PROBING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF VISUAL REPRESENTATIONS OF FUNCTION TRANSFORMATIONS: A MULIPLE SCRIPTING TASK APPROACH

James A. M. Álvarez, Theresa Jorgensen, Janessa Beach

The University of Texas at Arlington

In this paper, we use multiple scripting tasks as a research tool to investigate prospective secondary mathematics teachers' (PSMTs') mathematical knowledge of function transformations and their inclination to connect multiple representations of functions. Mathematically similar scripting tasks focused on visual representations of function transformations were given at three intervals during a 15-week semester in an undergraduate mathematics course on functions for PSMTs in the United States. Participant responses to these scripting tasks were analysed, and four prevalent themes were identified that reveal initial tendencies to disregard visual observations posed by students in the scripting tasks and limited use of their mathematical knowledge to connect multiple representations of functions.

INTRODUCTION

Prospective secondary mathematics teachers (PSMTs) will be expected to teach mathematics for which the concept of function is a fundamental component. However, Ponte and Chapman (2008) identify "lack of a good understanding of functions" as a consistent issue with the knowledge of PSMTs highlighted in the research literature (p. 227). For example, Even (1993) found that a limited conception of function influenced PSMTs pedagogical reasoning. Also, Hitt (1998) links a group of practicing secondary mathematics teachers' conceptual knowledge to difficulties in passing from one representation of function to another. With the prevalence of graphing technology, visual representations of functions can be easily generated and used in the classroom. However, this may expose ways in which PSMTs' limited understandings of the connections between representations may deter future teachers' capacity to address student understandings and leverage their own understandings.

In this study, we used three mathematically similar scripting tasks to explore any changes in the PSMTs' understanding that may have been influenced by inquiry-based lessons focused on functions and function patterns. Our research questions are: (1) To what aspect of the mathematics in the scripting task do PSMTs choose to attend? In particular, how do they connect different representations or attempt to make mathematical connections to resolve the student's question? (2) How do PSMTs choices to resolve the student's question incorporate or validate the student's mathematical thinking?

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 3-10). PME.

BACKGROUND AND THEORETICAL PERSPECTIVE

Script writing in the context of a mathematics course for preservice teachers can be a useful research tool to investigate mathematical knowledge and understanding for teachers (Zazkis & Zazkis, 2014). A scripting task typically begins with a hypothetical conversation between a teacher and a student, or between multiple students, which is then continued by the PSMT in a written dialogue. Script writing tasks provide PSMTs an opportunity to prepare a well-considered reply to a student, rather than an on-the-fly, in-the-moment response. Scripting tasks allow researchers a written window into the mathematical thinking of the PSMT, together with a view of how the PSMT chooses to address a cognitive conflict as expressed by a student, and their pedagogical sensitivity in assisting students (Kontorovich & Zazkis, 2016).

In a student-centred mathematics classroom, researchers have supported models of effective mathematics instruction in which a teacher fosters students' ability to consider various mathematical solutions (Hiebert et al. 1997). To do this, a teacher must use their own mathematics knowledge flexibly to draw out the important representations, ideas, and conceptions embedded in students' mathematical thinking. Teachers who lack this flexible knowledge of mathematics and student thinking may be more inclined toward ritualized "show-and-tell" (Silver et al., 2005). Ball's (1990) study exhibits this inclination when she probed ten elementary and nine secondary prospective teachers' understanding of division and found that the prospective teachers at both levels tended to search for the particular rules rather than focusing on underlying meanings. "They seemed to assume that stating a rule was tantamount to settling a mathematical question" (p. 141). In 2008, Ball et al. further categorized mathematical knowledge unique to the work of teaching or mathematical knowledge for teaching (MKT). The domains of MKT proposed by Ball et al. (2008) map to two categories – subject matter knowledge and pedagogical content knowledge. In the context of this study, subject matter knowledge is at the core of the sequence of scripting tasks completed by the PSMTs.

