTINUOUS FUNCTION

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Definitions are an integral aspect of mathematics. In particular, they form the backbone of deductive reasoning and facilitate precision in mathematical communication. However, when an agreed-upon definition is not established, its ability to serve these purposes can be called into question. While ambiguity can be productive, the existence of multiple non-equivalent definitions for the same term can make the truth value of certain mathematical statements unclear. In this study, we asked mathematics educators to determine the truth of a definitionally ambiguous mathematical claim. Based on their responses, we identified several factors that influenced the teachers' choice of definitions. Finally, we consider the pedagogical implications of employing such a task in teacher preparation programs.

INTRODUCTION

In mathematics, definitions are paramount. As Edwards and Ward (2008) write, "the words of the formal definition embody the essence of and completely specify the concept being defined" (p. 223). Mathematics fixates on definitions for their importance in logical argumentation and proof. To make conclusive statements about mathematical objects, it is necessary that "we do not leave the meaning of a term to contextual interpretation; we *declare* our definition and expect there to be no variance in its interpretation in that particular work" (ibid., p. 224, emphasis in original).

Despite the widely acknowledged significance of definitions in mathematics, different definitions often exist for the same term. Ideally, these definitions are equivalent and any one of them may be chosen as "the" definition from which the others follow as theorems (Winicki-Landman & Leikin, 2000). Sometimes, however, the same term has different definitions that do not encompass the same class of objects. This introduces ambiguity into mathematical tasks. For example, the recent work of Mirin et al. (2021) discusses two different definitions of *function*, both acceptable in the mathematics community, that lead to opposite conclusions when one must decide whether a given function is invertible. In this paper, we present multiple, mathematically acceptable definitions of *continuous function* that can likewise lead to ambiguity. We then present the results of a study in which we asked teachers to decide on the truth value of a statement concerning this term, including the considerations they attended to when making their decisions.

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DEFINITIONS AND DEFINITIONAL AMBIGUITY IN MATHEMATICS

On the importance of definitions and their features.

Mathematicians and mathematics educators alike acknowledge the importance of definitions in teaching, learning, and exploring mathematics. One important feature of definitions is that they facilitate communication within a mathematical community; that is, they specify how a term is used in order to assure that interlocutors refer to the same concept when using that term (e.g., Borasi, 1992). Mathematical definitions are used to introduce new objects, to determine properties of what was defined and to assess the validity of statements related to the defined objects (Martín-Molina et. al, 2018). As such, mathematical definitions serve as a basis for mathematical proofs (e.g., Weber, 2002). Importantly, mathematical definitions are also used to classify—to distinguish between what is or is not a particular entity (e.g., Zaslavsky & Shir, 2005).

Within the disciplinary practice of mathematics, definitions are dynamic and adaptive and may undergo refinements in light of counterexamples and further developments (e.g., Martín-Molina et. al, 2018). However, in school, students are either presented with precisely worded existing definitions (e.g., Edwards & Ward, 2004) or work with mathematical notions in the absence of any provided definitions. To account for these two cases, drawing on the work of philosophers and lexicographers, Edwards and Ward (2004, 2008) distinguished between *extracted* definitions and *stipulated* definitions. Extracted definitions are deduced from the inspection of a body of evidence. Stipulated definitions are handed down to learners from a knowledgeable expert. This distinction is eloquently summarized by Edwards and Ward (2008) when they observe that "extracted definitions report usage while stipulated definitions create usage" (p. 224).

According to Leikin and Winicki-Landman (2000), *equivalent* definitions generate the same set of objects that satisfy the definition. However, when one set of objects satisfied by Definition-A is a proper subset of objects satisfied by Definition-B, then the two definitions are *consequent* definitions. Other times, when the sets of objects generated by two definitions have a nonempty intersection, but neither is a proper subset of the other, Leikin and Winicki-Landman (ibid.) refer to the definitions as *competing*.

Van Dormolen and Zaslavsky (2003) specify that *a criterion of equivalence* is necessary for equivalent definitions to be a fundamental part of a deductive system. That is,

> when one gives more than one formulation for the same concept, one must prove that they are equivalent. In practice this means that one has to choose one of the formulations as the definition and consider the other formulations as theorems that have to be proved. (p. 95).

However, we find no explicit direction for how, in practice, non-equivalent definitions of the same concept are to be handled. When consequent or competing definitions exist for the same mathematical term, the truth value of statements related to that term may become ambiguous. The focus of our study is on teachers' mathematical decision-making when faced with such ambiguity.

On ambiguity and definitional ambiguity

According to Byers (2007), "ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (p. 2). Byers considered ambiguity in mathematics as a source of creative development and argued against the popular perception that the logical structure of mathematics is definitive. Building on Byers' definition but interpreting it in the context of teaching and learning mathematics, Foster (2011) argued that *productive ambiguity* is an essential component of learners' engagement with mathematics. In particular, "ambiguity is necessary for ideas to move forward because it creates an instability in what is currently known that allows the formation of new knowledge" (p. 3). Foster also categorized different appearances of ambiguity. He distinguished between symbolic ambiguity, multiple-solution ambiguity, paradigmatic ambiguity, linguistic ambiguity and definitional ambiguity; the latter is of our interest in this study.

