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The class of *c*-almost periodic functions defined on vertical strips in the complex plane

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Abstract
In this paper, we develop the notion of c-almost periodicity for functions defined on vertical strips in the complex plane. As a generalization of Bohr's concept of almost periodicity, we study the main properties of this class of functions which was recently introduced for the case of one real variable. In fact, we extend some important results of this theory which were already demonstrated for some particular cases. In particular, given a non-null complex number c, we prove that the family of vertical translates of a prefixed c-almost periodic function defined in a vertical strip U is relatively compact on any vertical substrip of U, which leads to proving that every c-almost periodic function is also almost periodic and, in fact, c^m-almost periodic for each integer number m.

Keywords Almost periodic functions · c-almost periodic functions · Almost anti-periodic functions · Functions of a complex variable

Mathematics Subject Classification 42A75 · 43A60 · 30Axx · 30D20

1 Introduction

Let E be a complex Banach space and $c \in \mathbb{C}\setminus\{0\}$. It is said that a continuous function $f : \mathbb{R} \to E$ is c-periodic if there exists $w > 0$ such that $f(x + w) = cf(x)$ for all

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$x \in \mathbb{R}$. This concept, which was proposed in [1], extends the more known notions of anti-periodicity (with $c = -1$) and Bloch periodicity (with c depending on w in the form $c = e^{ikw}$, $k \in \mathbb{R}$), which constitute variants of the usual periodicity with practical relevance for engineering science (especially condensed matter physics). See also [2, 10, 13, 14] for more information on these spaces of functions. In particular, it is easy to see that any anti-periodic function is also periodic (because of the fact that $f(x+2w) = f((x+w)+w) = -f(x+w) = f(x)$ for all $x \in \mathbb{R}$), and the class of Bloch (w, k)-periodic functions is equal to the class of anti-periodic functions when $kw = \pi$. In fact, if $kw \in \mathbb{Q}$ (put $kw = \frac{p}{q}$, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\setminus\{0\}$), then a Bloch $(2\pi w, k)$-periodic function $f(x)$ is also periodic because of the fact that

$$\begin{aligned} f(x+qw) &= f((x+(q-1)w)+w) = e^{2\pi ikw} f(x+(q-1)w) \\ &= e^{2\pi ikw} f(x+(q-2)w+w) = e^{2\pi i2kw} f(x+(q-2)w) \\ &= \cdots = e^{2\pi iqkw} f(x) = e^{2\pi ip} f(x) = f(x). \end{aligned}$$

As a generalization of purely periodic functions defined on the set of the real numbers, H. Bohr introduced the concept of almost periodicity during the 1920's. A continuous function $f : \mathbb{R} \to E$, where E is a complex Banach space whose norm is denoted by $\|\cdot\|$, is said to be almost periodic (in Bohr's sense) if for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any open interval of length l contains a number τ satisfying $\|f(x+\tau) - f(x)\| \leq \varepsilon$ for all real numbers x. The number τ described above is called an ε-almost period or an ε-translation number of $f(x)$ and, equivalently, the property of almost periodicity means that the set of all ε-translation numbers of $f(x)$ is relatively dense on the real line. We will denote as $AP(\mathbb{R}, \mathbb{C})$ the space of almost periodic functions in the sense of this definition (Bohr's condition). As in classical Fourier analysis, it can be seen that every almost periodic function is bounded and is associated with a Fourier series with real frequencies (see for example [3, 5]). Shortly after its development, this theory acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations.

In connection with the notion of almost periodicity, Khalladi et al. [12] have recently considered the following generalization, which is called c-almost periodicity: a continuous function $f : \mathbb{R} \to E$ (where E is a complex Banach space whose norm is denoted by $\|\cdot\|$) is said to be c-almost periodic if for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any open interval of length l contains a number τ satisfying $\|f(x+\tau) - cf(x)\| \leq \varepsilon$ for all x. It is clear that any c-periodic function is also c-almost periodic. The number τ described above is now called an (ε, c)-almost period or an (ε, c)-translation number of $f(x)$. Hence c-almost periodicity means that the set of all (ε, c)-translation numbers of $f(x)$ is relatively dense on the real line. We will denote as $AP_c(\mathbb{R}, E)$ the space of c-almost periodic functions in the sense of this definition. Note that the case $c = 1$ leads to the space $AP(\mathbb{R}, E)$. Moreover, if $c = -1$ the functions in the space $AP_{-1}(\mathbb{R}, E)$ are called almost anti-periodic (in this respect, Cheban considered this notion in two papers [6, 7] written around 1980). See also [2, 10, 13, 14] for more information on these spaces of functions defined on the real line.

In this paper, we will extend the concept of c-almost periodicity to the case of functions defined on vertical strips of the complex plane (see Definition 1). For the case $c = 1$, this notion corresponds with the set $AP(U, \mathbb{C})$ of almost periodic functions defined on a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. This space, which was theorized in [4], has been widely studied in the literature as an extension of the real case (see for example Chapter 3 of the books [3, 8] and the references [5, 9, 11, 16]). As it can be seen, many properties of this space of functions are closely related to the theory of analytic functions.

Given an arbitrary $c \in \mathbb{C}\setminus\{0\}$, in comparison with the real case and the almost periodicity, we will establish in Sect. 2 the main properties of the space of c-almost periodic functions in a vertical strip U (see particularly Propositions 1, 3, 4 and Corollary 1). It is worth noting that, in general, this space of functions is not a vector space with the usual operations (see Example 2). However, many other properties associated with the class of almost periodic functions (such as Propositions 1, 2, 4, 7, Corollary 1 or Theorems 1, 2) are also true for this new class of c-almost periodic functions.

In Sect. 3, we will demonstrate that the set of all the values of a c-almost periodic function on any vertical substrip included in its domain is relatively compact in $\mathbb{C}$ (see Proposition 7), which represents an extension of [8, Theorem 6.5]. Moreover, we will prove that the family of vertical translates of a prefixed c-almost periodic function defined in U is relatively compact on the vertical substrips of U (see Theorem 2), which represents an extension of the necessary condition of [8, Theorem 6.6]. This will allow us to prove that every c-almost periodic function defined in U is also almost periodic and, in fact, c^m-almost periodic for each $m \in \mathbb{Z}$ (see Proposition 3, point i), and Corollary 3). The proof of this result can be immediately adapted to the real case in order to state that $AP_c(\mathbb{R}, \mathbb{C})$ is included in $AP(\mathbb{R}, \mathbb{C})$ for any $c \in \mathbb{C}\setminus\{0\}$, which constitutes an extension of a result proved by Khalladi et al. for the case $|c| = 1$ (see [12, Proposition 2.11 and comments above]).

2 The property of *c*-almost periodicity for the case of functions of a complex variable

Our starting point is the following definition which is based on the concept of c-almost periodicity for the case of functions defined on $\mathbb{R}$ (see [12, Definition 2.1]) and Bohr's notion of almost periodicity for the case of functions of a complex variable (see for example [16, p. 246], [11, p. 311], [8, pp. 83,86] or [3, pp. 141,142]).

Definition 1 Let $f : U \to \mathbb{C}$ be a complex function continuous in a strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$, and $c \in \mathbb{C}\setminus\{0\}$. Given $\varepsilon > 0$ and a reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$), a real number τ satisfying the condition

$$|f(z + i\tau) - cf(z)| \leq \varepsilon, \quad \text{for all } z \in U_1,$$

is called an (ε, c)-*translation number of* $f(z)$ (or a *c-translation number belonging to* ε) associated with U_1. The set of all (ε, c)-translation numbers of $f(z)$ associated with a reduced strip $U_1 \subset U$ will be denoted by $E_c\{f(z), U_1, \varepsilon\}$ or simply $E_c\{f(z), \varepsilon\}$ (we will omit U_1 in the notation when it is clear from the context).