Developing a deep understanding of function transformations at the secondary level can require the learner to reconcile multiple representations of function, including graphical, tabular, and symbolic representations (c.f. Eisenberg & Dreyfus, 1994). Oehrtman et al. (2008) "recommend that school curricula and instruction provide more opportunities for students to experience diverse function types emphasizing multiple representations of the same functions" (p. 153). Dynamic visualization software can be a robust tool for students to make these connections as they explore the effect of different transformations (Villarreal, 2000). However, visual information can sometimes negatively influence misconceptions held by the learner (Aspinwall et al., 1997). For example, Álvarez et al. (2020) described a task for practicing teachers in which the teachers struggled to explain an apparent discrepancy between the dynamic visual representation of a vertical dilation of the linear function $f(x) = x$ and a rotation of the graph of $y = x$ about the origin. In addition, Moore and Thompson

(2015) use the study of shape thinking "to offer a new perspective on multiple representations by enabling researchers to be clearer about what a graph represents *to a student*, and thus what students understand multiple representations to be representations of" (p. 788).

METHODS

This study took place at a large, public university in the southwestern United States with an enrolment of over 42,000 students. The university is recognized as one of the most diverse national universities in the United States. Participants in this study consisted of 27 PSMTs enrolled in 2018 fall semester, second-year mathematics content course for PSMTs with a second-semester calculus prerequisite. Twelve participants self-identified as male and 15 self-identified as female.

The mathematics course implemented a unit developed by the *Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers* Project, *Explorations on Functions and Equations* (*EFE*). The *EFE* materials consist of 11 research-based lessons with an objective of deepening and broadening PSMTs function-related mathematical content knowledge from school algebra to calculus by exploring relevant topics in an inquiry-based learning environment. In the 15-week fall 2018 semester, the *EFE* materials spanned the first 10 weeks of the course approximately.

This study centres around three scripting tasks related to two lessons within the *EFE* materials: "Functions Arising from Patterns" and "Indistinguishable Function Transformations and Function Patterns." Zazkis and Zazkis (2014) advocate that scripting tasks "serve as a window for researchers to investigate participant's understanding of mathematics" (p. 68). The scripting tasks in this study are intended to reveal PSMTs' MKT. In particular, MKT related to connections between function transformations and their visual representations. During the eighth week of implementation of the *EFE*, students completed Scripting Task 1 (ST1). This served as a baseline for evaluating participants' MKT, and it was completed outside of class before the lesson on "Functions Arising from Patterns." ST1 (see **¡Error! No se encuentra el origen de la referencia.**) provides a fictional interaction between a teacher and a student, Grace, in which Grace questions the teacher about the horizontal compression she perceives in the transformation $g(x) = a \cdot f(x)$ as a varies dynamically where $f(x) = x^3$ and $a > 1$.

Over the next three 80-minute class meetings, students engaged in the lesson "Functions Arising from Patterns" and then "Indistinguishable Function Transformations and Function Patterns." For the former, PSMTs investigated patterns in the domain of given data sets and resulting patterns in the corresponding range data sets. Specifically, students explored the domain-range patterns within data sets arising from linear, quadratic, power, exponential, and logarithmic functions. They identified patterns such as an addition-product pattern for logarithmic functions by noticing that adding c to subsequent domain values results in a pattern of multiplying the corresponding range values by a constant k (that depends on c).

In the second part of the "Functions Arising from Patterns" lesson, students work with general forms of the functions to verify algebraically that the identified domain-range patterns apply to certain transformations on functions of the same type. For example, they verify the product-addition pattern for logarithmic functions.

> You are given the beginning of an interaction between a teacher and a student. Your task is to continue this imaginary conversation in the form of a dialogue between the teacher and student (or several students).
>
> T: Today we will continue our exploration of transformations of functions. Consider for example $g(x) = a \cdot f(x)$ where $f(x) = x^3$ and $a > 1$. How does the graph of g compare to the graph of f? Yes, Grace?
>
> S: So you said yesterday that multiplying by a is a vertical stretch.
>
> T: That is correct as long as $a > 1$.
>
> S: But my graph of g looks like a horizontal compression of f.
>
> T: Ok.
>
> S: So a on the outside means it's both a vertical stretch and a horizontal compression? Or what?
>
> T: ...

