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Mixing solutions for claims problems[☆]

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ABSTRACT

The literature on solutions for claims problems mainly orbits on three canonical rules: The Proportional, the Constrained Equal Awards and the Constrained Equal Losses. Mixtures of these solutions have been proposed to design alternative approaches to solve claims problems. We consider piece-wise and convex mixtures as two relevant tools. Piece-wise mixture guarantees that each agent obtains a minimal reimbursement, when it is available, while the remaining is distributed according to an alternative distribution criterion. Convex mixture shares the relevance of each distributive criterion according to an exogenously given weight. In this framework we explore which properties are preserved by mixed solutions. Moreover, we propose to design mixed solutions according to the compromising degree, an endogenous parameter capturing the relative relevance of the rationing that agents have to share collectively. We characterize the Proportional solution as the piece-wise mixture of any two solutions. The convex mixture of the Constrained Equal Awards and the Constrained Equal Losses solutions is explored from a normative point of view.

1. Introduction

The analysis of claims problems, and the proposal of 'equitable' solutions for them, is one of the most ancient (not yet incontrovertibly solved) problem since the homo œconomicus straighten. It involves a given amount of a (perfectly divisible) good that has to be distributed among a given group of claimants. The problem emerges whenever the available amount of good is not enough to cover the aggregate claim, and thus the agents have to be rationed.

Despite that there are several solutions fulfilling some equity features,[1] each of them can be objected by some agents because they feel damaged related to how (some of) his rivals have been treated. This allows to consider two kinds of solutions. On the one hand, what we call canonical solutions, in which it is easily identifiable which are the agents that are expected to object against them. On the other hand, compromise solutions, obtained as a mixture of the canonical ones, which try to reduce the conflict that agents exhibit. The most prominent examples for the first category are the Proportional, the Constrained Equal Awards and the Constrained Equal Losses solutions. For the second category, the most prominent example is the Talmud solution (Aumann and Maschler, 1985).

The aim of this paper is twofold. On the one hand we explore the properties that a compromise solution inherits from the canonical ones used to build it. On the other hand, we introduce an (endogenous) measure of the likelihood to reach a compromise on the solution to be adopted. This allows us to propose the use of the compromise convex solution that is highly sensitive to this measure.

As we illustrate in this Introduction, most of the classic compromise solutions provided by the literature can be described as a composition – or mixture – of two canonical solutions. Then, the original problem is split into two sub-problems, each one being solved according to one of these canonical solutions. Therefore, the solution for the original problem is obtained by adding the (canonical) solutions for these two sub-problems. To illustrate the 'mixing' procedure above, let us derive the Talmud solution as an accurate mixture of two well-known solutions.

A classical canonical solution for claims problems is the Constrained Equal Awards solution (CEA henceforth) suggested by Maimonides in the 12th Century (Aumann and Maschler, 1985). It prescribes equal division of the available amount of good among the agents, with the restriction that none of them is allowed an amount exceeding his claim. Alternatively, any claims problem can be analyzed from a dual perspective. This approach considers that any solution from a claims problem implicitly yields a prescription of how claimants should be rationed or, similarly, how the shortage associated to this problem should be distributed.

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1 The reader is addressed to the recent book by Thomson (2019), that supplies a nice, complete up-to-date survey of the literature.

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This allows to describe the Constrained Equal Losses solution, CEL henceforth, which is build so that all the claimants are (in absolute terms) equally rationed under the assumption that none of them is allowed a negative amount of good.[2] The Talmud solution prescribes to solve any claims problem by mixing the CEA and the CEL solutions as follows. Split the initial claims problem in two sub-problems in which agents' claims coincide – and thus each agent's claim at any problem is equal to one half of his initial claim –. Then split the available resource in an asymmetric way. The first part is one half of the aggregate claim – if such an amount is available – while the second one is precisely the remaining resource, if any. To compute the Talmud solution we proceed as follows: (1) solve the first sub-problem according to the CEA solution; (2) solve the second sub-problem according to the CEL solution; and (3) add the two sub-solutions above.

The idea of solving claims problems by mixing two well-behaved solutions has also been explored by some authors yielding different prescriptions. For instance, Piniles (1861) suggests to solve claims problems according to the following procedure. Given a problem, build two sub-problems in a similar way as the Talmud solution does. Then, apply the CEA solution to the two sub-problems and add these sub-solutions. Observe that, since no solution for claims problems is additive (Bergantiños and Méndez-Naya, 2001), this differs from the CEA solution. The Reverse Talmud solution (Chun et al., 2001) can be introduced as the Talmud solution by exchanging the roles of the CEA and the CEL solutions. Moreno-Ternero and Villar (2006) propose the study of a family of solutions whose central element is the Talmud solution. They consider that a given share θ of the aggregate claim is distributed according to the CEA solution whenever it is available, while the remaining resource (if any) is distributed according to the CEL solution. As in the Talmud solution, when determining how agents' claims are associated to each sub-problem, a share θ of each agent claim is linked to the first sub-problem, while the remaining share goes to the second sub-problem.

The common features of the above solutions yield to describe what we call 'piece-wise' solutions as follows. Select two canonical solutions, to be called the primary and the secondary solutions, respectively. For a given claims problem, divide it in two sub-problems. Each agent's claim at the first problem is a proportion, say θ, of his total claim. His claim for the second problem is the remaining $(1-\theta)$ share of his total claim. The allowable amount of resource associated to the first problem is the aggregate claim for this problem, unless it exceeds the total available amount of good, while the remaining amount of resource (if any) is associated to the second problem. Then, solve the first sub-problem according to the primary solution prescription, while the second sub-problem is solved following the secondary solution. Finally, add the solutions for the two sub-problems.

Note that in the piece-wise mixture of solutions, as above described, the secondary solution only comes into play when the available amount of resource is high enough. This contrast with the alternative mixture of the two canonical solutions proposed by Thomson and Yeh (2006). These authors suggest that the secondary solution should always have a relevance even when the amount of available good is low. This alternative approach yields to explore mixed solutions obtained by a convex combination of canonical solutions.

Most of the (mixed) solutions proposed by the literature to solve claims problems have been conceived to give an (exogenously determined) weight to each canonical solution. In most of the cases (e.g. the Talmud, or the Piniles solutions) this weight is $\theta = 0.5$, while other solutions (as the ones belonging to the TAL-family) are built by using alternative weights. The rigidity on how these weights are selected allows to find, associated to each mixed solution, atypical examples helping to criticize any solution.

In this paper we adopt a different approach on how these weights should be selected. In our opinion, once the canonical solutions have been agreed, the weights selected for each specific problem should be endogenously determined. This helps to capture the specific characteristics of the problem to be solved. In particular, we propose to weight the primary solution according to the 'compromising degree' of the problem to be solved.

Within this framework two approaches are studied. In both cases the CEA solution is the primary solution while the secondary solution is the CEL. We first found that, under piece-wise mixing, this Compromise solution coincides with the Proportional solution. This provides a connection between the aim of equalizing reimbursements and/or losses in absolute terms (CEA and CEL solutions) or in relative terms (Proportional solution). Then, we concentrate on exploring, from a normative perspective, the solution obtained by a convex mixture of the canonical solutions. This allows to define the Compromise Convex solution that, among other properties, fulfills equity, continuity and, for high levels of compromising degree, incentive compatibility.

The rest of the paper is organized as follows. Section 2 introduces the model and some relevant solutions. Section 3 introduces the notion of mixed solution as a compromise by the agents on how to distribute the available resource. Additionally, we explore which properties are preserved when mixing solutions and discuss the notion of duality for mixed solutions. Finally, Section 4 introduces the compromising degree as a way to select the right value of the parameter θ for each given problem. This allows to describe two mixing compromise solutions: The Compromise Piece-wise Solution which coincides with the Proportional solution; and the Compromise-Convex solution.

2. Preliminaries

A claims problem involves a set of agents $\mathcal{N} = \{1, \dots, i, \dots, n\}$ that have to share an amount $E \geq 0$ of a perfectly divisible good, called the estate.[3] Each agent has a claim $c_i \geq 0$ on this good, and the aggregate claim $C = \sum_{i=1}^{n} c_i$ exceeds the available amount of good; i.e. $C \geq E$. A (claims) problem is represented through a pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}_+^n$.