If the sets $E_c\{f(z), U_1, \varepsilon\}$ are relatively dense for every prefixed $\varepsilon > 0$ and reduced strip $U_1 \subset U$ (which means that there corresponds a number $l > 0$ such that every open interval of length l contains points in $E_c\{f(z), U_1, \varepsilon\}$), then the function $f(z)$ will be called *c-almost periodic in the strip U*. We will denote as $AP_c(U, \mathbb{C})$ the set of c-almost periodic functions in the sense of this definition.

The case $c = 1$ of the set of the almost periodic functions defined on U will be also denoted as $AP(U, \mathbb{C})$.

In the particular case that $c = -1$, it is said that $f(z)$ is *almost anti-periodic in the strip U*.

Remark 1 Given $c \in \mathbb{C}\setminus\{0\}$, $f \in AP_c(U, \mathbb{C})$ and $\varepsilon > 0$, note that the sets $E_c\{f(z), U_1, \varepsilon\} \subset \mathbb{R}$ are closed for every reduced strip $U_1 \subset U$. Indeed, fixed such a strip $U_1 \subset U$ and $\{\tau_n\}_{n\geq 1} \subset E_c\{f(z), U_1, \varepsilon\}$ a sequence converging to a value $\tau \in \mathbb{R}$, then

$$|f(z + i\tau_n) - cf(z)| \leq \varepsilon, \quad \text{for all } z \in U_1 \text{ and } n \in \mathbb{N},$$

and we deduce from the continuity of $f(z)$ that

$$|f(z + i\tau) - cf(z)| \leq \varepsilon, \quad \text{for all } z \in U_1,$$

which yields that $\tau \in E_c\{f(z), U_1, \varepsilon\}$.

Remark 2 Let $f : U \to \mathbb{C}$ be a complex function continuous in a strip U, $c \in \mathbb{C}\setminus\{0\}$, and take $\varepsilon > 0$ and $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$. If $\tau_1 \in E_c\{f(z), U_1, \frac{\varepsilon}{2|c|}\}$ and $\tau_2 \in E_c\{f(z), U_1, \frac{\varepsilon}{2}\}$, then $\tau_1 + \tau_2 \in E_{c^2}\{f(z), U_1, \varepsilon\}$. Indeed, we have

$$\begin{aligned}\left|f(z + i(\tau_1 + \tau_2)) - c^2 f(z)\right| &\leq |f(z + i(\tau_1 + \tau_2)) - cf(z + i\tau_1)| \\ &\quad + \left|cf(z + i\tau_1) - c^2 f(z)\right| \leq \varepsilon, \quad \text{for all } z \in U_1.\end{aligned}$$

If $c \neq 1$ and U is an arbitrary vertical strip of the complex plane, it is straightforward to see that the non-zero constant functions are not c-almost periodic in U (but they are almost periodic in U). However, if c is a unitary complex number, the exponential monomials of the form $ae^{\lambda z}$, with $a \in \mathbb{C}$ and $\lambda \in \mathbb{R}\setminus\{0\}$, are representative examples of c-almost periodic functions in U.

Example 1 Let $c \in \mathbb{C}$ be a complex number so that $|c| = 1$ (put $c = e^{i\theta_c}$ with $\theta_c \in (-\pi, \pi]$). Given an arbitrary vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$, $a \in \mathbb{C}$ and $\lambda \in \mathbb{R}\setminus\{0\}$, consider the function $f(z) = ae^{\lambda z}$, $z \in U$. Then $f(z) \in AP_c(U, \mathbb{C})$. Indeed, given $\varepsilon > 0$ and $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$, note that the choice $\tau = \frac{\theta_c}{\lambda} + \frac{2\pi k}{\lambda}$, $k \in \mathbb{Z}$, satisfies

$$|f(z + i\tau) - cf(z)| = |a||e^{\lambda(z+i\tau)} - ce^{\lambda z}| = |a||e^{i\theta_c}e^{\lambda z} - e^{i\theta_c}e^{\lambda z}| = 0 < \varepsilon,$$

for all $z \in U_1$. Hence the sets of the (ε, c)-translation numbers of $f(z)$ (associated with every reduced strip of U) are relatively dense.

Given $c \in \mathbb{C}\backslash\{0\}$, we next show that the sets $AP_c(U, \mathbb{C})$ and $AP_{1/c}(U, \mathbb{C})$ are equal.

Lemma 1 *Let $c \in \mathbb{C}\backslash\{0\}$ and $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ (with $-\infty \leq \alpha < \beta \leq \infty$). Then $f(z) \in AP_c(U, \mathbb{C})$ if and only if $f(z) \in AP_{1/c}(U, \mathbb{C})$.*

Proof Given $\varepsilon > 0$, $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$, $\tau \in \mathbb{R}$ and $z \in U_1$, note that

$$\left| f(z - i\tau) - \frac{1}{c} f(z) \right| = \frac{1}{|c|} |cf(z - i\tau) - f(z)| = \frac{1}{|c|} |f(z) - cf(z - i\tau)| \,.$$

This means that $\tau \in E_c\{f(z), U_1, |c|\varepsilon\}$ if and only if $-\tau \in E_{1/c}\{f(z), U_1, \varepsilon\}$, which proves the result. □

As usual, put $z = x + iy$. Note that a c-almost periodic function $f(z)$ in a certain vertical strip U is also a c-almost periodic function of the real variable y on any vertical line included in U. That is, for every $\varepsilon > 0$ and an arbitrary $x_0 \in (\alpha, \beta)$, there corresponds a number $l = l(\varepsilon) > 0$ such that any open interval of length l contains a number τ satisfying $|f(x_0 + i(y + \tau)) - cf(x_0 + iy)| \leq \varepsilon$ for all y. We next analyse the converse of this property.

Proposition 1 *Let $f : U \to \mathbb{C}$ be an analytic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ (with $-\infty \leq \alpha < \beta \leq \infty$) which is bounded in any substrip $\{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. Given $c \in \mathbb{C} \setminus \{0\}$, suppose that $f(x_0 + iy)$ is a c-almost periodic function of the real variable y for some $x_0 \in (\alpha, \beta)$. Then $f(z)$ is c-almost periodic in U.*

Proof Take $c \in \mathbb{C} \setminus \{0\}$ and $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\}$ a reduced vertical strip of U, with $\alpha_1 < x_0 < \beta_1$. Let $U_2 = \{z \in \mathbb{C} : \alpha_2 \leq \operatorname{Re} z \leq \beta_2\} \supset U_1$, with $\alpha < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < \beta$. By hypothesis, we can assure the existence of a positive number M such that $|f(z)| \leq M \; \forall z \in U_2$, which yields that

$$|f(z + i\tau_1) - cf(z)| \leq (1 + |c|)M, \quad \text{for all } \; z \in U_2, \tag{1}$$

for any $\tau_1 \in \mathbb{R}$. Also, by c-almost periodicity of f on the vertical line $x = x_0$ (included in the interior of U_1), we know that for every $\delta_1 > 0$ there exists a relatively dense set of (δ_1, c)-translation numbers of $f(x_0 + iy)$. Let τ be an (δ_1, c)-translation number of $f(x_0 + iy)$, which means that

$$|f(x_0 + i(y + \tau)) - cf(x_0 + iy)| \leq \delta_1, \quad \text{for all } \; y \in \mathbb{R}. \tag{2}$$