Figure 1: ST1

The "Indistinguishable Function Transformations and Function Patterns" lesson examines function patterns that may seemingly produce a dynamic process that defies the algebraic rules previously learned about transformations of functions. PSMTs encounter four scenarios in which a particular transformation represented algebraically simultaneously appears also to correspond to a different type of transformation. They are invited to use appropriate technology in their exploration. The following is an excerpt of one of the scenarios:

> For a given function f, we define a new function $g(x) = f(x + c)$ where $c > 0$. The graph of the new function g is a horizontal translation (shift) of the graph of f, but it also appears to be a vertical translation (shift) of the graph of f. In order to observe this, which function pattern must f have? Explain your reasoning. Identify the function type for which this observation would apply.

Directly after completing these explorations on function patterns and transformations, students were given Scripting Task 2 (ST2) to be completed outside of class. This task is then intended to elicit participants' MKT after they have had the opportunity to delve into these ideas within the *EFE* lessons. Like ST1, ST2 presents a fictional conversation between a student, Isaac, and his teacher. In this conversation though, Isaac asks why he sees a vertical stretch in the transformation $g(x) = f(x + c)$, where $f(x) = 3^x$. Students were asked to carry out this dialogue between the teacher and Isaac. Finally, Scripting Task 3 (ST3) was presented to students as a part of their end-of-course final exam. This task asks students complete another discussion between a teacher and the student, Isaac, where Isaac asks about the vertical stretch he perceives in the transformation $g(x) = f(cx)$, where $f(x) = x^3$ and $c > 0$. Between ST2 and

ST3, students completed explorations outside the *EFE* lessons that centred on ideas of statistical regression, the polar coordinate system, and complex numbers. Thus, ST3 is intended to reveal the MKT that persisted over time.

Following the completion of all three scripting tasks, participant responses were de-identified and linked to a participant number. We then coded responses to identify the ways in which PSMTs leveraged their understandings of function transformations and representations to attend to student questions posed in the scripting tasks. Separately, we each generated initial codes for all the scripting task data. All initial codes were then reviewed and triangulated by the research team, organized into common reactions, defined, and named. We then examined the prevalence of these reactions.

RESULTS

Each scripting task posed ended with a scripting-task-student (STS) question arising from the situation such as "S: So does adding inside a function give you both a horizontal shift and a vertical stretch? Or what?" from ST2. Four dominant reactions were identified when examining participant responses to STS questions across all three scripting tasks. These reactions were applying form-dependent reasoning, directing visual observations, comparing representations, and focusing on algebraic equivalence.

PSMTs' use of form-dependent reasoning involved directly appealing to a rule to redirect the STS claim. For example, on ST3, one PSMT explains, "In this case, it appears as if it is a vertical stretch, but it is not. When the 'c' is larger or smaller, it will appear to look more like a horizontal stretch. Just remember the rules because looks can be deceiving." On ST2, another PSMT also says, "No, it may be perceived that way, but when we have a constant added inside the function than you will always get a vertical or horizontal shift." On ST1, 45% of the PSMTs appealed to the rule or form only whereas 25% and 30% did so on ST2 and ST3, respectively.

The reaction of directing visual observations was identified when PSMTs' explanations directed students to attend only to the changes related to the form of the expression such as the following participant answer on ST1.

T: Well, what exactly is a vertical stretch?

S: It's when the y-values in the graph are bigger than the y-values of the parent function's graph?

T: So, it has nothing to do with the x-values?

S: No, the x-values stay the same.

T: Then, if your x-values are the same, but your y-values are bigger, what does the graph look like?

S: Tall and skinny.

T: Exactly. It looks tall and skinny because the y-values changed, but there is not actually a horizontal compression.

S: Oh, that makes sense. It's the scale of my x-axis that makes it look like a horizontal compression.

On Scripting Tasks 1 and 2, 31% of the PSMTs directed STSs in this way whereas on ST3 only 4% did this.

PSMTs used comparing representations most on ST1 (32%), but then this dropped to 6% on ST2 and increased again to 27% on ST3. For example, on ST2 a PSMT draws a student's attention to a tabular representation to illustrate the transformation but does not validate *why* the student is observing the apparent contradiction to the learned rule.

Focusing on the algebraic representation as a way to explain the apparent contradiction in the STS claim or question only appeared in less than 5% of the responses on ST1, 19% of the responses in ST2, and 38% of the responses on ST3. These responses involved the PSMT showing how the algebraic representation may help illuminate why there is an apparent contradiction to the learned rules.

