A CASE STUDY ON STUDENTS' APPROACH TO EUCLIDEAN PROOF IN THE RATIONALITY PERSPECTIVE

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A Rational Mathematical Template (RMT) is the couple consisting of a mathematical entity (definition, proof, etc.) and a rational (according to Habermas) process aimed at producing an instance of that entity. In this report we develop research on RMTs by a teaching experiment on the RMT of proof in two 10th grade classes. The design of the teaching experiment and the analysis of one student's productions were occasions to focus on the relationships between the rational process and its product and on the role of awareness as condition for the mediating role of RMTs in the classroom.

INTRODUCTION

In the last three decades, attention has been addressed in different disciplines to routines, particularly in the sciences of administration, organization and labor (see Feldman & Pentland, 2003). In Mathematics Education, Lavie, Steiner and Sfard (2019) move from "the thesis that repetition is the gist of learning" to consider routine "as the basic unit of analysis in the study of learning" (p. 153) and to the definition of task and procedure, and of routine as a "task-procedure pair": "a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task" (p. 161). They further elaborate the notion of discursive routines by distinguishing between 'process-oriented discursive routines' (called rituals), and 'product-oriented discursive routines' (called 'explorations'). They claim that discursive routines (guided by the question "How do I proceed?") are expected to undergo gradual de-ritualization until they become explorations (guided by the question "What is it that I want to get?").

In spite of the common interest for the invariant aspects of the activity in similar task situations, the discourse developed in Boero & Turiano (2020), Boero (2022) and in this paper on the RMT construct develops according to motifs that differentiate it from Lavie, Steiner and Sfard' construct. The original motif of the elaboration of our construct was to characterize, in an educational perspective, the components of the "intersubjectively shared lifeworld" in the ideal characterization of communicative rationality proposed by Habermas:

> Communicative rationality is expressed in the unifying force of speech oriented towards understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world. (Habermas, 1998, p. 315)

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 83-90). PME.

By interpreting Habermas' text in an educational perspective, our research problem was: what may allow students to share problems and solutions when moving, at the individual and collective level, from what they have already experienced to new challenges on a given subject, at the same time nurturing and developing their "intersubjectively shared lifeworld"? RMT, defined as a couple (mathematical entity; rational process aimed at producing one instance of the entity), was conceived as a possible solution for this problem.

Since the beginning, the RMTs were intended as dynamic-evolving objects of teaching and learning (with an ideal reference to the mathematical culture witnessed by the teacher) in order to meet two needs inherent in this expected individual and collective evolution: the need for common references in each phase of the classroom work, suitable to inform the individual and collective activities of production and reflection on the product; and the need for mediators between the students, the students and the teacher, and the students and the culture (see Boero & Turiano, 2020), in the perspective of progressive evolution of the mastery of the entities and of the related processes towards the learning goals of the teacher.

The initial elaboration on the RMTs needed and still needs further developments in order to become an effective tool for designing and analysing teaching in the perspective of rationality. A first contribution was offered by the analysis of the progressive construction (mediated by the teacher) of the RMT of definition (Boero & Turiano, 2020): the RMT tool worked as analytical tool to analyse the progressive evolution of the mastery of definitions by 8^{th}-grade students through classroom discussions "orchestrated" by the teacher. In Boero (2022) the reported study concerns the RMT of counter-example. Focus is on the evolution of the three components of the rational process in two classroom discussions and their contribution to the development of students' rationality. Attention is paid to the conditions that allowed such construction: general and specific knowledge, and the already existing, positive relationships between students and with the teacher.

The case study reported in this paper had the ambitious aim to answer the following research questions: Is it possible to exploit the RMT of proof as a tool to design and analyse the progressive development in the classroom of the mastery of proof (as a mathematical entity) and of proving (as a rational process)? What about the aspects of the RMT of proof, which may allow it to play the role of mediator between the students, the students and the teacher, and the students and the culture on proof? And what about the aspects of the RMT of proof, which may allow it to play a cultural role in order to develop students' (and teacher's) well-being in the classroom?.

THEORETICAL FRAMEWORK

Habermas' construct of rationality

Habermas' construct of rationality (Habermas, 1998) concerns discursive practices that satisfy epistemic, teleological and communicative requirements: conscious checking of the truth of statements and the validity of reasonings according to shared criteria in a given cultural context (*epistemic rationality*); evaluation of strategies developed to attain the aim of the activity, in the perspective of possibly adopting them in similar, future circumstances (*teleological rationality*); choice of suitable communication tools to reach the others in a given social context (*communicative rationality*), the three components being strictly interconnected.