A solution for claims problems is a function φ associating each problem a distribution of the estate among the agents so that no agent is assigned more than his claim (claims boundedness) nor a negative amount (non-negativity), and the estate is exactly distributed (efficiency); i.e., $\varphi : \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ is such that for each (E, c), (a) $0 \leq \varphi_i(E, c) \leq c_i$ for each $i \in \mathcal{N}$; and (b) $\sum_{h=1}^{n} \varphi_h(E, c) = E$.

Additionally, and associated to the cooperative approach of solutions introduced by O'Neill (1982), through this paper we assume that solutions satisfy the following Limited Responsibility condition.

Definition 1. We say that solution φ satisfies the Limited Responsibility Condition, LRC thereafter, if for each problem (E, c) and any $0 \leq E' < E$, $\varphi_i(E, c) - \varphi_i\left(E', c\right) \leq E - E'$ for each agent $i \in \mathcal{N}$.

[2] Section 2 provides formal, analytical descriptions for all the solutions mentioned in this section.

[3] In order to avoid technical issues, we include the two degenerate cases, in which the estate is null, $E = 0$, or it equals the aggregate claim $E = C$. In both cases, the notion of solution for claims problems determines what each agent receives.

Remark 1. The Limited Responsibility Condition is a very mild requirement that prevents some agents from increasing their allocation beyond the estate increase (at the expense of other agents whose allocation is reduced). Note that Resource Monotonicity, a condition that fulfill most of the rules, implies LRC.

As we mentioned in the Introduction, a solution attracting much attention of some authors is the Constrained Equal Awards solution, denoted as φ^{CEA}. It distributes equally the estate among the agents with the only restriction that none of them is assigned more than his claim. Therefore, for each problem (E, c), and any agent $i \in \mathcal{N}$, $\varphi_i^{CEA}(E, c) = \min\{\lambda, c_i\}$, where λ is the unique solution to $\sum_{h=1}^{n} \min\{\lambda, c_h\} = E$.

An alternative approach to solving claims problems comes from a dual perspective. We say that solutions φ^{α} and $\varphi^{D(\alpha)}$ are dual whenever for each problem (E, c),

$$\varphi^{\alpha}(C - E, c) + \varphi^{D(\alpha)}(E, c) = c. \tag{1}$$

Note that for a given problem (E, c), $L = C - E$ can be interpreted as the rationing that, in aggregate, claimants should bear when solving the problem (E, c). Therefore, for any agent $i \in \mathcal{N}$, $\varphi_i^{\alpha}(C - E, c) = \varphi_i^{\alpha}(L, c)$ determines i's rationing according to solution φ^{α}, and thus $c_i - \varphi_i^{\alpha}(L, c)$ is the amount allocated to agent i when he is rationed according to φ^{α}. Thus, Eq. (1) establishes an equivalence between solving any given problem according to solution $\varphi^{D(\alpha)}$ and doing it indirectly by determining each agent's rationing throughout φ^{α}.

The CEL solution is dual to the CEA solution. It associates to each problem (E, c) and any agent $i \in \mathcal{N}$, the amount $\varphi_i^{CEL}(E, c) = \max\{0, c_i - \mu\}$, where μ is the unique solution to $\sum_{h=1}^{n} \max\{0, c_h - \mu\} = E$.

To conclude, and for completeness purposes, we mention that, in parallel to the CEA and CEL solutions that equalize awards and losses in absolute terms, the Proportional solution attributed to Aristotle, equalizes awards (and losses) in relative terms, so for each problem (E, c), and any agent $i \in \mathcal{N}$, it fits the expression

$$\varphi_i^{P}(E, c) = \frac{E}{C} c_i. \tag{2}$$

2.1. CEA, CEL and some mixed solutions

The dual relationship between the CEA and the CEL solutions has some interpretative implications. In general, as Claim 1 reports, agents with a low claim prefer to be reimbursed according to the CEA solution, while agents with high claim rather prefer to be reimbursed according to the CEL solution.

Claim 1. *For each problem (E, c) and any given agent, say i,*

(a) if $\varphi_i^{CEA}(E, c) > \varphi_i^{CEL}(E, c)$, then $\varphi_h^{CEA}(E, c) > \varphi_h^{CEL}(E, c)$ for each h such that $0 < c_h \le c_i$; and
(b) if $\varphi_i^{CEA}(E, c) < \varphi_i^{CEL}(E, c)$, then $\varphi_j^{CEA}(E, c) < \varphi_j^{CEL}(E, c)$ for each j such that $c_j \ge c_i$.

Note that condition (a) is only possible if $\varphi_i^{CEL}(E, c) = 0$, or $\lambda > c_i - \mu$. In both cases, the same is true for each h such that $0 < c_h \le c_i$. Regarding condition (b), it is only possible if $\varphi_i^{CEL}(E, c) = c_i - \mu$, $\varphi_i^{CEA}(E, c) = \lambda$, and $\lambda < c_i - \mu$, which remains valid for each j such that $c_j \ge c_i$.

The conflict of interests pointed out by the above statement invites to resort to more neutral solutions, build as a mixture of the (extreme) CEA and CEL solutions. This is, among others, the case of the Talmud solution that prioritizes the CEA solution until each agent is granted one half of his claim, and then distributes the remaining (if any) according to the CEL solution. To be precise, the Talmud solution associates to each problem (E, c) the following distribution of the estate

$$\varphi^{\tau}(E, c) = \varphi^{CEA}\left(\min\left\{E, \frac{1}{2}C\right\}, \frac{1}{2}c\right) + \varphi^{CEL}\left(\max\left\{0, E - \frac{1}{2}C\right\}, \frac{1}{2}c\right). \tag{3}$$

The Talmud solution is the mixture of the CEA and CEL rules most studied in the literature. Nevertheless, it is not the only mixture of these canonical rules attracting the attention of some researchers. The remain of this section is devoted to introduce some of these 'mixing' solutions.

Chun et al. (2001) suggest the employ of the Reverse Talmud solution $\varphi^{R\tau}$ that is described in a similar way as the Talmud rule by exchanging the (primary/secondary) roles of the CEA and the CEL rules. It associates to each problem (E, c) the distribution of the estate

$$\varphi^{R\tau}(E, c) = \varphi^{CEL}\left(\min\left\{E, \frac{1}{2}C\right\}, \frac{1}{2}c\right) + \varphi^{CEA}\left(\max\left\{0, E - \frac{1}{2}C\right\}, \frac{1}{2}c\right). \tag{4}$$

The Piniles' solution φ^{π} coincides with the Talmud solution in problems with a low estate, while it coincides with the reverse Talmud solution for problems with a high estate. To be precise, for each given problem (E, c), it distributes the estate according to the expression

$$\varphi^{\pi}(E, c) = \varphi^{CEA}\left(\min\left\{E, \frac{1}{2}C\right\}, \frac{1}{2}c\right) + \varphi^{CEA}\left(\max\left\{0, E - \frac{1}{2}C\right\}, \frac{1}{2}c\right). \tag{5}$$

For completeness we define the Reverse Piniles' solution $\varphi^{R\pi}$, that coincides with the reverse Talmud solution in problems with a low estate, while it coincides with the Talmud solution for problems with a high estate:

$$\varphi^{R\pi}(E, c) = \varphi^{CEL}\left(\min\left\{E, \frac{1}{2}C\right\}, \frac{1}{2}c\right) + \varphi^{CEL}\left(\max\left\{0, E - \frac{1}{2}C\right\}, \frac{1}{2}c\right). \tag{6}$$

As to extend the Talmud solution, Moreno-Ternero and Villar (2006) study a family of solutions, they call the TAL-family, where (at most) a given share of the aggregate claim is allocated according to the CEA solution, while the remaining share of the estate (if any) is allocated according to the CEL solution. That is, for a given $\theta \in (0, 1)$,

$$\varphi^{\theta\tau}(E, c) = \varphi^{CEA}(\min\{E, \theta C\}, \theta c) + \varphi^{CEL}(\max\{0, E - \theta C\}, (1 - \theta)c). \tag{7}$$

Finally, van den Brink et al. (2013) explore the design of a family of solutions 'reversing' the TAL-family. For a given $\theta \in (0, 1)$, it associates to a problem (E, c) the distribution of the estate

$$\varphi^{R\theta\tau}(E, c) = \varphi^{CEL}(\min\{E, \theta C\}, \theta c) + \varphi^{CEA}(\max\{0, E - \theta C\}, (1 - \theta)c). \tag{8}$$

3. Compromising through mixed solutions

A deeper analysis of the different solutions explored in the literature yields to a standard (generic) procedure to solve claims problems by mixing canonical solutions. It can be described according to the following sequential procedure.