Definitional ambiguity, according to Foster (2011), arises "where there is more than one way of interpreting the meaning of a mathematical term." His example is the term "radius," which may refer to a geometric object or its length. In these cases, whether the reference is to a geometric object (as in a construction) or its size (as in the task, find the radius of a circle with a circumference of 5π cm) is clear in context. However, there are also situations in which definitional ambiguity is the result of different but non-equivalent definitions. We wondered how teachers resolve such situations. This led to the following research question: *What guides teachers' decision making in cases of definitional ambiguity*?

Definitional ambiguity: the case of "continuous function"

When searching for a definition of continuous function, either online or in calculus books, the most common results are definitions of continuity at a point or continuity on an interval. From these stipulated definitions, a possible extracted definition of a continuous function is "a function that is continuous everywhere." However, the meaning of "everywhere" can be interpreted differently and depends on which stipulated definitions this definition is extracted from.

Definition-1: A continuous function is a function that is continuous on all the points of the function domain.

Definition-2: A continuous function is a function that is continuous on all the real numbers.

We purposefully do not comment here on which definition we consider as correct. We do note that, using Definition-2, $f(x) = 1/x$ is not a continuous function as there is a discontinuity at $x = 0$. This interpretation corresponds to the naïve concept image of a continuous function that requires it to be drawn without lifting pen from paper. Using

Definition-1, $f(x)$ is a continuous function as it is continuous at all points of its domain, which excludes $x = 0$. Jayakody and Zazkis (2015) elaborated in detail on the inconsistent conclusions that can be reached by examining definitions of continuity in different sources. In particular, they noted inconsistency in referring to discontinuity at points where a function is not defined.

THEORETICAL UNDERPINNING: CONDITIONAL CONSTRUALS

Milewski et al. (2021) introduced the notion of *conditional construals* to describe teacher decision making in ambiguous situations that arose in mathematics classrooms. Conditional construals are described as "moments when teachers require additional context in order to judge whether a given teaching action is appropriate." Milewski et al. (ibid.) used linguistic indicators, such as "it depends," to identify instances of conditional construal. We note that, in these instances, the provided examples attended to teachers' pedagogical decisions related to pedagogical scenarios. For example, in the exemplified responses, teachers conditioned their choices as depending on time constraints, the instructional sequence, or their familiarity with students.

We extend the notion of conditional construal to cases where a mathematical decision depends on implicit mathematical assumptions. To illustrate, consider the following statement: In division of 13 by 5, the quotient is 2. Do you agree? Your decision depends on your definition of a quotient, which in turn depends on the kind of division you consider. The statement is true when the division is of whole numbers, which implies a whole number quotient and remainder. The statement is not true if the division is of rational numbers, and the definition of quotient is taken to be the result of that division. The conditional construal is mathematical in nature. One may argue that this conditional construal also requires pedagogical context—however, we note that conversations about both whole number and rational division might occur in the same pedagogical context: a middle school classroom.

METHODS

Participants in this study were prospective teachers in the last term of their teacher certification program and practicing teachers enrolled in a professional development course ($n = 29$, referred to as T-1 to T-29). They were asked to respond, in writing, to the claim that $f(x) = 1/x$ is a continuous function. This response required the teachers to indicate their evaluation of whether the claim is true or false; to provide a justification, indicating any sources that informed their decisions; and to provide any hypothetical arguments that might be used by someone who disagreed with their evaluation. These responses served as a starting point to initiate a subsequent classroom discussion on definitions in mathematics.

Analysis of the written responses was conducted using the phases of reflexive thematic analysis. In particular, an inductive thematic analysis allowed for coding and theme development to be directed by the content of the data (Braun et al., 2019). In the first phase of analysis, each member of the research team familiarized themselves with the data. That is, they read and re-read the teachers' responses in order to become immersed in and intimately familiar with how they qualified both their justifications and any hypothetical disagreements. Then, each response was coded by multiple members of the research team to identify the conditional construals used as respondents conditioned their decisions. Initial codes were primarily semantic in that their creation was instigated by a teacher's explicit language choice—for example, the use of linguistic markers for conditionality such as "it depends." Later, these semantic codes were supplemented with latent codes that captured those instances in which conditional construals were implicit in the text (Braun et al., 2019). Members of the research team met regularly to discuss the generation and application of codes.

Next, the research team identified collections of codes—and, in some cases, especially prevalent single codes—that might constitute themes. These preliminary themes were examined in light of their ability to both answer the research question and meaningfully describe the dataset. Throughout this process, the research team members collaborated to refine ambiguous themes, merge redundant themes, and otherwise ensure that each theme contributed to the narrative of the data.