One salient aspect of Habermas' elaboration on rationality is the fact that the three components of rationality are described as ideal characteristics of discursive practices, while human behaviours are considered rational even in the case that they are only purposefully oriented towards that ideal horizon (see Boero & Planas, 2014). This remark looks important in order to adapt Habermas' elaboration on rationality in mathematics education for both analyzing and comparing rationalities inherent in the different domains of mathematics, and designing and analyzing students' and teachers' activities. In particular, since 2006 some researchers in our group and outside it tried to adapt Habermas' construct in mathematics teacher education (one of the studies is reported in Guala & Boero, 2017) and to plan teaching aimed at developing and analyzing students' rational behaviors (see Boero & Planas, 2014 for a general account and a presentation of five studies).

The Rational Mathematical Template of proof

RMT of proof is characterized by specific epistemic, teleological and communicative aspects: the process is aimed at producing a text with the specific logical and communicative requirements of proof, according to the different methods of proof (direct, by contradiction, by contraposition, by induction…). The process of proving may be considered "rational" when its different phases (exploration, construction of the reasoning, writing the proof text – not necessarily in this linear order) are consciously developed and evaluated according to the aim of the activity, attention being paid to epistemic and communicative requirements inherent in the product.

THE TEACHING EXPERIMENT

We will consider a teaching experiment on Euclidean proof, which involved two 10^{th} grade classes of scientific and technological oriented high school, with 19 and 25 students each. The activities were performed in the period November, 17, 2017 - May, 11, 2018, with two hours each week, for most of the school weeks in the period, for a total of 36 hours, in parallel with other activities on algebra, analytic geometry and probability. The activities were preceded (in grade IX, with the same teacher, and at the beginning of grade X) by some preliminary activities in plane geometry on the nature of definitions, and on some statements of theorems already met by students in comprehensive school, with a few easy proofs utilizing them. We will focus on a situation of conjecturing and proving (and related activities) and on the productions of a student that we will name Mario, which took place at the beginning of March, 2018. We have chosen Mario's productions due to the fact that Mario was one of the students who moved from a low level of performances at the beginning of the sequence (and in Mathematics in general), to an over the average level at the end.

The general design of the sequence of activities on the approach to Euclidean proof in Geometry took into account the fact that geometric constructions (with related theoretical justifications) and theorems alternate in Euclid's Elements. This choice allowed a smooth approach to generality and precision of the discourse on geometric figures (through comparisons of construction texts produced by students for "construction tasks") and to proving (through theoretical justifications of constructions). Students' acquired familiarity with geometric constructions allowed them to produce suitable geometric drawings for conjecturing and for proving tasks.

The classroom activities (a couple of tasks for each two hours) included, from the beginning, tasks of individual geometric construction, with related verbal description. They concerned the bisector of a given angle (with related theoretical justification), a circle tangent to two assigned straight lines, a circle of given radius tangent to two intersecting straight lines, the circles inscribed in, and circumscribed to, a given triangle. Each of them was followed by oral (through a classroom discussion) or written individual revision of constructions produced by some schoolfellows and selected by the teacher. Revisions included checking the generality of the construction and the identification of lacking details and erroneous verbal expressions. Tasks of theoretical, written individual justifications of the construction (based on known statements) were proposed for each construction. They were followed by individual comparison and/or individual revision and/or classroom discussion of theoretical justifications produced by some schoolmates. Concerning theorems, conjecturing and proving activities related to geometric figures, and then proving activities of statements proposed by the teacher, started at the beginning of March, 2018 (Mario's proof text reported below concerns the first activity of this kind). Like for the other activities, systematic individual and/or classroom revisions, comparisons, discussions of proof texts followed each individual proving activity, attention being paid to the key elements of the produced statements and proofs (particularly as concerns the expression of the hypothesis and the thesis, and the necessity of a complete and not redundant proof text). Other activities were proposed, starting from January, 2018: individual cloze activities (followed by a classroom discussion) to complete a theoretical justification of a construction, which was provided by the teacher, by choosing the kind of justification of some steps (by construction; by hypothesis; by definition of…; by theorem…); identification of the proof strategy in the proof text of a schoolmate, with search for possible lacks and mistakes and of theorems and definitions needed to get the proof according to that strategy. The alternation of individual productions (or revisions) and classroom comparisons and discussions was aimed at implementing the RMT of proof as a mediator between the students, the students and the teacher, and the students and the culture (see Boero & Turiano, 2020, p. 145).