(1) There is a general consensus on assigning a secure share of each agent's claim, when the estate is high enough. This secured claim, that depends on c, is denoted by s, and satisfies that, for each agent i, $s_i \leq c_i$. For each given problem (E, c), secured claim s allows to split it into two sub-problems $\left(E^1, s\right)$ and $\left(E^2, c-s\right)$.
(2) There is a primary solution, say φ^1, that determines how agents are rationed when the estate is not enough to allow each one his 'secure claim'. This primary solution applies to the primary sub-problem $\left(E^1, s\right)$, where $E^1 = \min\left\{E, \sum_{h=1}^{n} s_h\right\}$.
(3) There is a secondary solution, say φ^2, that becomes involved only when the level of estate allows to assign each agent his secure share s_i. This secondary solution is applied to the residual sub-problem $\left(E^2, c-s\right) = \left(E-E^1, c-s\right)$.
(4) Finally, for a given problem (E, c), the description of the mixed solution, say φ^m, is

$$\varphi^m(E, c) = \varphi^1\left(E^1, s\right) + \varphi^2\left(E-E^1, c-s\right). \tag{9}$$

For instance, in the case of the Talmud solution, the above expression translates into (1) $s = c/2$, (2) $\varphi^1 = \varphi^{CEA}$, and (3) $\varphi^2 = \varphi^{CEL}$.

Note that in the above description of how canonical solutions are mixed along the literature there are two main elements deserving additional comments. The first one is the determination of the secured claim s. What it is common in the literature is to consider a close relation between the original claims vector c and the secured claim s, so that even though the latter is not exogenously given, its determination is not affected by the estate. What the literature points out is that a 'popular' approach to the selection of the secured claims vector s is by resorting to a proportional relation between s and the claims vector c, say $\theta \in (0, 1)$. Note that the selection of a given proportion θ not only guarantees that, for each agent i, s_i never surpasses his claim c_i, but also ensures that the determination of the secured claims is done according to an anonymous – and thus equitable – procedure. The second one is that any problem is solved by dividing it into two sub-problems, each one being treated according to a specific criterion.

The ingredients above yield to define the piece-wise combination of solutions as follows.

Definition 2. For any two solutions for claims problems, say φ^1 and φ^2, and each given proportionality factor $\theta \in (0, 1)$, we define their *piece-wise* combination as the solution for claims problems φ^ω associating to each problem (E, c) the allocation

$$\varphi^\omega(E, c) = \varphi^1(\min\{E, \theta C\}, \theta c) + \varphi^2(\max\{0, E-\theta C\}, (1-\theta)c). \tag{10}$$

In such a case, to simplify notation, we write $\varphi^\omega \equiv \Omega^\theta\left(\varphi^1, \varphi^2\right)$.

As an illustrative example, recall that the Talmud solution can be expressed as $\varphi^\tau \equiv \Omega^{1/2}\left(\varphi^{CEA}, \varphi^{CEL}\right)$. Similarly, the Piniles' solution can also be expressed as $\varphi^\pi \equiv \Omega^{1/2}\left(\varphi^{CEA}, \varphi^{CEA}\right)$.

Thomson and Yeh (2006) argue that the solutions conceived as piece-wise combinations of canonical solutions exhibit the interpretative controversy that the secondary solution is relegated to problems where the estate is high enough.[4] According to the description above on how the original problem is split into two sub-problems, each of them being solved according to one canonical solution each, we define the solutions conceived as 'convex combination' of two canonical solutions as follows.

[4] To be precise, Thomson and Yeh (2006, p. 7) write: "When two rules express opposite viewpoints on how to solve a claims problem, it is natural to compromise between them by averaging".

Definition 3. For any two solutions for claims problems, say φ^1 and φ^2, and each given proportionality factor $\theta \in (0, 1)$, we define their *convex* combination as the solution for claims problems φ^κ associating to each problem (E, c) the allocation

$$\varphi^\kappa(E, c) = \varphi^1(\theta E, \theta c) + \varphi^2((1-\theta)E, (1-\theta)c). \tag{11}$$

In such a case, to simplify notation, we write $\varphi^\kappa \equiv \mathcal{K}^\theta\left(\varphi^1, \varphi^2\right)$.

Remark 2. Note that our definition for the convex combination of solutions differs from what Thomson and Yeh (2006) call the *weighted average of rules*:

$$\mathcal{C}^\theta\left(\varphi^1, \varphi^2\right) = \theta\varphi^1 + (1-\theta)\varphi^2 \qquad \theta \in (0, 1).$$

This is because, while these authors study the combination of the solutions we are concerned on how the original problem is split in two sub-problems whose solutions are added. Nevertheless, when the canonical solutions are scale invariant[5] both definitions coincide. As mentioned in Marchant (2008), "we can easily construct a scale invariant rule starting from any rule and, hence, scale invariance is a very weak condition". So, we will not distinguish between our convex operator and the weighted average of rules.

3.1. Preserved properties under mixture

According to a large tradition of exploring the properties satisfied by different solutions (see, e.g., Thomson, 2019) we now concentrate on the study of which properties are preserved when mixing solutions.[6]

We say that operator Θ preserves property Y whenever for any two solutions φ^1 and φ^2 satisfying such a property and any scalar $\theta \in (0, 1)$ solution $\Theta^\theta\left(\varphi^1, \varphi^2\right)$ also satisfies property Y.

For the case of their weighted average of solutions Thomson and Yeh (2006) find that some important properties are not preserved. This is summarized in Table 1.

A similar analysis related to the piece-wise operator Ω, summarized in Table 2, is captured by Theorem 1.

Theorem 1. *Let φ^1 and φ^2 two solutions for claims problems, and $\theta \in (0, 1)$ a given parameter. Then,*

(1) The piece-wise operator $\Omega^\theta\left(\varphi^1, \varphi^2\right)$ preserves resource monotonicity, order preservation, anonymity, continuity, claim monotonicity, *and* consistency. *Moreover, if φ^1 and φ^2 are resource monotonic,* population monotonicity *is also preserved.*
(2) The piece-wise operator $\Omega^\theta\left(\varphi^1, \varphi^2\right)$ fails to preserve invariance under claims truncation, minimal rights first, self-duality, composition down and composition up.

Remark 3. As mentioned in Thomson (2003, p. 270), "most of the rules that have been considered in the literature, and all of the rules that we have formally defined, are resource monotonic". Therefore the last statement in part (1) of Theorem 1 might conclude by claiming that *population monotonicity is always preserved for any pair of reasonable solutions.*

[5] A solution for claims problems, say φ, satisfies scale invariance (also known as homogeneity) if for each problem (E, c) and each $\lambda > 0$, $\varphi(\lambda E, \lambda c) = \lambda\varphi(E, c)$..
[6] For exposition convenience, we relegate the introduction of these standard properties to the Appendix.

Table 1
Convex operator: When applied to scale invariant solutions, if both solutions φ^1 and φ^2 satisfy the mentioned property, then for each given θ, $\mathcal{K}^\theta\left(\varphi^1,\varphi^2\right)$ also satisfies this property: true (✓), false (×).

Invariance under Claims Truncation	✓	Order Preservation	✓	Anonymity	✓
Continuity	✓	Claim Monotonicity	✓	Resource Monotonic	✓
Minimal Rights First	✓	Composition Down	×	Composition Up	×
Self Duality	✓	Population Monotonic	✓	Consistency	×

Table 2
Piece-wise operator: If both solutions φ^1 and φ^2 satisfy the mentioned property, then for each given θ, $\Omega^\theta\left(\varphi^1,\varphi^2\right)$ also satisfies this property: true (✓), false (×).

Invariance under Claims Truncation	×	Order Preservation	✓	Anonymity	✓
Continuity	✓	Claim Monotonicity	✓	Resource Monotonic	✓
Minimal Rights First	×	Composition Down	×	Composition Up	×
Self Duality	×	Population Monotonic	✓	Consistency	✓

Proof.

(1) It is obvious that equal treatment of equals, order preservation and anonymity are preserved. On the other hand, as mentioned, the piece-wise operator has been defined in a continuous way, so that if two solutions are continuous, their piece-wise combination is continuous too.

To see that the piece-wise operator preserves resource monotonicity, let us consider two resource monotonic solutions φ^1 and φ^2. For a given claims vector c, and thus aggregate claim C, consider two different levels of estate E and E' such that $0 \le E < E' \le C$. We consider the following three cases, that exhaust all the possibilities.