For every value τ defined as above, by (1) and (2) the function given by $g_\tau(z) := f(z + i\tau) - cf(z)$, $z \in U$, is analytic in U and it satisfies the following inequalities: $|g_\tau(z)| \leq (1 + |c|)M$ for all $z \in U_2$ and $|g_\tau(x_0 + iy)| \leq \delta_1$ for all $y \in \mathbb{R}$. Consequently, we can apply [3, p. 138] (or [8, Theorem 3.5]) in order to assure that to any $\varepsilon > 0$ (with $\varepsilon < (1 + |c|)M$) corresponds $\delta > 0$ satisfying

$$|g_\tau(z)| = |f(z + i\tau) - cf(z)| \leq \varepsilon, \quad \forall z \in U_1,$$

provided that $|g_\tau(x_0+iy)| = |f(x_0+i(y+\tau)) - cf(x_0+iy)| \leq \delta$ for all $y \in \mathbb{R}$. However, by taking $\delta_1 = \delta$ in (2), it is clear that this last inequality is true for a relatively dense set of (δ, c)-translation numbers of $f(x_0+iy)$. Consequently, the set of (ε, c)-translation numbers of $f(z)$ is also a relatively dense set of the real line, and $f(z)$ is c-almost periodic in U. □

As a consequence of the result above, we can establish in a simple way certain properties of the c-almost periodic analytic functions defined on vertical strips by using the corresponding properties of the c-almost periodic functions of a real variable.

In any case, we next show that every function in $AP_c(U, \mathbb{C})$ is bounded on any vertical substrip included in U.

Proposition 2 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then $f(z)$ is bounded in any strips $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and $\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b\}$ with $\alpha < a < b < \beta$.*

Proof Take $U_2 = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\} \subset U$ with $\alpha < a < b < \beta$. As $f(z)$ is continuous on $\operatorname{cl} U_2$, it is bounded on any compact subset included in $\operatorname{cl} U_2$. Now, take $\varepsilon = 1$ and $U_1 = \operatorname{cl} U_2$. By c-almost periodicity, there exists a number $l > 0$ such that every open interval of length l contains an (ε, c)-translation number of $f(z)$ associated with U_1. If $z = x+iy$ is an arbitrary complex number in U_1, we can assure the existence of an (ε, c)-translation number $\tau \in \mathbb{R}$ of $f(z)$ so that $y+\tau \in [0, l]$. Also, let $M = \max\{|f(w)| : w = x+iy \text{ with } y \in [0, l] \text{ and } x \in [a, b]\}$. Consequently,

$$|cf(z)| \leq |f(z+i\tau) - cf(z)| + |f(x+i(y+\tau))| \leq 1 + M,$$

which yields that

$$|f(z)| \leq \frac{1+M}{|c|}.$$

This proves that $f(z)$ is bounded in U_1 (and hence also in U_2). □

Remark 3 In general, the boundedness of the function is not true in the whole strip U. For example, the function $f(z) = \frac{1}{\sinh z}$ is in $AP_1(U_r, \mathbb{C})$, where U_r is of the form $U_r = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < r\}$ for any $r > 0$, but it is not bounded in U_r (see [8, p. 86]).

As a consequence of the definition of c-almost periodicity and the proposition above, we obtain the following results.

Corollary 1 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Also, consider an arbitrary reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$, with $\alpha < \alpha_1 < \beta_1 < \beta$. Then the following properties are satisfied:*

(i) *The function $f(z)$ is uniformly continuous in U_1.*
(ii) *If $f(z)$ is analytic in U, then all its derivatives are uniformly continuous in U_1.*

Proof (i) Fixed $\varepsilon > 0$ and a reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$, take l the length associated with $\varepsilon_1 = \frac{|c|\varepsilon}{3}$ according to the definition of c-almost periodicity of $f(z)$ in U_1, which means that every open interval of length $l > 0$ contains an (ε_1, c)-translation number of $f(z)$ associated with U_1. As $f(z)$ is continuous on U, we know that $f(z)$ is uniformly continuous on the compact set $K = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1,\ -1 \le \operatorname{Im} z \le 1 + l\}$. Let $\delta < 1$ be the positive number associated with ε_1 according to the property of uniform continuity of $f(z)$ in K, i.e.

$$|f(z_1) - f(z_2)| \le \varepsilon_1 \quad \text{when} \quad z_1, z_2 \in K \quad \text{with} \quad |z_1 - z_2| < \delta.$$

In this way, consider $z_1, z_2 \in U_1$ with $|z_1 - z_2| < \delta$, and take a number $\tau \in (-\operatorname{Im} z_1, -\operatorname{Im} z_1 + l)$ which is an (ε_1, c)-translation number of $f(z)$ associated with U_1. Hence $\operatorname{Im} z_1 + \tau \in [0, l] \subset [-1, 1 + l]$ (i.e. $z_1 + i\tau \in K$) and $|\operatorname{Im}(z_1 - z_2)| \le |z_1 - z_2| < \delta < 1$, which yields that $\operatorname{Im} z_2 + \tau \in [-1, 1 + l]$ (i.e. $z_2 + i\tau \in K$). Then it is satisfied

$$\begin{aligned}|cf(z_1) - cf(z_2)| \le |cf(z_1) &- f(z_1 + i\tau)| \\ &+ |f(z_1 + i\tau) - f(z_2 + i\tau)| \\ &+ |f(z_2 + i\tau) - cf(z_2)| \le 3\varepsilon_1 = |c|\varepsilon,\end{aligned}$$

which means that f is uniformly continuous on U_1.

(ii) Since $f(z)$ is analytic in U and it is bounded in any substrip $\{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$ (by Proposition 2), this result is now a direct consequence of [8, Theorem 3.7]. □

Corollary 2 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Then for every $\varepsilon > 0$ and reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ there exist two positive numbers l and δ with the property that any interval of length l (of the real line) contains a subinterval of length δ whose points are (ε, c)-translation numbers of $f(z)$ associated with U_1.*

Proof Fix $\varepsilon > 0$ and take an arbitrary reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. By Corollary 1 there exists $\delta_1 > 0$ such that, under the condition $|h| < \delta_1$, it is accomplished that

$$|f(z + ih) - f(z)| < \frac{\varepsilon}{2|c|}, \quad \text{for every } z \in U_1. \tag{3}$$

Moreover, by c-almost periodicity, there exists a number $l_1 > 0$ such that every open interval of length l_1 contains an $(\frac{\varepsilon}{2}, c)$-translation number of $f(z)$ associated with U_1, i.e. a real number τ satisfying

$$|f(z + i\tau) - cf(z)| \le \frac{\varepsilon}{2}, \quad \text{for every } z \in U_1. \tag{4}$$

If $\tau \in [r, r+l_1]$ for some $r \in \mathbb{R}$, then $\tau + h \in [r - \delta_1, r + l_1 + \delta_1]$ for any real number h such that $|h| < \delta_1$. Thus we deduce from (3) and (4) that

$$\begin{aligned}|f(z+i(\tau+h)) - cf(z)| &\le |f(z+i(\tau+h)) - cf(z+ih)| \\ &\quad + |cf(z+ih) - cf(z)| \\ &< \frac{\varepsilon}{2} + |c|\frac{\varepsilon}{2|c|} = \varepsilon,\end{aligned}$$

for every $z \in U_1$. This means that every $\tau + h$ with $|h| < \delta_1$ is an (ε, c)-translation number of $f(z)$ associated with U_1. Thus the result holds by taking $l = l_1 + 2\delta_1$ and $\delta = 2\delta_1$. □

If $f \in AP_c(U, \mathbb{C})$ and $h \in \mathbb{R}$, it is clear from the definition of c-almost periodicity that the function $f(z + ih)$ is also in $AP_c(U, \mathbb{C})$. Among other properties, we next analyse the c-almost periodicity of the product by scalars, the complex conjugate, the real and imaginary part, and the multiplicative inverse (or reciprocal) of c-almost periodic functions.