Mario's texts and their analysis

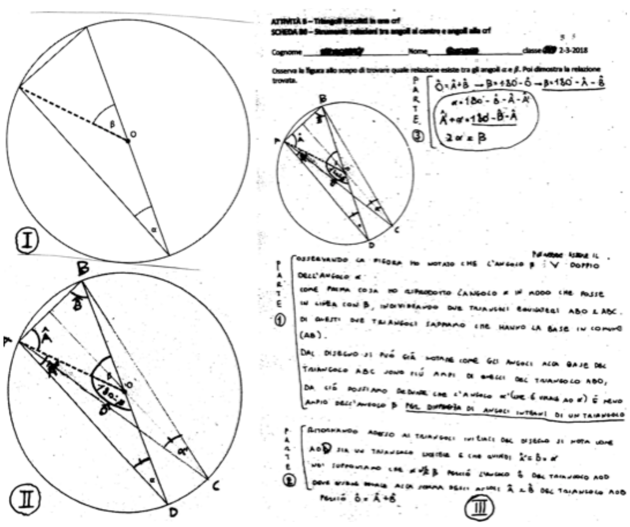

Fig 1: Original figure (I); Mario's figure (II) and text (III)

(PART 1) By observing the figure, I noticed that angle β might be the double of angle α. As first thing, I reproduced the angle α in such a way that it was aligned with β, by finding two equilateral triangles ABO and ABC. These two triangles have their base in common (AB). From the drawing, we may already notice how the angles adjacent to the base of the triangle ABC are wider than those of the triangle ABO, from which we may deduce that the angle α' (that is equal to α) is less wide that the angle β by difference of internal angles of a triangle.

(PART 2) Now, by coming back to the initial triangles of the figure, we notice how AOD is an isosceles triangle and then $\hat{A}'=\widehat{D}=\alpha$. We suppose that $\alpha=½\beta$ thus the angle $\hat{O}$ of the triangle AOD must be equal to the sum of the angles $\hat{A}$ and $\widehat{B}$ of the triangle AOB, hence $\hat{O}=\hat{A}+\widehat{B}$.

(PART 3)

$\hat{O}=\hat{A}+\widehat{B} \rightarrow \beta=180°-\hat{O} \rightarrow \underline{\beta=180°-\hat{A}-\widehat{B}}$

$$\alpha=180°-\widehat{B}-\hat{A}-\hat{A}'$$
$$\hat{A}'+\alpha=\underline{180°-\widehat{B}-\hat{A}}$$
$$2\alpha=\beta$$

From the teleological point of view, Mario looks aware of the different phases of his conjecturing and proving process (the spatial organization of the text and their labels PARTE 1, PARTE 2, PARTE 3 shows three distinguished steps; within the third step Mario puts the core of the proof into evidence, like in the above quote). Moreover, also his revision of his proof text confirms a high level of awareness:

> In this revision I realized that this worksheet well represents my way of reasoning. A gradual reasoning in which, first, I observe the figure and I notice some possible conjectures, then I try to develop the first thoughts, like that of aligning the triangles ABO and ABC. Thanks to this idea I succeeded in finding the basis of my reasoning (…).

In the following analysis of Mario's text some weaknesses on the epistemic and communicative ground will be put into evidence by the use of *italic*.

Mario moves from an initial, possible conjecture ("the angle β might be the double of the angle α"; the initial writing was "the angle β is the double of the angle α") to an exploration of the situation. We may notice a *communication mistake* ("equilateral triangles" instead of "isosceles triangles") and the *lack of justification* of *isosceles triangles*. Then Mario exploits the familiarity with geometric constructions to get a suitable figure, and finally he gets the justification of a weaker statement ($\alpha < \beta$) through visual evidence, a theoretical justification ("by difference of internal angles of a triangle") *implicitly* based on the theorem that the sum of the internal angles is the same for any triangle, and an *unjustified claim* ($\alpha' = \alpha$).

In the second part of his reasoning, by exploring the original figure of the worksheet, Mario notices that the angle α is equal to the angle $\hat{A}'$ (by a theoretical, explicit reason related to the fact that the triangle AOD is *isosceles*; however, the *theoretical justification of it is lacking* – only visual evidence is put on the fore. At that point he foresees how to get the proof: he comes back to the initial possible conjecture, that now is expressed as a *hypothesis to derive what follows*, but probably plays the role of a *hypothesis* to be verified, which results in an *abduction*. This is the starting point of a piece of text of difficult interpretation (at the end of part 2 and at the beginning of part 3), in particular it is not clear the meaning of the two arrows. Mario seems to feel the need to work on the angle $\hat{O}$ of the triangle AOD, which *must be* equal to the sum of the angles $\hat{A}$ and $\hat{B}$ in order to find some relationships that are needed to get the proof. It is clear that Mario works on already considered properties of the triangles (the sum of the internal angles, and the congruence of the angles of isosceles triangles) but *explicit justifications are lacking*. This phase seems to play a heuristic role to get the underlined formula: $\underline{\beta = 180° - \hat{A} - \hat{B}}$. At that point Mario starts a sequence of algebraic expressions that bring to the conclusion. From the surrounding line it is clear that Mario considers what is inside as the proof. The *lack of verbal comments and of some intermediate algebraic expressions* (e.g. the recall of $\underline{\beta = 180° - \hat{A} - \hat{B}}$ and of $\hat{A}' = \alpha$) do not prevent the reader from interpreting Mario's reasoning, also thanks to the spatial disposition of the lines.