(i) θ is such that $E' \le \theta C$. Then, solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ satisfies that for each agent $i \in \mathcal{N}$,

$$\varphi_i^\omega(E,c) = \varphi_i^1(E,\theta c) \le \varphi_i^1\left(E',\theta c\right) = \varphi_i^\omega\left(E',c\right);$$

(ii) θ is such that $E \ge \theta C$. Then, solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ satisfies that for each agent $i \in \mathcal{N}$,

$$\begin{aligned}\varphi_i^\omega(E,c) &= \theta c_i + \varphi_i^2(E-\theta C,(1-\theta)c) \le \\ &\le \theta c_i + \varphi_i^2\left(E'-\theta C,(1-\theta)c\right) = \varphi_i^\omega\left(E',c\right);\end{aligned}$$

(iii) θ is such that $E < \theta C < E'$. Then, solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ satisfies that for each agent $i \in \mathcal{N}$,

$$\begin{aligned}\varphi_i^\omega(E,c) &= \varphi_i^1(E,\theta c) \le \theta c_i \le \theta c_i \\ &\quad + \varphi_i^2\left(E'-\theta C,(1-\theta)c\right) \\ &= \varphi_i^\omega\left(E',c\right).\end{aligned}$$

To explore claim monotonicity, let us consider two given problems differing only in agent i's claim. That is these problems are (E,c) and $\left(E,c'\right)$, where $c_i' > c_i$, and $c_j' = c_j$ for all $j \ne i$. Assume that solutions φ^1 and φ^2 are claim monotonic. Consider the following three cases, that exhaust all the possibilities.

(i) θ is such that $E \le \theta C$. Then solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ fulfills that for agent i,

$$\varphi_i^\omega(E,c) = \varphi_i^1(E,\theta c) \le \varphi_i^1\left(E,\theta c'\right) = \varphi_i^\omega\left(E,c'\right).$$

(ii) θ is such that $\theta C < E < \theta C'$. Define $c_i'' = \frac{1}{\theta}\left(E - \sum_{j\ne i}\theta c_j\right)$, and denote by c'' the claims vector whose ith component is c_i'', while $c_j'' = c_j$ for $j \ne i$. Note that $\theta C'' = E$ and for any solution, say φ, $\varphi\left(E,\theta c''\right) = \theta c''$. Then solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ satisfies that for agent i,

$$\begin{aligned}\varphi_i^\omega(E,c) &= \theta c_i + \varphi_i^2(E-\theta C,(1-\theta)c) \le \theta c_i + (E-\theta C) = \\ &= \theta c_i'' = \varphi_i^1\left(E,\theta c''\right) \le \varphi_i^1\left(E,\theta c'\right) = \varphi_i^\omega\left(E,c'\right).\end{aligned}$$

(iii) θ is such that $E \ge \theta C'$. Then, for agent i, solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ satisfies that

$$\varphi_i^\omega(E,c) = \theta c_i + \varphi_i^2(E-\theta C,(1-\theta)c).$$

By LRC we have that

$$\theta c_i + \varphi_i^2(E-\theta C,(1-\theta)c) \le \theta c_i' + \varphi_i^2\left(E-\theta C',(1-\theta)c\right).$$

Since φ^2 is claim monotonic,

$$\theta c_i' + \varphi_i^2\left(E-\theta C',(1-\theta)c\right) \le \theta c_i' + \varphi_i^2\left(E-\theta C',(1-\theta)c'\right),$$

and thus $\varphi_i^\omega(E,c) \le \varphi_i^\omega\left(E,c'\right)$.

We now deal with the analysis of population monotonicity. Given a set of agents $\mathcal{N}$, and a problem for them (E,c), consider an external agent $i \notin \mathcal{N}$, a claim for such an agent c_i and the problem $\left(E,c'\right)$ obtained from embodying i in problem (E,c). Note that the aggregate claim in the two problems is $C = \sum_{j\in\mathcal{N}} c_j$ when i is not taken into account and $C' = C + c_i$ otherwise. Consider two resource monotonic and population monotonic solutions, φ^1 and φ^2, a parameter $0 < \theta < 1$, and solution $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$. Consider the following three cases, that exhaust all the possibilities.

(i) $E \le \theta C$. Then, for each $h \in \mathcal{N}$,

$$\varphi_h^\omega(E,c) = \varphi_h^1(E,c) \ge \varphi_h^1\left(E,c'\right) = \varphi_h^\omega\left(E,c'\right).$$

(ii) θ is such that $\theta C < E < \theta C'$. Then, for each $h \in \mathcal{N}$,

$$\begin{aligned}\varphi_h^\omega\left(E,c'\right) &= \varphi_h^1\left(E,c'\right) \le \theta c_h \le \theta c_h \\ &\quad + \varphi_h^2(E-\theta C,(1-\theta)C) \\ &= \varphi_h^\omega(E,c).\end{aligned}$$

(iii) $\theta C' \le E$. Then, for each agent $h \in \mathcal{N}$,

$$\begin{aligned}\varphi_h^\omega(E,c) - \varphi_h^\omega\left(E,c'\right) &= \varphi_h^2(E-\theta C,(1-\theta)c) \\ &\quad - \varphi_h^2\left(E-\theta C',(1-\theta)c'\right).\end{aligned}$$

Since φ^2 is population monotonic,

$$\begin{aligned}\varphi_h^\omega(E,c) - \varphi_h^\omega\left(E,c'\right) &\ge \varphi_h^2\left(E-\theta C,(1-\theta)c'\right) \\ &\quad - \varphi_h^2\left(E-\theta C',(1-\theta)c'\right) \\ &\ge 0,\end{aligned}$$

where the last inequality follows because φ^2 is resource monotonic.

To conclude, we show that the piece-wise operator preserves consistency. Let us consider two consistent solutions, say φ^1 and φ^2, and a given parameter θ. Let $\varphi^\omega = \Omega^\theta\left(\varphi^1,\varphi^2\right)$ denote the piece-wise combination of the solutions above. For a given coalition $\emptyset \ne \mathcal{S} \subset \mathcal{N}$ and any problem (E,c), let denote $E^{\mathcal{S}}=\sum_{i\in\mathcal{S}}\varphi_i^{\omega}(E,c)$, and $C^{\mathcal{S}}=\sum_{i\in\mathcal{S}}c_i$. First, it is important to note that $E^{\mathcal{S}}\leq\theta C^{\mathcal{S}}$ if and only if $E\leq\theta C$. Consider the following cases, that exhaust all the possibilities.

(i) $E\leq\theta C$. Then, $\varphi^{\omega}(E,c)=\varphi^{1}(E,\theta c)$. Therefore consistency is immediately derived from consistency of φ^1.

(ii) $E\geq\theta C$. Then, $\varphi^{\omega}(E,c)=\theta c+\varphi^{2}(E-\theta C,(1-\theta)c)$ and thus consistency is immediately derived from consistency of φ^2.

(2) To prove that the piece-wise operator fails to preserve invariance under claims truncation, minimal rights first, self-duality, composition down and composition up, let us consider the following counterexamples.

(i) **Invariance under Claims Truncation**. Assume $\varphi^1=\varphi^2=\varphi^{CEA}$, and $\theta=\frac{1}{2}$ so $\varphi^{\omega}=\Omega^{\frac{1}{2}}\left(\varphi^{CEA},\varphi^{CEA}\right)$. Consider the three-agent problem $(E,c)=(202,(60,100,220))$. Then, $\varphi^{\omega}(E,c)=(34,54,114)$.
On the other hand, $\bar{c}=(60,100,202)$ and then $\varphi^{\omega}(E,\bar{c})=(37,57,108)$, so invariance under claims truncation is not preserved.

(ii) **Minimal Rights First**. Assume $\varphi^1=\varphi^2=\varphi^{CEL}$, and $\theta=\frac{1}{2}$ so $\varphi^{\omega}=\Omega^{\frac{1}{2}}\left(\varphi^{CEL},\varphi^{CEL}\right)$. Consider the three-agent problem $(E,c)=(42,(12,24,54))$. Note that the minimal right for agents is $m(E,c)=(0,0,6)$. Since $E<\theta C$, $\varphi^{\omega}(E,c)=\varphi^{1}(E,c/2)=\varphi^{CEL}(42,(6,12,27))=(5,11,26)$.
On the other hand, $m(E,c)+\varphi^{\omega}\left(E-\sum_i m_i(E,c),\ c-m(E,c)\right)=(0,0,6)+\varphi^{CEL}(36,(6,12,24))=(0,0,6)+(4,10,22)=(4,10,28)\neq\varphi^{\omega}(E,c)$, and thus minimal rights first is not preserved.