Proposition 3 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Then the following properties are satisfied:*

(i) *$f(z)$ is also a c^m-almost periodic function in U for each $m \in \mathbb{Z}\setminus\{0\}$.*
(ii) *$\overline{f}(z)$ is a $\overline{c}^m$-almost periodic function in U for each $m \in \mathbb{Z}\setminus\{0\}$.*
(iii) *$f(\overline{z})$ is a $\frac{1}{c}$-almost periodic function in U (and hence a c^m-almost periodic function in U for each $m \in \mathbb{Z}\setminus\{0\}$).*
(iv) *$\lambda f(z)$ is a c^m-almost periodic function in U for each $m \in \mathbb{Z}\setminus\{0\}$ and $\lambda \in \mathbb{C}$.*
(v) *$f^2(z) := (f(z))^2$ is a c^{2k}-almost periodic function in U for each $k \in \mathbb{Z}\setminus\{0\}$.*
(vi) *If $|f(z)| \ge m > 0\ \forall z \in U$, then $\frac{1}{f}(z) := \frac{1}{f(z)}$ is a $\frac{1}{c}$-almost periodic function in U (and hence a c^m-almost periodic function in U for each $m \in \mathbb{Z}\setminus\{0\}$).*
(vii) *If $|c| = 1$, then $|f| : U \to [0, \infty)$ is an almost periodic function in U.*
(viii) *If $c \in \mathbb{R}$, then the functions* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are c-almost periodic in U.*

Proof Fix $c \in \mathbb{C}\setminus\{0\}$ and $f \in AP_c(U, \mathbb{C})$. We first recall that, for every arbitrary $\varepsilon > 0$ and reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$, the set $E_c\{f(z), \varepsilon\}$ (of all (ε, c)-translation numbers of $f(z)$ associated with U_1) is relatively dense. So, fix $\varepsilon > 0$ and such a reduced strip $U_1 \subset U$.

(i) Fixed $m \in \mathbb{N}$, consider the value $a = 1 + |c| + \ldots + |c|^{m-1}$ $= \begin{cases} m & \text{if } |c| = 1 \\ \left(\frac{1-|c|^m}{1-|c|}\right) & \text{if } |c| \ne 1 \end{cases}.$

Note that every $\tau \in E_c\{f(z), \varepsilon/a\}$ satisfies

$$\begin{aligned}\left|f(z+im\tau)-c^m f(z)\right| &\le |f(z+im\tau)-cf(z+i(m-1)\tau)| \\ &\quad + \left|cf(z+i(m-1)\tau)-c^2 f(z+i(m-2)\tau)\right| + \cdots + \\ &\quad + \left|c^{m-1}f(z+i\tau)-c^m f(z)\right| \\ &\le \frac{\varepsilon}{a}\left(1+|c|+\ldots+|c|^{m-1}\right), \quad \forall z \in U_1.\end{aligned}$$

This shows that $m\tau \in E_{c^m}\{f(z), \varepsilon\}$. Hence $f(z)$ is a c^m-almost periodic function in U. Finally, by Lemma 1, $f(z)$ is also $\frac{1}{c^m}$-almost periodic function in U. This proves the result.

(ii) Note that every $\tau \in E_c\{f(z), \varepsilon\}$ satisfies

$$\left|\overline{f}(z+i\tau)-\overline{c}\,\overline{f}(z)\right| = |f(z+i\tau)-cf(z)| \le \varepsilon, \quad \forall z \in U_1,$$

which yields that $\tau \in E_{\overline{c}}\{\overline{f}(z), \varepsilon\}$. This shows that $\overline{f} \in AP_{\overline{c}}(U, \mathbb{C})$. Finally, we deduce from i) that $\overline{f} \in AP_{\overline{c}^m}(U, \mathbb{C})$ for any $m \in \mathbb{Z}\backslash\{0\}$.

(iii) Let $g(z) := f(\overline{z})$, $z \in U$. Note that every $\tau \in E_c\{f(z), |c|\varepsilon\}$ satisfies

$$\begin{aligned}\left|g(z+i\tau)-\frac{1}{c}g(z)\right| &= \left|f(\overline{z+i\tau})-\frac{1}{c}f(\overline{z})\right| = \left|f(\overline{z}-i\tau)-\frac{1}{c}f(\overline{z})\right| \\ &= \frac{1}{|c|}\,|f(\overline{z})-cf(\overline{z}-i\tau)| \le \frac{|c|\varepsilon}{|c|} = \varepsilon, \quad \forall z \in U_1,\end{aligned}$$

which yields that $\tau \in E_{\frac{1}{c}}\{g(z), \varepsilon\}$. This shows that $g(z) \in AP_{1/c}(U, \mathbb{C})$ and, by (i), $g(z) \in AP_{c^m}(U, \mathbb{C})$ for any $m \in \mathbb{Z}\backslash\{0\}$.

(iv) Fixed $\lambda \in \mathbb{C}\backslash\{0\}$ (the case $\lambda = 0$ is trivial), note that every $\tau \in E_c\{f(z), \frac{\varepsilon}{|\lambda|}\}$ satisfies

$$|\lambda f(z+i\tau)-c\lambda f(z)| = |\lambda|\,|f(z+i\tau)-cf(z)| \le |\lambda|\,\frac{\varepsilon}{|\lambda|} = \varepsilon, \quad \forall z \in U_1,$$

which, jointly with (i), proves the result.

(v) As f is bounded in any reduced strip $U_1 \subset U$ (by Proposition 2), there exists $M > 0$ such that $|f(z)| \le M$ $\forall z \in U_1$. Now, note that every $\tau \in E_c\{f(z), \frac{\varepsilon}{M(1+|c|)}\}$ satisfies

$$\begin{aligned}\left|f^2(z+i\tau)-c^2f^2(z)\right| &= |(f(z+i\tau)-cf(z))(f(z+i\tau)+cf(z))| \\ &\le |f(z+i\tau)-cf(z)|\,M(1+|c|) \\ &\le \frac{\varepsilon}{M(1+|c|)}M(1+|c|) = \varepsilon, \quad \forall z \in U_1,\end{aligned}$$

which proves that $\tau \in E_{c^2}\{f^2(z), \varepsilon\}$. This shows that $f^2(z) \in AP_{c^2}(U, \mathbb{C})$. Finally, we deduce from i) that $f^2(z) \in AP_{c^{2k}}(U, \mathbb{C})$ for any $k \in \mathbb{Z}\backslash\{0\}$.