Mario's text represents an intermediate step in his approach to the RMT of proof; in the classroom, it looks as a (relatively) high level performance, as concerns the mastery of the whole process (from exploration to proof construction), in comparison with most of his mates' productions. However, Mario's text also reveals some weaknesses (which were rather common in the classroom, at that stage of the construction of the RMT of proof), as we have put into evidence in the above analysis. For all these reasons, Mario's text has been proposed to the class as an object of an individual revision task: "Why this proof has been considered in a positive way by the teacher, in spite of lacks and mistakes in part 2 and part 3? How to correct and improve it?", in the perspective of a discussion to share and discuss what students had discovered, and thus to focus on crucial aspects of the proving process and the proof text. The activity helped Mario to identify an important mistake. In his revision he writes:

> Thanks to comparisons with my schoolmates I realized that at the end of Part 2 and at the beginning of Part 3 my reasoning starts with $\beta = 2\alpha$, which is not a hypothesis but the thesis to be proven, while the hypothesis is that the triangle in inscribed in a circle with one side as a diameter (…).

In another individual activity on the same conjecturing and proving task, students were required to correct, complete and re-write the proofs after identifying and maintaining the authors' reasoning, and to put hypotheses, thesis, and theorems and definitions into evidence. This excerpt from Mario's text under this task well represents the high level of awareness already developed by a consistent number of students (about one half of them), and (as the previous excerpt) the climate created through the need of analysing and improving the schoolmates' texts according to the shared rationality criteria:

> In the first solution I realized that there was an unusual reasoning, different from those we had considered in the discussion, but it is correct. However, some points should be improved: the fact that $\alpha = \partial$ does not result from the definition of an isosceles triangle, but from a theorem. There is an important lacking point: the proof that BE is parallel to DO. It is needed to use the theorems on alternate angles.

CONCLUSIONS AND DISCUSSION

Through the description of the sequence of activities and the analysis of Mario's productions we have tried to put into evidence how the use of the RMT of proof may serve the planning and the analysis of classroom activities aimed at student's approach to proving and proof in grade X. As an analytical tool related to Habermas' rationality, the RMT of proof was used to identify weak points of Mario's proof text. They needed (and allowed) interventions (through revision tasks and related discussions) to develop awareness, in particular, of crucial epistemic and communicative aspects of proof.

We may observe how in the planning of the teaching experiment awareness (of the requirements of the product of the process and of the organization of the process) played a crucial role through several specific tasks; this looks necessary to ensure the rationality of the process and the epistemic and communicative quality of its product. The analysis of Mario's productions shows how the role of awareness in the planning of the teaching experiment results directly in the mastery of his personal process and in the revision of the epistemic aspects of his schoolmate's proof text, and indirectly in the climate of the work in the classroom, through the acknowledgment of the contributions of his schoolmates to overcome an important weakness in his text, and the mature, constructive approach to his schoolmate's production. Mario's productions are representative examples of what happened in the two classrooms during the teaching experiment. In particular, the systematic work on students' awareness of the requirements of rationality through the revision, cloze and identification tasks seems to have a supportive, double function on the cultural ground: for the development of a collaborative style of work in the classroom (thus contributing to the well-being of all the involved people), and to ensure the role of mediation that the RMT of proof plays in the long term development of students' proving. This double function looks to be not limited to the case of the RMT of proof and should be elaborated in general, thanks to other teaching experiments on complex, demanding RMTs (like that of analytical model of physical phenomena, or that of probabilistic model of stochastic phenomena).

From the theoretical point of view, the content of the previous Section puts into evidence the distance between the RMT construct and the construct by Lavie, Steiner & Sfard, 2019 beyond what concerns the motifs of the constructs (see Introduction). In particular, in the classroom long term construction of the RMT of proof that we have described it is not possible to distinguish a ritual phase from an exploration phase. Indeed, for intrinsic reasons due to the necessity of developing awareness (a crucial requirement of Habermas' rationality), since the very beginning students are engaged in both productive and reflective activities on accessible tasks, which gradually evolve trough a conscious mastery of more and more complex situations.

However, the definition of RMT still needs an in-depth work, if we want to move from an extensive definition (i. e. a definition concerning a set of assigned "entities", with specifications for the components of the rational process aimed at the production of "instances" of those individual "entities"), to an intensive definition (i.e. a definition based on a characteristic, common property of the "mathematical entities", with specification of the general aspects of the rational process that result from the entity).

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