(iii) **Self-duality**. Assume that $\varphi^1=\varphi^2=\varphi^{\tau}$, and $\theta=\frac{1}{3}$ so $\varphi^{\omega}=\Omega^{\frac{1}{3}}(\varphi^{\tau},\varphi^{\tau})$. Consider the three-agent problem $(E,c)=(36,(12,24,36))$. Then,

$$\begin{aligned}\varphi^{\omega}(E,c) &= \varphi^{\tau}(24,(4,8,12))+\varphi^{\tau}(12,(8,16,24))=\\ &= (4,8,12)+(4,4,4)\ =\ (8,12,16)\,.\end{aligned}$$

The problem dual to (E,c) is $(E,c)^d=(C-E,c)=(36,(12,24,36))$. Self-duality of φ^{ω} implies that $c-\varphi^{\omega}(C-E,c)=\varphi^{\omega}(E,c)$. Nevertheless,

$$\begin{aligned}c-\varphi^{\omega}(C-E,c) &= (12,24,36)-(8,12,16)\\ &= (4,12,20)\neq(8,12,16)\\ &= \varphi^{\omega}(E,c)\,.\end{aligned}$$

(iv) **Composition Up** and **Composition Down**. As Thomson (2003) reports, solutions φ^{CEA} and φ^{CEL} satisfy Composition Down and Composition Up. Nevertheless $\varphi^{\tau}=\Omega^{\frac{1}{2}}\left(\varphi^{CEA},\varphi^{CEL}\right)$ does not fulfill these properties. ■

Remark 4. Note that when comparing both operators, the piece-wise operator fails to preserve minimal rights first and self-duality, that are preserved by the convex operator. On the contrary, consistency is preserved by the piece-wise operator while it is not preserved by the convex operator. None of them preserves composition up or down.

3.2. Duality under mixture

A property that has particularly worried to some authors is to prescribe solutions satisfying self-duality. This is because under these solutions, the distribution of the estate and the aggregate loss is conducted according to a common criterion. As mentioned in Thomson (2003) "the problem of dividing 'what is available' and the problem of dividing 'what is missing' should be treated symmetrically". The aim of this section is to seed some light about how our approach of mixing solutions could help to build self-dual solutions. Here we propose two general results – each one related to any of the mixtures explored in this paper – allowing to describe two mixing procedures yielding self-dual solutions.

Related to the convex operator, the next result points out that it commutes with the dual operator.

Proposition 1. *Let φ^1 and φ^2 two solutions, and $0<\theta<1$ a given proportionality factor. Define $\varphi^{\kappa}=\mathcal{K}^{\theta}\left(\varphi^1,\varphi^2\right)$. Then*

$$\varphi^{D(\kappa)}=\mathcal{K}^{\theta}\left(\varphi^{D(1)},\varphi^{D(2)}\right).$$

Proof. Note that, for each given problem (E,c),

$$\begin{aligned}\varphi^{D(\kappa)}(E,c) &= c-\varphi^{\kappa}(C-E,c)=\\ &= c-\varphi^{1}(\theta(C-E),\theta c)\\ &\quad -\varphi^{2}((1-\theta)(C-E),(1-\theta)c)=\\ &= \left[\theta c-\varphi^{1}(\theta(C-E),\theta c)\right]+\left[(1-\theta)c\right.\\ &\quad \left.-\varphi^{2}((1-\theta)(C-E),(1-\theta)c)\right]=\\ &= \varphi^{D(1)}(\theta E,\theta c)+\varphi^{D(2)}((1-\theta)E,(1-\theta)c)=\\ &= \mathcal{K}^{\theta}\left(\varphi^{D(1)},\varphi^{D(2)}\right)(E,c)\,.\quad ■\end{aligned}$$

Corollary 1. *For each solution φ^1 the convex combination $\mathcal{K}^{1/2}\left(\varphi^1,\varphi^{D(1)}\right)$ is self-dual.*

A parallel analysis for the piece-wise operator yields the following result.

Proposition 2. *Let φ^1 and φ^2 two solutions, and $0<\theta<1$ a given proportionality factor. Define $\varphi^{\omega}=\Omega^{\theta}\left(\varphi^1,\varphi^2\right)$. Then*

$$\varphi^{D(\omega)}=\Omega^{1-\theta}\left(\varphi^{D(2)},\varphi^{D(1)}\right).$$

Proof. Note that, for any given problem (E,c), $\varphi^{D(\omega)}(E,c)=c-\varphi^{\omega}(C-E,c)$. Let us consider the following two cases, that exhaust all the possibilities.

(i) $C-E\leq\theta C$. Then,

$$\varphi^{D(\omega)}(E,c)=c-\varphi^{\omega}(C-E,c)=c-\varphi^{1}(C-E,\theta c)\,.$$

Since $\varphi^{1}(C-E,\theta c)=\theta c-\varphi^{D(1)}(E-(1-\theta)C,\theta c)$, it follows that

$$\begin{aligned}\varphi^{D(\omega)}(E,c) &= (1-\theta)c+\varphi^{D(1)}(E-(1-\theta)C,\theta c)=\\ &= \varphi^{D(2)}((1-\theta)C,(1-\theta)c)+\varphi^{D(1)}(E-(1-\theta)C,\theta c)\,.\end{aligned}$$

(ii) $C-E>\theta C$. Then,

$$\begin{aligned}\varphi^{D(\omega)}(E,c) &= c-\varphi^{\omega}(C-E,c)=\\ &= c-\left[\theta c+\varphi^{2}((1-\theta)C-E,(1-\theta)c)\right] =\\ &= (1-\theta)c-\varphi^{2}((1-\theta)C-E,(1-\theta)c) =\\ &= \varphi^{D(2)}(E,(1-\theta)c)\,. \quad ■\end{aligned}$$

Corollary 2. *For each solution φ^1, the piece-wise combination $\Omega^{1/2}\left(\varphi^1,\varphi^{D(1)}\right)$ satisfies self-duality.*

Remark 5. *Proposition 2 is useful to compute the dual solution for each one belonging to the TAL family (Moreno-Ternero and Villar, 2006). Since $\varphi^{D(CEA)}=\varphi^{CEL}$ and $\varphi^{D(CEL)}=\varphi^{CEA}$, for each $0<\theta<1$, $\varphi^{D(\theta\tau)}=\varphi^{(1-\theta)\tau}$. Note that, in particular, this implies that the unique self-dual solution belonging to the TAL family is the Talmud solution.*

4. An accurate selection of θ and two solutions

The analysis performed in Section 3 assumes that the mixing parameter θ is exogenously determined. Nevertheless, when the selection of a specific solution is influenced by the claimants' opinion, a conflict of interest often occurs. Note that, for a given problem, agents with a low claim prefer the adoption of the CEA rather than the CEL solution. On the contrary, agents with the highest claim prefer to be rationed according to the CEL rather than adopting the CEA as a way to solve the problem. This suggests that the selection of θ, when mixing these two canonical solutions, needs to be related to the specific problem to be solved. This is why we propose a solution obtained by mixing the CEA and its dual solution, the CEL. In this mixture the weight associated to the CEA solution, for each given problem, is its compromising degree, which is very related to the relative level of rationing that agents should suffer.

Definition 4. For a given problem $\mathbb{P} = (E, c)$ we define its compromising degree as the ratio

$$\theta(E, c) \equiv \theta^{\mathbb{P}} = \frac{E}{C}.$$

Note that for each given problem $\mathbb{P} = (E, c)$, $0 \leq \theta^{\mathbb{P}} \leq 1$. The lower conflict appears the higher E is. In addition the ability to reach an agents' compromise to accept a solution is higher for low levels of conflict. The mixture of two solutions according to the compromising degree yields the description of the following compromise composition.