(vi) If $|f(z)| \geq m_1 > 0\ \forall z \in U$, note that every $\tau \in E_c\{f(z), \varepsilon|c|m_1^2\}$ satisfies

$$\left|\frac{1}{f(z+i\tau)} - \frac{1}{c}\frac{1}{f(z)}\right| = \left|\frac{f(z) - \frac{1}{c}(f(z+i\tau))}{f(z)f(z+i\tau)}\right| = \left|\frac{f(z+i\tau) - cf(z)}{cf(z)f(z+i\tau)}\right|$$
$$\leq \frac{\varepsilon|c|m_1^2}{|c|m_1^2} = \varepsilon,$$

for all $z \in U_1$, which proves that the multiplicative inverse, or reciprocal, of $f(z)$ is in $AP_{1/c}(U, \mathbb{C})$. Therefore, we deduce from i) that it is also in $AP_{c^m}(U, \mathbb{C})$ for each $m \in \mathbb{Z}\backslash\{0\}$.

(vii) Note that every $\tau \in E_c\{f(z), \varepsilon\}$ satisfies

$$\begin{aligned} ||f(z+i\tau)| - |f(z)|| &= ||f(z+i\tau)| - |cf(z)|| \\ &\leq |f(z+i\tau) - cf(z)| \leq \varepsilon, \quad \forall z \in U_1, \end{aligned}$$

which proves that $\tau \in E_1\{|f(z)|, \varepsilon\}$. This shows the result.

(viii) Note that every $\tau \in E_c\{f(z), \varepsilon\}$ satisfies

$$\begin{aligned} |\mathrm{Re}(f(z+i\tau)) - c\,\mathrm{Re}(f(z))| &= |\mathrm{Re}\,(f(z+i\tau) - cf(z))| \\ &\leq |f(z+i\tau) - cf(z)| \leq \varepsilon, \end{aligned}$$

for all $z \in U_1$, which proves that $\tau \in E_c\{\mathrm{Re}(f(z)), \varepsilon\}$. The case $\mathrm{Im}\, f$ is analogous. □

Remark 4 Let $c \in \mathbb{C}$ be a complex number so that $|c| = 1$. Consider the vertical strip $U = \{z \in \mathbb{C} : \alpha < \mathrm{Re}\, z < \beta\}$, with $-\infty < \alpha < \beta < \infty$. By Example 1, the function $f(z) = e^z$, $z \in U$, is in $AP_c(U, \mathbb{C})$. However, if $c \neq \pm 1$, the function $\mathrm{Re}\, f(x+iy) = e^x \cos(y)$ is not c-almost periodic in U. Indeed, if $\tau \in \mathbb{R}$, note that

$$\begin{aligned} |\,\mathrm{Re}\, f(z+i\tau) - c\,\mathrm{Re}\, f(z)| &= |e^x \cos(y+\tau) - ce^x \cos(y)| \\ &= e^x |\cos(y+\tau) - c\cos(y)| \end{aligned}$$

for all $z = x + iy \in U$. Consequently, $\mathrm{Re}\, f(x+iy)$ is c-almost periodic in U if and only if the function $g(y) = \cos(y)$ is in $AP_c(\mathbb{R}, \mathbb{C})$. But we know that $g(y)$ is in $AP_c(\mathbb{R}, \mathbb{C})$ if and only if $c = \pm 1$ (see [12, Example 2.15]). In fact, given $\varepsilon > 0$ and $c = e^{i\theta_c} \neq \pm 1$ (with $\theta_c \in \mathbb{R}$), the definition of c-almost periodicity leads to the existence of real values τ satisfying $|\cos(x+\tau) - c\cos x| < \varepsilon$ for every $x \in \mathbb{R}$. In particular, we have

$$|\cos\tau - e^{i\theta_c}| < \varepsilon.$$

However, since $|\cos\tau - e^{i\theta_c}| = |\cos\tau - \cos\theta_c - i\sin\theta_c| \geq |\sin\theta_c| \neq 0$, the choice $0 < \varepsilon < |\sin\theta_c|$ represents a contradiction.

We next prove that the set $AP_c(U, \mathbb{C})$ is closed with respect to the topology of uniform convergence on the reduced strips of the vertical strip U.

Proposition 4 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f_n : U \to \mathbb{C}$ be a sequence of c-almost periodic functions in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. If $\{f_n(z)\}_{n\ge 1}$ converges uniformly on the reduced strips of U to a function $f : U \to \mathbb{C}$, then $f(z)$ is a c-almost periodic function in U.*

Proof By hypothesis, given $\varepsilon > 0$ and a reduced strip $U_1 \subset U$, there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \min\left\{\frac{\varepsilon}{3}, \frac{\varepsilon}{3|c|}\right\} \quad \text{for each } n \ge n_0 \text{ and for all } z \in U_1.$$

Therefore, every τ in the set of $(\frac{\varepsilon}{3}, c)$-translation numbers of $f_{n_0}(z)$ associated with U_1 satisfies

$$\begin{aligned}|f(z+i\tau) - cf(z)| &\le |f(z+i\tau) - f_{n_0}(z+i\tau)| + |f_{n_0}(z+i\tau) - cf_{n_0}(z)| \\ &\quad + |cf_{n_0}(z) - cf(z)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |c|\frac{\varepsilon}{3|c|} = \varepsilon, \quad \text{for all } z \in U_1,\end{aligned}$$

which shows that τ in the set of (ε, c)-translation numbers of $f(z)$ associated with U_1. Thus the result holds. □

Given an arbitrary $c \in \mathbb{C}\setminus\{0\}$, we already know that the set $AP_c(U, \mathbb{C})$ is closed with respect to product by scalars. However, generally speaking, we next show that it is not closed with respect to addition or multiplication and, therefore, it is not a vector space with the usual operations (except for the vector space $AP(U, \mathbb{C})$ which corresponds with the case $c = 1$).

Example 2 Take $|c| = 1$ (with $c \ne 1$) and $\lambda \in \mathbb{R}\setminus\{0\}$. Then the functions $f_1(z) = e^{\lambda z}$ and $f_2(z) = e^{-\lambda z}$ are in $AP_c(U, \mathbb{C})$ (see Example 1). However, it is clear that the function $f_1(z) \cdot f_2(z) \equiv 1$ is not in $AP_c(U, \mathbb{C})$.

Now, take $c = -1$ and $U = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$, with $a < 0 < b$, and consider the function $g(z) = g_1(z) + g_2(z)$, $z \in U$, where $g_1(z) = \frac{1}{2}e^{4z}$ and $g_2(z) = 2e^{2z}$. Note that, by Example 1, $g_1(z)$ and $g_2(z)$ are in $AP_{-1}(U, \mathbb{C})$. Suppose that $g(z) \in AP_{-1}(U, \mathbb{C})$. Thus for every $\varepsilon \in (0, 1)$ and $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$, with $\alpha_1 < 0 < \beta_1$, we can find $\tau \in \mathbb{R}$ such that

$$|g(z+i\tau) - cg(z)| = \left|\frac{1}{2}e^{4(z+i\tau)} + 2e^{2(z+i\tau)} + \frac{1}{2}e^{4z} + 2e^{2z}\right| \le \varepsilon, \quad \text{for all } z \in U_1. \tag{5}$$

In particular, for $z = it \in U_1$ we get

$$\begin{aligned}|g(z+i\tau)+g(z)| &= \left|\frac{1}{2}e^{4i(t+\tau)}+2e^{2i(t+\tau)}+\frac{1}{2}e^{4it}+2e^{2it}\right| \\ &\geq \left|\mathrm{Re}\left(\frac{1}{2}e^{4i(t+\tau)}+2e^{2i(t+\tau)}+\frac{1}{2}e^{4it}+2e^{2it}\right)\right| \\ &= \left|\frac{1}{2}\cos(4(t+\tau))+2\cos(2(t+\tau))+\frac{1}{2}\cos(4t)+2\cos(2t)\right| \\ &= \left|4\cos^4(t+\tau)-\frac{3}{2}+4\cos^4 t-\frac{3}{2}\right|, \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

Indeed, for any $x \in \mathbb{R}$, we have that

$$\begin{aligned}\frac{1}{2}\cos(4x)+2\cos(2x) &= \cos^2(2x)+2\cos(2x)-\frac{1}{2} \\ &= (2\cos^2 x-1)^2+2\cos(2x)-\frac{1}{2} \\ &= 4\cos^4 x+\frac{1}{2}-4\cos^2 x+2(2\cos^2 x-1) = 4\cos^4 x-\frac{3}{2}.\end{aligned}$$

Therefore, for $z = \pi i$ it is accomplished that

$$|g(z+i\tau)+g(z)| \geq 4\cos^4\tau+1 \geq 1,$$

which is a contradiction with (5). Consequently, $g(z)$ is not in $AP_{-1}(U, \mathbb{C})$. This example is inspired by [12, Example 2.15].