In addition, we noted that PSMTs were much more likely to validate the STS claims on the final scripting task when compared to the previous tasks. That is, 46% of the participants validated the STS claim on the ST3 versus 19% on ST2 and 14% on ST1. Validating a STS claim did not preclude a participant from then evoking form dependent reasoning, directing visual observations, comparing representations, or appealing to algebraic equivalence in attempts to complete the scripting task. Thus, in most instances, participants were not attending to *why* the student was seeing what they were seeing, but only addressing how they *should be* seeing it. Their responses would continue with "this is why it is not…"

DISCUSSION

To address our research questions, we employed repeated use of related scripting tasks as a research tool. The codes identified suggest that our PSMT participants held views similar to Ball's (1990) prospective elementary and secondary mathematics teachers that "stating a rule was tantamount to settling a mathematical question" (p. 141).

Although the lessons attend to multiple representations of function, PSMTs displayed uneven abilities to connect different representations and use this knowledge to attend to student thinking. Although reliance on "rule following" decreased from 45% to 25% from ST1 to ST2, the persistence of "rule following" indicates further revisions to the lessons or refinements to the facilitation of the tasks is warranted.

The tendency for PSMTs to have the teacher in the script direct the scripting task student's attention to the transformation that they "should see," decreased dramatically to only 4% on ST3. This may have been due to group discussions and review before the final exam in which PSMTs viewed animations directing their attention to seeing these simultaneous transformations does occur and that simply redirecting attention to the "correct" transformation does not help a student understand why they see what they see. This relates Moore and Thompson's (2015) idea that we may not clearly understand what the dynamic situations represent to the PSMT and how PSMTs' understanding enables them to make connections among different representations. PSMTs may question their understanding of function transformations when confronted with conflicting visual information causing visual imagery that interferes with their understanding as seen in Aspinwall, et al. (1997). Development of the MKT to untangle this conflicting visual information was not present in our participants.

Our findings related to PSMTs validating student thinking, but then explaining "why it's not…" give some insight into how PSMTs may have an underdeveloped understanding of representations. The use of multiple scripting tasks to track PSMTs understanding in this way reveals that while the PSMTs overwhelmingly validated student thinking on ST3 more work is needed to help them attend to answering the student's "why" question and not only superficially acknowledge their thinking to move to a standard explanation. We continue to investigate how scripting tasks, used in this manner, can inform curriculum development as well as provide formative assessment on appropriate mathematical concepts.

Acknowledgement

This research is based upon work partially supported by the National Science Foundation (NSF) under grant number DUE-1612380. Any opinions, findings, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the NSF.

References

Álvarez, J.A.M., Rhoads, K., & Jorgensen, T. (2020). Mathematics teacher educators learning from efforts to facilitate the learning of key mathematics concepts while modelling evidence-based teaching practice. In K. Beswick & O. Chapman (Eds.), *The International Handbook of Mathematics Teacher Education*: *Volume 4* (2nd Ed.) (pp. 305-341). Koninklijke Brill NV, Leiden.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, *33*, 301-317.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher education*, 59(5), 389-407.

Eisenberg, T., Dreyfus, T. (1994). On Understanding How Students Learn to Visualize Function Transformations. In E. Dubinsky, A. Schoenfeld, J. Kaput (Eds), *Research in Collegiate Mathematics Education, I,* 45-68, CBMS Issues in Mathematical Education 4.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*, 94-116.

J. Hiebert, T.P. Carpenter, E. Fennema, K.C. Fuson, D. Wearne, H. Murray, et al. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann, Portsmouth, NH.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123-134.

Kontorovich, I. & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, 93(2), 223-243.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782-789). Pittsburgh, PA: West Virginia University

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational Reasoning Abilities that Promote Coherence in Students' Understanding of Function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Ponte, J.P. & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. D. English (Ed.), *Handbook of international research in mathematics education: Directions for the 21st century* (2nd Ed., pp. 223-261). New York: Routledge.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. T. F. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior*, 24(3), 287-301. doi:10.1016/j.jmathb.2005.09.009.

Villarreal, M. (2000). Mathematical thinking and intellectual technologies: the visual and the algebraic. *For the learning of mathematics*, *20*(2), 2-7.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in mathematics education, 16*(1), 54-70.