Definition 5. The **Compromise Piece-wise Composition** of two solutions, say φ^1 and φ^2, is the solution for claim problems associating to each problem $\mathbb{P} = (E, c)$ the awards distribution

$$\Omega^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)(E, c) = \varphi^1\left(\min\left\{E, \theta^{\mathbb{P}}C\right\}, \theta^{\mathbb{P}}c\right) + \varphi^2\left(\max\left\{0, E - \theta^{\mathbb{P}}C\right\}, \left(1-\theta^{\mathbb{P}}\right)c\right).$$

Definition 6. The **Compromise Convex Composition** of two solutions, say φ^1 and φ^2, is the solution for claim problems associating to each problem $\mathbb{P} = (E, c)$ the awards distribution

$$\mathcal{K}^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)(E, c) = \varphi^1\left(\theta^{\mathbb{P}}E, \theta^{\mathbb{P}}c\right) + \varphi^2\left(\left(1-\theta^{\mathbb{P}}\right)E, \left(1-\theta^{\mathbb{P}}\right)c\right).$$

Remark 6. The difference regarding the piece-wise and convex compositions is that the parameter θ depends on each particular problem. Then, for problems with low conflict, $\theta^{\mathbb{P}} \approx 1$, the proposal made by φ^1 has the main effect in the compromise solution, whereas for problems with high conflict, $\theta^{\mathbb{P}} \approx 0$, the important proposal is the one made by φ^2. It is clear that an alternative to the use of $\theta^{\mathbb{P}}$ is to measure the relative losses in a problem; that is, to define

$$\sigma(E, c) \equiv \sigma^{\mathbb{P}} = \frac{C - E}{C}.$$

This reverses the effect of φ^1 and φ^2 in low and high conflict problems. Most of the results we obtain here can be replicated by replacing $\theta^{\mathbb{P}}$ by $\sigma^{\mathbb{P}}$.

Just for illustrative purposes, Example 1 describes the estate distribution for two related problems, according to some relevant solutions, when we consider $\varphi^1 = \varphi^{CEA}$ and $\varphi^2 = \varphi^{CEL}$.

Example 1. Consider two four-agent instances differing only in the estate to be distributed. Agents' claims are $c = (750, 1500, 2250, 3000)$, and thus $C = 7500$. The first problem refers to the case where $E = 5400$ and thus it exhibits a high value of $\theta^{\mathbb{P}} = 0.72$. The second one corresponds to situations with strong rationing needed. We assume that the estate is $E' = 2100$, and thus $\theta^{\mathbb{P}'} = 0.28$. Table 3 describes, for each of the above problems, its CEA, CEL, Talmud, $\Omega^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$ and $\mathcal{K}^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$ solutions.

Table 3
Comparison of solutions.

(a) High $\theta^{\mathbb{P}}$ (low conflict).

	φ^{CEA}	φ^{CEL}	φ^{τ}	$\Omega^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$	$\mathcal{K}^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$
φ_1	750	225	375	540	603
φ_2	1500	975	925	1080	1353
φ_3	1575	1725	1675	1620	1617
φ_4	1575	2475	2425	2160	1827

(b) Low $\theta^{\mathbb{P}}$ (high conflict).

	φ^{CEA}	φ^{CEL}	φ^{τ}	$\Omega^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$	$\mathcal{K}^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$
φ_1	525	0	375	210	147
φ_2	525	0	575	420	147
φ_3	525	675	575	630	633
φ_4	525	1425	575	840	1173

If we observe the results in Example 1, in both cases the proposal made by the Compromise Piece-wise Composition, $\Omega^{\mathbb{P}}\left(\varphi^{CEA}, \varphi^{CEL}\right)$, coincides with the celebrated Proportional solution described in Eq. (2). Our next result shows that this is always true, for any pair of solutions we combine.

Proposition 3. *Let φ^1 and φ^2 two solutions for claims problems. Then, for each problem $\mathbb{P} = (E, c)$,*

$$\Omega^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)(E, c) = \varphi^P(E, c).$$

Proof. Note that, by construction, for each problem $\mathbb{P} = (E, c)$, its compromising degree $\theta^{\mathbb{P}}$ satisfies that $E = \sum_{i\in\mathcal{N}} \theta^{\mathbb{P}} c_i$. This implies that for each problem (E, c),

$$\begin{aligned}\Omega^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)(E, c) &= \varphi^1\left(\min\left\{E, \theta^{\mathbb{P}}C\right\}, \theta^{\mathbb{P}}c\right) \\ &\quad + \varphi^2\left(\max\left\{0, E - \theta^{\mathbb{P}}C\right\}, \left(1-\theta^{\mathbb{P}}\right)c\right) = \\ &= \varphi^1\left(\theta^{\mathbb{P}}C, \theta^{\mathbb{P}}c\right) + \varphi^2\left(0, \left(1-\theta^{\mathbb{P}}\right)c\right) \\ &= \theta^{\mathbb{P}}c = \varphi^P(E, c). \quad \blacksquare\end{aligned}$$

The description above is useful to provide a new interpretation of the Proportional solution as a compromising solution for claims problems. For the sake of completeness we introduce the following table extracted from Thomson (2019) summarizing how the $\Omega^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)$ solution behaves related to the properties highlighted in Section 3 (see Table 4).

Remark 7. Note that the conclusions of Proposition 3 are not still valid when we remplace $\theta^{\mathbb{P}}$ by parameter $\sigma^{\mathbb{P}}$. In this case, we obtain

$$\Upsilon^{\mathbb{P}}\left(\varphi^1, \varphi^2\right) = \varphi^1\left(\min\left\{E, \sigma^{\mathbb{P}}C\right\}, \sigma^{\mathbb{P}}c\right) + \varphi^2\left(\max\left\{0, E - \sigma^{\mathbb{P}}C\right\}, \left(1-\sigma^{\mathbb{P}}\right)c\right) =$$

$$= \begin{cases} \varphi^1\left(E, \sigma^{\mathbb{P}}c\right) & \text{if} \quad E \leq \frac{C}{2} \\ \sigma^{\mathbb{P}}c + \varphi^2\left(2E - C, \left(1-\sigma^{\mathbb{P}}\right)c\right) & \text{if} \quad E \geq \frac{C}{2} \end{cases}$$

When applied to CEA and CEL solutions, this proposal has the flavor of the Talmud solution.

The convex mixture of the CEA and CEL solutions according to the compromising degree yields the description of the following compromise solution.

Table 4
Properties satisfied by the $\Omega^{\mathbb{P}}\left(\varphi^1, \varphi^2\right)$ solution: true (✓), false (×).

Invariance under Claims Truncation	×	Order Preservation	✓	Anonymity	✓
Continuity	✓	Claim Monotonicity	✓	Resource Monotonic	✓
Minimal Rights First	×	Composition Down	✓	Composition Up	✓
Self Duality	✓	Population Monotonic	✓	Consistency	✓

Definition 7. The **Compromise Convex Solution**, CC in short, is the solution for claim problems φ^{CC} associating to each problem $\mathbb{P} = (E, c)$ the awards distribution

$$\varphi^{CC}(E,c) = \varphi^{CEA}\left(\theta^{\mathbb{P}}E, \theta^{\mathbb{P}}c\right) + \varphi^{CEL}\left(\left(1-\theta^{\mathbb{P}}\right)E, \left(1-\theta^{\mathbb{P}}\right)c\right). \quad (12)$$

The remainder of this section is devoted to explore how the CC solution behaves from a normative viewpoint.

4.1. The compromise convex solution and its reverse

We now concentrate on the analysis of which, among the standard properties, are fulfilled by the CC solution. Note that some of the properties satisfied by both the CEA and CEL solutions might not be preserved because θ is endogenously determined. This affects particularly to claim monotonicity. The two canonical solutions (i.e., CEA and CEL) are claim monotonic. Nevertheless, when comparing two problems differing uniquely in the claim of an agent, it might be the case that such an agent is worse off when his claim is high, in comparison with the case where it was lower. This only happens in problems $\mathbb{P} = (E, c)$ whose associated parameter $\theta^{\mathbb{P}}$ is low, and thus a collective compromise is hardly reachable.

Table 5 summarizes the statement of Theorem 2 that explores which properties are satisfied by the CC solution.

Theorem 2. *The Compromise Convex solution φ^{CC} fulfills the following properties:* Order preservation, anonymity, continuity, resource monotonicity, self-duality *and* population monotonicity. *For problems $\mathbb{P} = (E, c)$ such that $\theta^{\mathbb{P}} > 0.5$, this solution is* claim monotonic *too. Invariance under claims truncation, minimal rights first, composition down, composition up and consistency are not fulfilled by φ^{CC}. Finally, for problems $\mathbb{P} = (E, c)$ such that $\theta^{\mathbb{P}} < 0.5$, this solution fails to fulfill claim monotonicity.*

Proof. First, note that both the CEA and CEL solutions fulfill order preservation, anonymity and continuity, and the parameter θ continuously varies with E and c. Therefore φ^{CC} also satisfies the three properties above.