3 Close connections with Bohr's almost periodicity

We next show some important inclusions which extend some results incorporated in [12, Corollary 2.10 and Proposition 2.11].

Proposition 5 *Let c be a non-zero complex number such that $\frac{\arg c}{2\pi} \in \mathbb{Q}$. Take $U = \{z \in \mathbb{C} : \alpha < \mathrm{Re}\, z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then*

$$AP_c(U, \mathbb{C}) \subset AP_{|c|^q}(U, \mathbb{C}),$$

where $q \in \mathbb{N}$ is so that $\frac{\arg c}{2\pi} = \frac{p}{q}$ for a certain $p \in \mathbb{Z}$ such that $(p, q) = 1$. In particular, under the same condition, the case $|c| = 1$ yields the inclusion $AP_c(U, \mathbb{C}) \subset AP(U, \mathbb{C})$.

Proof Put $\arg c = \frac{2\pi p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ so that $(p, q) = 1$ (i.e. the greatest common divisor of p and q is equal to 1). Then $c^q = |c|^q e^{qi \arg c} = |c|^q e^{2p\pi i} = |c|^q$. Let $f \in AP_c(U, \mathbb{C})$. By Proposition 3(i), it is accomplished that $f \in AP_{c^q}(U, \mathbb{C}) = AP_{|c|^q}(U, \mathbb{C})$, which proves the result. ☐

Proposition 6 *Let c be a non-zero complex number such that $\frac{\arg c}{\pi} \notin \mathbb{Q}$ and $|c| = 1$. Take $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Then $AP_c(U, \mathbb{C}) \subset AP(U, \mathbb{C})$.*

Proof Let $f \in AP_c(U, \mathbb{C})$, and fix $\varepsilon > 0$ and a reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$. We recall that, by Proposition 2, there exists $M > 0$ such that $|f(z)| < M$ for all $z \in U_1$. Suppose that $\arg c$ is not a rational multiple of π, which yields that $e^{ni \arg c} \neq 1$ for all $n \in \mathbb{N}$. Now, choose $n_1, n_2 \in \mathbb{N}$ such that $|e^{n_2 i \arg c} - e^{n_1 i \arg c}| < \frac{\varepsilon}{2M}$ (note that the existence of n_1 and n_2 is assured in virtue of $\{e^{ni \arg c} : n \in \mathbb{N}\} \subset \{z \in \mathbb{C} : |z| = 1\}$ and the length of the unit circumference is finite). Hence

$$|e^{(n_2 - n_1)i \arg c} - 1| = |e^{n_2 i \arg c} - e^{n_1 i \arg c}| < \frac{\varepsilon}{2M}.$$

Take $m_\varepsilon = n_2 - n_1$. Then, by Proposition 3(i), it is accomplished that $f \in AP_{c^{m_\varepsilon}}(U, \mathbb{C})$. Consequently, every $\tau \in E_{c^{m_\varepsilon}}\{f(z), U_1, \varepsilon/2\}$ satisfies

$$|f(z + i\tau) - f(z)| \le \left| f(z + i\tau) - c^{m_\varepsilon} f(z) \right| + \left| c^{m_\varepsilon} f(z) - f(z) \right| \le \varepsilon, \quad \forall z \in U_1,$$

which proves the result. □

We already know some conditions under which it can be assured that the sets of c-almost periodic functions are also almost periodic in some vertical strips of the complex plane. However, we will improve these results in the subsequent development (see Corollary 3 at the end of this section).

For this purpose, we next show a result concerning the set of values of a c-almost periodic function on a vertical line, which extends [3, p. 144, 9°. Theorem]. In this respect, given a function $f(z)$ defined in a vertical strip U and a real number $\sigma_0 \in U$, consider the notation $\operatorname{Img}(f(\sigma_0 + it)) = \{w \in \mathbb{C} : \exists t \in \mathbb{R}$ such that $w = f(\sigma_0 + it)\}$, i.e. $\operatorname{Img}(f(\sigma_0 + it))$ is the set of values assumed by $f(z)$ on the straight line $\operatorname{Re} z = \sigma_0$.

Theorem 1 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be an analytic c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Consider $\sigma_0 \in (\alpha, \beta)$ and take $\operatorname{Img}(f(\sigma_0 + it)) = \{w \in \mathbb{C} : \exists t \in \mathbb{R}$ such that $w = f(\sigma_0 + it)\}$. Then the function $f(z)$ assumes all the values in the set of the accumulation points of $\operatorname{Img}(f(\sigma_0 + it))$ in any vertical strip of the form $\{z \in \mathbb{C} : \sigma_0 - \delta < \operatorname{Re} z < \sigma_0 + \delta\}$, with $\delta > 0$.*

Proof Let w_0 be an accumulation point of the set $\operatorname{Img}(f(\sigma_0 + it))$, and consider the function $g(z) := f(z) - w_0$, $z \in U$. If $g(z)$ vanishes at $z = \sigma_0 + it$ for some $t \in \mathbb{R}$, then the result is straightforward. Suppose that $g(z)$ does not vanish at the vertical line $x = \sigma_0$. Thus there exists a sequence of points of the form $\{\sigma_0 + it_n\}_{n \ge 1}$ such that $\{g(\sigma_0 + it_n)\}_{n \ge 1}$ tends to 0 as n goes to ∞ (hence $\{cg(\sigma_0 + it_n)\}_{n \ge 1}$ also tends to 0 as n goes to ∞). Now, take $y_0 \in \mathbb{R}$ such that $\varepsilon = |f(\sigma_0 + iy_0) - z_0| > 0$, where $z_0 = cw_0$ (if y_0 did not exist, the set $\operatorname{Img}(f(\sigma_0 + it)) = \{z_0\}$ would not have accumulation points). By c-almost periodicity, the set $E_c\{f(z), \varepsilon/2\}$ of the $(\varepsilon/2, c)$-translation numbers of $f(z)$ associated with the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = \sigma_0\}$ is relatively dense, which yields that there exists $l_\varepsilon > 0$ such that any interval of length l_ε in the vertical line $x = \sigma_0$ contains a point τ at which

$$|f(\sigma_0 + it + i\tau) - cf(\sigma_0 + it)| \leq \frac{\varepsilon}{2}, \quad \text{for all } t \in \mathbb{R}.$$

Thus

$$\begin{aligned} |cg(\sigma_0 + iy_0 - i\tau)| &= |cf(\sigma_0 + iy_0 - i\tau) - z_0| \\ &= |cf(\sigma_0 + iy_0 - i\tau) - f(\sigma_0 + iy_0) + f(\sigma_0 + iy_0) - z_0| \\ &\geq \varepsilon - |f(\sigma_0 + iy_0) - cf(\sigma_0 + iy_0 - i\tau)| \\ &\geq \frac{\varepsilon}{2} > 0. \end{aligned}$$

Furthermore, given $\delta > 0$, take $U_\delta = \{z \in \mathbb{C} : \sigma_0 - \delta < \operatorname{Re} z < \sigma_0 + \delta\}$. We know that $f(z)$ is bounded in U_δ (by Proposition 2). Consequently, by [3, p. 139, theorem], the function $g(z) = f(z) - w_0$ has zeros in U_δ, which proves the result. □

We next show that the set of all the values of a c-almost periodic function (in $AP_c(U, \mathbb{C})$) on any vertical substrip of U is relatively compact in $\mathbb{C}$, which represents an extension of [8, Theorem 6.5].