FINDINGS

A total of 12 out of 29 respondents identified the claim as a true statement, whereas 14 identified it as false. The final 3 respondents remarked that the claim could be interpreted as either true or false depending on additional assumptions made by the reader. Respondents' conditional construals were primarily centered on choosing a domain over which the continuity of the function should be considered. This decision was sometimes, but not always, tied to their choice of definition.

Choice of domain is dependent on the definition

Most often, participants chose a domain by choosing one definition of continuous function over another. To make this choice, many participants first chose a definition for continuity at a point, from which they extracted a definition of continuous function; this extracted definition tended to inherit its domain from the chosen stipulated definition. The definition would then prompt them to attend to either the entire real line or only those points where $f(x)$ is defined, in line with either Definition-2 or Definition-1 described above. Regardless of which definition they chose, respondents almost always acknowledged the alternative view as part of a hypothetical counterargument. For example, T-23 began her explanation of why the claim is false by "presuming that by continuous function we mean an everywhere continuous function." She later acknowledged that another reader might come to the opposite conclusion if they do not consider continuity at $x = 0$.

Choice of domain is dependent on mathematical convention

When deciding on a domain, some participants attempted to align with what they perceived to be mathematical convention. For example, T-5 first presented a naïve conceptualization of continuity as a single unbroken line—but added that "we usually look at the domain (x-axis values) and or the range (y-axis values) of the function." Consequently, T-5 argued that the claim was true because $f(x)$ could be drawn as a single unbroken curve on each half of its domain. Of note is the fact that participants who appealed to a standard mathematical consensus sometimes disagreed about what exactly that consensus is. T-24 argued that the claim was false unless one disregards the discontinuity at zero, but that "by convention we do not restrict the domain in this manner, unless explicatively stated." T-25 made a similar assessment, adding that "since the domain in the claim is unspecified, it is assumed that we are talking about all real numbers." However, when considering hypothetical counterarguments to his conclusion that the claim was true, T-2 explained that only "purists would argue that all points $-\infty$ to ∞ should be shown to be continuous for a function to be continuous."

Choice of definition is dependent on personal preference

Some respondents selected from possible stipulated definitions based off of an underlying personal belief of what constitutes a continuous function. For example, T-11 examined multiple textbook definitions related to continuity. He admitted that he does not "like a definition of a continuous function that allows functions that are not continuous at all points," and ultimately rejected the Definition-1 as "overly-accepting." In contrast, T-9 chose Definition-1 because "I don't believe it makes sense to consider properties of functions when they are not defined." Finally, T-10 stated that "my understanding of a continuous function is that the function is continuous in its domain," but that someone might disagree because, "from their perspective, a continuous function must be continuous everywhere."

Choice of definition is dependent on visual intuition

Prevalent in responses to the claim were participants' underlying intuitions about what a continuous function should look like; such as when T-2 described a continuous function as "a function that does not have any abrupt changes in value across its domain." More often, participants described the naïve conceptualization of a continuous function as one that can be drawn without lifting one's pencil—although they did not often hold this conceptualization themselves, and instead acknowledged it as a hypothetical argument someone else might employ. For example, both T-13 and T-15 concluded that the claim was true but recognized that a counterargument might stem from the perspective that "it is obvious to the eyes of the reader that the function is not 'connected.'"

T-13 noted that the naïve conceptualization of continuity is "often an instructional language used by teachers and online to try and help students decide whether a function is continuous or not." Similarly, T-12 recognized that "the determination of continuity by drawing without lifting your pencil is an informal, practical way to determine the continuity of a function." Despite initially using this method herself, T-12 later used Definition-1 to argue that the claim is true. She found this to be "a more precise mathematical method which lends mathematical rigor to backing up the truth of the claim."

DISCUSSION AND IMPLICATIONS

Definitions are a pillar of mathematics, yet the notion of definitional ambiguity has not yet received significant attention in mathematics education research. Lack of an agreed-upon, formal definition can lead to cases of definitional ambiguity. In this study we focused on the existence of non-equivalent definitions for continuous function that could be extracted from related stipulated definitions for continuity at a point. The following observation made by T-28 summarizes, in part, the pedagogical implications from our study:

> As we were discussing a lot about how there is no agreed upon definition for many math claims and that different definitions can come up depending on where you are located for your learning. I never thought about this before. I always thought math was the one thing that was the same everywhere. But I am now seeing that math definitions change over time and location.

Participants reflected on their involvement with the task as an "eye-opening" experience, which, for some, changed their perceptions of mathematics. Several participants reported on their search for a "correct" definition, and their dissatisfaction with the ambiguity that they instead discovered.

As noted in previous studies (Foster, 2011; Marmur & Zazkis, 2021), productive ambiguity can be used to foster learners' knowledge and enrich classroom discussions. Involving teachers with cases of productive ambiguity, such as in the task described in this study, is a valuable pedagogical activity that can expand teachers' knowledge as well as enrich their appreciation of mathematics as a discipline. It can be used not only as a prelude for clarifying definitions and the importance of definitions in mathematical activity, but also lead up to a discussion on the nature of mathematics as a human endeavor and on ambiguity as a driving force in mathematical creativity.

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