To explore the behavior of φ^{CC}, let λ be the unique value satisfying $\sum_{h\in\mathcal{N}} \min\{\lambda, c_h\} = E$. Note that such λ is essential to compute, for each agent i, $\varphi_i^{CEA}(E,c) = \min\{\lambda, c_i\}$. Similarly, let μ be the unique value for which $\sum_{h\in\mathcal{N}} \max\{0, c_h - \mu\} = E$. In this case, parameter μ is essential to compute, for each agent i, $\varphi_i^{CEL}(E,c) = \max\{0, c_i - \mu\}$. Then,

(a) For $\theta^{\mathbb{P}} > 0.5$, by Theorem 1 in Alcalde and Peris (2017), $\lambda > \mu$, which determines the explicit form of the CC solution:

$$\varphi_i^{CC}(E,c) = \begin{cases} \theta^{\mathbb{P}}c_i & \text{if } c_i \le \mu \\ \theta^{\mathbb{P}}c_i + \left[1-\theta^{\mathbb{P}}\right](c_i-\mu) = c_i - \left[1-\theta^{\mathbb{P}}\right]\mu & \text{if } \mu < c_i < \lambda \\ \theta^{\mathbb{P}}\lambda + \left[1-\theta^{\mathbb{P}}\right](c_i-\mu) & \text{if } \lambda \le c_i \end{cases}$$

(b) If $\theta^{\mathbb{P}} < 0.5$, by Theorem 2 in Alcalde and Peris (2017), $\lambda < \mu$, and in this case:

$$\varphi_i^{CC}(E,c) = \begin{cases} \theta^{\mathbb{P}}c_i & \text{if } c_i \le \lambda \\ \theta^{\mathbb{P}}\lambda & \text{if } \lambda < c_i < \mu \\ \theta^{\mathbb{P}}\lambda + \left[1-\theta^{\mathbb{P}}\right](c_i-\mu) & \text{if } \mu \le c_i \end{cases}$$

(c) Finally, if $\theta^{\mathbb{P}} = 0.5$, by Theorem 3 in Alcalde and Peris (2017), $\lambda = \mu$, and in this case $\varphi_i^{CC}(E,c) = \frac{1}{2}c_i$, for all $i \in \mathcal{N}$.

To show that φ^{CC} fulfills claim monotonicity for problems with $\theta^{\mathbb{P}} > 0.5$, consider a given problem $\mathbb{P} = (E, c)$ and let $\theta^{\mathbb{P}}$ be its compromising degree. If $\mathbb{P}' = \left(E, c'\right)$ is such that $c_i' = c_i + \varepsilon_i > c_i$ and $c_j' = c_j$, for all $j \in \mathcal{N}, j \ne i$, only one of the following situations holds[7]:

(i) $c_i \le \mu$. Then,

$$\varphi_i^{CC}(E,c) = \theta^{\mathbb{P}}c_i = \frac{E}{C}c_i \le \frac{E}{C+\varepsilon_i}(c_i+\varepsilon_i) = \theta^{\mathbb{P}'}c_i' \le \varphi_i^{CC}\left(E,c'\right).$$

(ii) $\mu < c_i < \lambda$. Then,

$$\varphi_i^{CC}(E,c) = c_i - \left[1-\theta^{\mathbb{P}}\right]\mu \le (c_i+\varepsilon_i) - \left[1-\theta^{\mathbb{P}'}\right](\mu+\varepsilon_i) \le \varphi_i^{CC}\left(E,c'\right).$$

(iii) $\lambda \le c_i$. Then,

$$\varphi_i^{CC}(E,c) = \theta^{\mathbb{P}}\lambda + \left[1-\theta^{\mathbb{P}}\right](c_i-\mu) \le$$
$$\le \theta^{\mathbb{P}'}\lambda + \left[1-\theta^{\mathbb{P}'}\right]((c_i+\varepsilon_i)-(\mu+\varepsilon_i)) \le \varphi_i^{CC}\left(E,c'\right).$$

So, in any case, the convex compromise solution φ^{CC} fulfills claim monotonicity for problems such that $\theta^{\mathbb{P}} > 0.5$.

The intuition why, for low values of $\theta^{\mathbb{P}}$, claim monotonicity fails can be explained as follows. For $\theta^{\mathbb{P}} \le 0.5$, according to Theorem 2 in Alcalde and Peris (2017), $\lambda \le \mu$. Then, for some problem there should be an agent i such that $\lambda < c_i \le \mu$. This agent would prefer to have a claim c_i' such that $\lambda \le c_i' < c_i \le \mu$. This is because

$$\varphi_i^{CEA}(E,c) = \varphi_i^{CEA}\left(E,c'\right) = \lambda > 0, \quad \varphi_i^{CEL}(E,c) = \varphi_i^{CEL}\left(E,c'\right) = 0,$$

while $\theta^{\mathbb{P}} < \theta^{\mathbb{P}'}$.

To illustrate the above situation, reconsider the "low compromising degree" instance described in Example 1, in which $E = 2100$ and $c = (750, 1500, 2250, 3000)$. In this case $\lambda = 525$, while $\mu = 1575$. Now consider that c_2 drops to $c_2' = 1000$. For problem $\left(E, c'\right) = (2100, (7500, 1500, 2250, 3000))$ we have that $\varphi_2^{CEA}\left(E,c'\right) = \varphi_2^{CEA}(E,c) = 525$, $\varphi_2^{CEL}\left(E,c'\right) = \varphi_2^{CEL}\left(E,c'\right) = 0$, while $\theta^{\mathbb{P}'} = 0.3 > 0.28 = \theta^{\mathbb{P}}$. Therefore, $\varphi_2^{CC}\left(E,c'\right) = 157.5 > 147 = \varphi_2^{CC}(E,c)$.

To prove resource monotonicity note that the Compromise Convex solution can be rewritten as:

$$\varphi_i^{CC}(E,c) = \theta^{\mathbb{P}}\min\{c_i,\lambda\} + \left(1-\theta^{\mathbb{P}}\right)\max\{0, c_i-\mu\}. \quad (13)$$

Given two claims problems $\mathbb{P} = (E, c)$ and $\mathbb{P}' = \left(E', c\right)$, $E < E' \le C$, it is clear that $\lambda < \lambda'$, whereas $\mu > \mu'$. Now, for each agent $i \in \mathcal{N}$, we consider the following situations, that exhaust all the possibilities.

(1) $\theta^{\mathbb{P}} \ge 0.5$
We know that in this case $\lambda \ge \mu$ and then, if

(i) $c_i \le \mu \le \lambda$, $\varphi_i^{CC}(E,c) = \theta^{\mathbb{P}}c_i \le \theta^{\mathbb{P}'}c_i \le \varphi_i^{CC}(E',c)$.

[7] Note that, regarding problems $\mathbb{P}$ and $\mathbb{P}'$, the values defining φ^{CEA} and φ^{CEL} fulfill $\lambda = \lambda'$ and $\mu < \mu'$.

Table 5
Properties fulfilled by φ^{CC}: true (✓), false (×), only for problems $\mathbb{P}=(E,c)$ with $\theta^{\mathbb{P}}>0.5$ (⊕).

Invariance under Claims Truncation	×	Order Preservation	✓	Anonymity	✓
Continuity	✓	Claim Monotonicity	⊕	Resource Monotonic	✓
Minimal Rights First	×	Composition Down	×	Composition Up	×
Self Duality	✓	Population Monotonic	✓	Consistency	×

(ii) $\mu < c_i \le \lambda$, $\varphi_i^{CC}(E,c)=\theta^{\mathbb{P}}c_i+\left(1-\theta^{\mathbb{P}}\right)(c_i-\mu)=$
$=c_i-\left(1-\theta^{\mathbb{P}}\right)\mu \le c_i-\left(1-\theta^{\mathbb{P}'}\right)\mu' \le \varphi_i^{CC}(E',c)$.

(iii) $\mu \le \lambda < c_i$, $\varphi_i^{CC}(E,c)=\theta^{\mathbb{P}}\lambda+\left(1-\theta^{\mathbb{P}'}\right)(c_i-\mu)\le$
$\le \theta^{\mathbb{P}'}\lambda'+\left(1-\theta^{\mathbb{P}}\right)(c_i-\mu')\le \varphi_i^{CC}(E',c)$.