Proposition 7 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then for any reduced vertical substrip $V \subset U$, the image set $\{w \in \mathbb{C} : \exists z \in V \text{ such that } w = f(z)\}$ is relatively compact in $\mathbb{C}$.*

Proof Let us demonstrate that for any $\varepsilon > 0$ the set of values of f in any vertical reduced substrip of U can be covered by finitely many balls of radius ε (recall that, in the Banach spaces, the relatively compact sets coincide with the precompact sets). By c-almost periodicity, for every reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$, there exists a number $l > 0$ such that every open interval of length l contains an (ε, c)-translation number of $f(z)$ associated with U_1. So, fix a reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\}$ with $\alpha < \alpha_1 < \beta_1 < \beta$. As $f(z)$ is continuous on the vertical strip U, recall first that the set of values of $f(z)$ on any compact subset included in U is compact. In this way, consider a finite amount of balls of radius $\frac{\varepsilon}{2|c|}$ which cover the set $\{f(z) : z = x + iy \text{ with } y \in [0, l] \text{ and } x \in [\alpha_1, \beta_1]\}$, and let $\{z_1, z_2, \ldots, z_p\}$ denote the centers of these balls. Now, if $z = x + iy$ is an arbitrary complex number in U_1, we can assure the existence of an $(\varepsilon/2, c)$-translation number $\tau \in \mathbb{R}$ (in the interval $[y - l, y]$) of $f(z)$ so that $y - \tau \in [0, l]$. Also, let z_j be the center of the ball of radius $\frac{\varepsilon}{2|c|}$ which contains $f(x + i(y - \tau))$. Therefore

$$|f(z) - cz_j| \leq |f(z) - cf(z - i\tau)| + |cf(z - i\tau) - cz_j| \leq \frac{\varepsilon}{2} + |c|\frac{\varepsilon}{2|c|} = \varepsilon.$$

This proves that the balls of centers $\{cz_1, cz_2, \ldots, cz_p\}$ and radius ε cover the set $\{f(z) : z \in U_1\}$. □

As a consequence of the result above, given a vertical strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\}$ (or $U_1 = \{z \in \mathbb{C} : \alpha_1 < \operatorname{Re} z < \beta_1\}$), with $\alpha < \alpha_1 < \beta_1 < \beta$, from any sequence $\{f(z_j) : z_j \in U_1\}_{j\geq 1}$ we can extract a subsequence which is convergent. In particular, for every $z \in U_1$, the sequence $\{f(z + ih_j)\}_{j\geq 1}$ admits a subsequence $\{f(z + ih_{j_k})\}_{k\geq 1}$ which is convergent.

If $c \in \mathbb{C}\setminus\{0\}$ and $h \in \mathbb{R}$, recall that the vertical translate $f_h(z) := f(z + ih)$ is in $AP_c(U, \mathbb{C})$ for any function $f \in AP_c(U, \mathbb{C})$. We next prove that the family of functions $\{f_h(z) : h \in \mathbb{R}\}$ is relatively compact on the vertical substrips of U (in the set of bounded continuous complex functions defined on U). This property, whose proof is similar to that for the case of almost periodic functions, represents an extension of the necessary condition of [8, Theorem 6.6].

Theorem 2 *Given $c \in \mathbb{C}\setminus\{0\}$, let $f : U \to \mathbb{C}$ be a c-almost periodic function in a certain vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then the family of vertical translates $\{f_h(z) : h \in \mathbb{R}\}$ (with $f_h(z) := f(z + ih)$, $z \in U$) is relatively compact on any reduced vertical strip of U.*

Proof Let $\{f_{h_k}(z)\}_{k\geq 1}$ be a sequence of vertical translates of f, and consider $S = \{s_n\}_{n\geq 1}$ a dense set in U (for example $S = (\mathbb{Q} + i\mathbb{Q}) \cap U = (\mathbb{Q} \cap (\alpha, \beta)) + i\mathbb{Q}$). In this way, from the sequence of complex values $\{f(s_1 + ih_k)\}_{k\geq 1}$ we can extract a subsequence $\{f(s_1 + ih_{k,j_1})\}_{j_1\geq 1}$ which is convergent in $\mathbb{C}$ (by Proposition 7). Analogously, from the sequence $\{f(s_2 + ih_{k,j_1})\}_{j_1\geq 1}$ we can extract a subsequence $\{f(s_2 + ih_{k,j_2})\}_{j_2\geq 1}$ which is convergent in $\mathbb{C}$, and so on. In general, by using a diagonal procedure of extraction, the sequence $\{f(s_k + ih_{k,j_k})\}_{k\geq 1}$ is convergent, and thus $\{f(z + ih_{k,j_k})\}_{k\geq 1}$ is convergent for every $z \in S$. In fact, let us demonstrate that the sequence $\{f(z + ih_{k,j_k})\}_{k\geq 1}$ is uniformly convergent in every reduced substrip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$, with $\alpha < \alpha_1 < \beta_1 < \beta$. Indeed, given $\varepsilon > 0$, by c-almost periodicity there exists a number $l > 0$ such that every open interval of length l contains a $(\frac{|c|\varepsilon}{5}, c)$-translation number of $f(z)$ associated with U_1, i.e. a real number τ satisfying the condition

$$|f(z + i\tau) - cf(z)| \leq \frac{|c|\varepsilon}{5}, \quad \text{for all } z \in U_1. \tag{6}$$

Also, by Corollary 1, we know that $f(z)$ is uniformly continuous in any reduced vertical substrip of U, which yields the existence of $\delta > 0$ such that

$$|f(z_j) - f(z_k)| < \frac{|c|\varepsilon}{5} \quad \text{for any } z_j, z_k \in U_1 \text{ with } |z_j - z_k| < \delta. \tag{7}$$

Now, take a subdivision of the square $C_l = [\alpha_1, \beta_1] \times [0, l]$ in such a way that $C_l = \bigcup_{j,k=1}^{n} C_{l,j,k}$, where every $C_{l,j,k} = [\gamma_{j-1}, \gamma_j] \times [\xi_{k-1}, \xi_k]$, $j, k = 1, \ldots, n$, is so that $|(\gamma_j + i\xi_k) - (\gamma_{j-1} + i\xi_{k-1})| < \delta$ (the length of the diagonal of every subsquare is less than δ). Choose a point of S in each subsquare $C_{l,j,k}$ of C_l, and denote the set of all them as $S_0 = \{p_1, p_2, \ldots, p_t\}$. Since S_0 is a finite set, the sequence $\{f(z + ih_{k,j_k})\}_{j\geq 1}$ is uniformly convergent with respect to $z \in S_0$. Hence, we can determine a positive integer number N such that for $n, m \geq N$ we have

$$|f(p_k + ih_{n,j_n}) - f(p_k + ih_{m,j_m})| < \frac{|c|\varepsilon}{5}, \quad k = 1, 2, \ldots, t. \tag{8}$$

Finally, take an arbitrary $z = x + iy \in U_1$, and let τ be an (ε, c)-translation number of $f(z)$ (associated with U_1) with $\tau \in [-y, -y + l]$, which yields that $y + \tau \in [0, l]$ and $z + i\tau$ belongs to a certain subsquare of C_l. Denote by p_j the complex number in S_0 (corresponding with this subsquare) such that $|z + i\tau - p_j| < \delta$. Consequently, if $n, m \geq N$ (recall that N does not depend on the point $z \in U_1$) then

$$\begin{aligned}
|c||f(z + ih_{n,j_n}) - f(z + ih_{m,j_m})| &= |cf(z + ih_{n,j_n}) - cf(z + ih_{m,j_m})| \\
&\leq |cf(z + ih_{n,j_n}) - f(z + ih_{n,j_n} + i\tau)| \\
&\quad + |f(z + ih_{n,j_n} + i\tau) - f(p_j + ih_{n,j_n})| \\
&\quad + |f(p_j + ih_{n,j_n}) - f(p_j + ih_{m,j_m})| \\
&\quad + |f(p_j + ih_{m,j_m}) - f(z + ih_{m,j_m} + i\tau)| \\
&\quad + |f(z + ih_{m,j_m} + i\tau) - cf(z + ih_{m,j_m})| \\
&< \frac{|c|\varepsilon}{5} + \frac{|c|\varepsilon}{5} + \frac{|c|\varepsilon}{5} + \frac{|c|\varepsilon}{5} + \frac{|c|\varepsilon}{5} \\
&= |c|\varepsilon,
\end{aligned}$$

where (6) was used in the first and the last terms of the strict inequality, (7) in the second and fourth terms, and (8) in the third term. Hence the last inequality shows that the sequence $\{f(z+ih_{n,j_n})\}_{n\geq 1}$ satisfies Cauchy's condition for uniform convergence on U_1 (in fact, also on $V = \{z \in \mathbb{C} : \alpha_1 < \operatorname{Re} z < \beta_1\}$ and any reduced strip of V). We deduce from Proposition 4 that the limit function is in $AP_c(V, \mathbb{C})$. Consequently, the initial sequence of vertical translates $\{f_{h_k}(z)\}_{k\geq 1}$ contains a subsequence that converges uniformly on every reduced strip (in $AP_c(V, \mathbb{C})$), and the result holds. □

It is known that Bohr's notion of almost periodicity of a function defined on $\mathbb{R}$ is equivalent to the relative compactness of the set of its translates with respect to the topology of the uniform convergence (see e.g. [3, 16, 17]). In the same terms, the property of relative compactness of the vertical translates, with respect to the topology of the uniform convergence on reduced strips, identifies the class of almost periodic functions defined on vertical strips of the complex plane. This means that the converse of Theorem 2 is only true for the case $c = 1$. Although this result is known, we provide their proof here for the sake of completeness.

Lemma 2 *Given a vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$, then $f \in AP(U, \mathbb{C})$ if and only if the family of vertical translates $\{f_h(z) : h \in \mathbb{R}\}$ (with $f_h(z) := f(z + ih)$, $z \in U$) is relatively compact on any reduced vertical strip of U.*

Proof The necessity is proved in Theorem 2 (which is also true for the case $c = 1$).

To prove the sufficiency, suppose by *reductio ad absurdum* that f is not an almost periodic function in the strip U. Thus there exists at least one real value $\varepsilon > 0$ and a reduced substrip $U_1 \subset U$ such that for any $l > 0$ we can determine an interval of length l in the real line which does not contain any ε-translation number of $f(z)$ (associated with U_1). Now, consider an arbitrary real number h_1 and an interval $(a_1, b_1) \subset \mathbb{R}$ of length greater than $2|h_1|$ which does not contain any ε-translation number of $f(z)$ (associated with U_1). If we take $h_2 = \frac{1}{2}(a_1 + b_1)$, then $h_2 - h_1 \in (a_1, b_1)$ and, consequently, $h_2 - h_1$ cannot be an ε-translation number of $f(z)$ (associated with U_1). In the same way, there exists an interval $(a_2, b_2) \subset \mathbb{R}$ of length greater than $2(|h_1| + |h_2|)$ which does not contain any ε-translation number of $f(z)$ (associated with U_1). If we take $h_3 = \frac{1}{2}(a_2 + b_2)$, this fact yields that $h_3 - h_1,\ h_3 - h_2 \in (a_2, b_2)$ and hence $h_3 - h_1$ and $h_3 - h_2$ are not ε-translation numbers of $f(z)$ (associated with U_1). If we reiterate this process in a similar manner, we construct a sequence of real numbers $h_1, h_2, h_3, \ldots$ such that none of the differences $h_i - h_j$, $i, j \geq 1$, is an ε-translation number of $f(z)$ (associated with U_1). Therefore, for any i and j we have

$$\begin{aligned} &\sup\{\left|f(z + ih_i) - f(z + ih_j)\right| : z \in U_1\} \\ &\quad = \sup\{\left|f(w + i(h_i - h_j)) - f(w)\right| : w \in U_1\} > \varepsilon. \end{aligned}$$

In fact, we know that there exists $w_0 \in U_1$ satisfying the inequality $\left|f(w_0 + i(h_i - h_j)) - f(w_0)\right| > \varepsilon$, which yields that the point $z_0 = w_0 - ih_j \in U_1$ satisfies $\left|f(z_0 + ih_i) - f(z_0 + ih_j)\right| > \varepsilon$. Consequently, the sequence of vertical translates $\{f(z + ih_n)\}_{n \geq 1}$ does not contain any subsequence uniformly convergent on U_1. This contradicts our hypothesis. Hence the result holds. □

Now, we formulate the following corollary of Theorem 2 and Lemma 2 which states that every c-almost periodic function in a vertical strip U is also almost periodic in U.

Corollary 3 *Let c be a non-null complex number and $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then $AP_c(U, \mathbb{C}) \subset AP(U, \mathbb{C})$.*

Remark 5 As a consequence of Corollary 3, we have proved that Proposition 3, point i), is also true for the case $m = 0$, i.e. if $f \in AP_c(U, \mathbb{C})$ then $f \in AP_{c^m}(U, \mathbb{C})$ for each $m \in \mathbb{Z}$.

It is worth noting that, *mutatis mutandi*, Theorem 2 can be analogously proved for the case of c-almost periodic functions $f(x)$ defined on $\mathbb{R}$ (with respect to the topology of the uniform convergence on $\mathbb{R}$). In this respect, the paper [12, p. 181] includes the statement that $AP_c(\mathbb{R}, \mathbb{C}) \subset AP(\mathbb{R}, \mathbb{C})$ is true for the case $|c| = 1$ (see [12, Proposition 2.11 and comments above]). In our case, we have extended this result for every value $c \in \mathbb{C}\setminus\{0\}$.

Corollary 4 *Let c be a non-null complex number. Then $AP_c(\mathbb{R}, \mathbb{C}) \subset AP(\mathbb{R}, \mathbb{C})$.*

Remark 6 During the refereeing process, a reviewer informed us of the notion introduced in the ArXiv paper [15, Definition 2.6] in connection with our Definition 1, where it is necessary to use the identification of vertical strips in the complex plane with the corresponding subsets of the Euclidean space $\mathbb{R}^2$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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