(2) $\theta^{\mathbb{P}}<0.5$
We know that in this case $\lambda<\mu$ and by using an analogous reasoning as in case (1) we obtain the required result.

To prove that φ^{CC} is self-dual, let $\varphi^{D(CC)}$ the solution dual to φ^{CC}. Then, for each problem (E,c), $\varphi^{D(CC)}(E,c)=c-\varphi^{CC}(L,c)$. Taking into account that for each problem $\mathbb{P}=(E,c)$, $\theta^{\mathbb{P}}=1-\theta^{\mathbb{P}^d}$, where $\mathbb{P}^d=\left(\sum_{h=1}^n c_h-E,c\right)=(L,c)$,

$$\begin{aligned}\varphi^{D(CC)}(E,c)&=c-\varphi^{CC}(L,c)=c-\theta^{\mathbb{P}^d}\varphi^{CEA}(L,c)-\left[1-\theta^{\mathbb{P}^d}\right]\varphi^{CEL}(L,c)=\\&=\theta^{\mathbb{P}^d}\left[c-\varphi^{CEA}(L,c)\right]+\left[1-\theta^{\mathbb{P}^d}\right]\left[c-\varphi^{CEL}(L,c)\right]\quad=\\&=\left[1-\theta^{\mathbb{P}}\right]\left[c-\varphi^{CEA}(L,c)\right]+\theta^{\mathbb{P}}\left[c-\varphi^{CEL}(L,c)\right]\quad=\\&=\left[1-\theta^{\mathbb{P}}\right]\varphi^{CEL}(E,c)+\theta^{\mathbb{P}}\varphi^{CEA}(E,c)=\varphi^{CC}(E,c),\end{aligned}$$

and thus φ^{CC} is self-dual.

To prove that φ^{CC} is population monotonic consider a given set of agents $\mathcal{N}$ and a problem $\mathbb{P}=(E,c)$ for this population. Now assume that $\mathcal{N}$ is enlarged by incorporating an additional agent $i\notin\mathcal{N}$ with claim $c_i>0$. This allows to study the "enlarged problem" involving agents in $\mathcal{N}^+=\mathcal{N}\cup\{i\}$ facing the problem $\mathbb{P}^+=\left(E,c^+\right)=(E,(c,c_i))$. Note that for $E=0$, population monotonicity is trivially fulfilled. Thus, assume $E>0$, which implies that

$$\theta^{\mathbb{P}^+}=\frac{E}{\sum_{j\in\mathcal{N}^+}c_j}<\frac{E}{\sum_{j\in\mathcal{N}}c_j}=\theta^{\mathbb{P}}. \tag{14}$$

Let μ^+ be the unique solution to $\sum_{j\in\mathcal{N}^+}\max\left\{0,c_j-\mu^+\right\}=E$, while λ^+ denotes the unique solution to $\sum_{j\in\mathcal{N}^+}\min\left\{c_j,\lambda^+\right\}=E$.

Let us fix agent $j\in\mathcal{N}$. Note again that when $c_j=0$, $\varphi_j^{CC}(E,c)=\varphi_j^{CC}\left(E,c^+\right)=0$. Therefore we concentrate on the case where $c_j>0$. Assume that $\theta^{\mathbb{P}^+}\ge 0.5$. Then, by (14), $\theta^{\mathbb{P}}>\theta^{\mathbb{P}^+}\ge 0.5$. Moreover it is also fulfilled that $\mu\le\mu^+\le\lambda^+\le\lambda$. Consider the following cases, that exhaust all the possibilities.

(a) $c_j\le\mu$. Then, $\varphi_j^{CC}(E,c)=\theta^{\mathbb{P}}c_j>\theta^{\mathbb{P}^+}c_j=\varphi_j^{CC}\left(E,c^+\right)$.
(b) $\mu<c_j\le\mu^+$. Then $\varphi_j^{CC}\left(E,c^+\right)=\theta^{\mathbb{P}^+}c_j<\theta^{\mathbb{P}}c_j\le\theta^{\mathbb{P}}c_j+\left[1-\theta^{\mathbb{P}}\right](c_j-\mu)=\varphi_j^{CC}(E,c)$.
(c) $\mu^+<c_j\le\lambda^+$. Then $\varphi_j^{CC}\left(E,c^+\right)=c_j-\left[1-\theta^{\mathbb{P}^+}\right]\mu^+<c_j-\left[1-\theta^{\mathbb{P}}\right]\mu=\varphi_j^{CC}(E,c)$.
(d) $\lambda^+<c_j\le\lambda$. Then, $\varphi_j^{CC}\left(E,c^+\right)=\theta^{\mathbb{P}^+}\lambda^++\left[1-\theta^{\mathbb{P}^+}\right](c_j-\mu^+)=c_j-\left[1-\theta^{\mathbb{P}^+}\right]\mu^++(\lambda^+-c_j)\theta^{\mathbb{P}^+}<c_j-\left[1-\theta^{\mathbb{P}^+}\right]\mu^+\le c_j-\left[1-\theta^{\mathbb{P}}\right]\mu=\varphi_j^{CC}(E,c)$.
(e) $\lambda<c_j$. Then, $\varphi_j^{CC}\left(E,c^+\right)=\theta^{\mathbb{P}^+}\lambda^++\left[1-\theta^{\mathbb{P}^+}\right](c_j-\mu^+)<\theta^{\mathbb{P}}\lambda+\left[1-\theta^{\mathbb{P}}\right](c_j-\mu)=\varphi_j^{CC}(E,c)$.

Now, if we assume that $\theta^{\mathbb{P}}\le 0.5$, we obtain that $\mu\ge\mu^+\ge\lambda^+\ge\lambda$. Therefore we can consider again the five cases above yielding the desired result.

Finally assume that $\theta^{\mathbb{P}^+}<0.5<\theta^{\mathbb{P}}$. In this case we have four possible configurations for the values of μ, μ^+, λ and λ^+:

(i) $\mu\le\lambda^+<\mu^+\le\lambda$
(ii) $\mu\le\lambda^+<\lambda<\mu^+$
(iii) $\lambda^+<\mu<\mu^+\le\lambda$
(iv) $\lambda^+<\mu<\lambda<\mu^+$

For each of this configurations we can consider again the five possible cases for the position of c_j, yielding the desired result.

We now show that φ^{CC} does not satisfy invariance under claims truncation. Let us consider the claims problem $(E,c)=(600,(480,720,1200))$. Then, $\varphi^{CC}(E,c)=(50,95,455)$. Nevertheless, as in the truncated claims problem $(E,\bar{c})=(600,(480,600,600))$ agents 2 and 3 have the same claim, they are allocated the same amount, which differs from the proposal for the initial problem.

To see that φ^{CC} does not satisfy minimal rights first, let us consider the claims problem $(E,c)=(5100,(750,1500,2250,3000))$. In this case $m(E,c)=(0,0,0,600)$. Therefore, $\varphi^{CC}(E,c)=(558,1274,1514,1754)$, which differs from

$$\begin{aligned}&m(E,c)+\varphi^{CC}\left(E-\textstyle\sum_{i\in\mathcal{N}}m_i(E,c),c-m(E,c)\right)\quad=\\&(0,0,0,600)+(541.3,1128.3,1389.1,1441.3)\\&\qquad=(541.3,1128.3,1389.1,2041.3).\end{aligned}$$

To show that φ^{CC} does not meet neither composition down nor composition up, we consider the "high compromising degree" instance described in Example 1, in which $E=2100$ and $c=(750,1500,2250,3000)$. Assume a drop in the estate from the initial $E=2100$ to $E'=1800$. Composition down implies that $\varphi^{CC}\left(E',c\right)=\varphi^{CC}\left(E',\varphi^{CC}(E,c)\right)$. Nevertheless, as $\varphi^{CC}(E,c)=(147,147,633,1173)$,

$$\begin{aligned}\varphi^{CC}\left(E',c\right)&=\varphi^{CC}(1800,(750,1500,2250,3000))\\&=(108,108,507,1077)\neq\\&\neq(136.3,136.3,622.3,905.1)\\&=\varphi^{CC}(1800,(147,147,633,1173))\\&=\varphi^{CC}\left(E',\varphi^{CC}(E,c)\right).\end{aligned}$$

Now, suppose that the estate increases from the initial $E=2100$ to $E''=2400$. Composition up implies that $\varphi^{CC}\left(E'',c\right)=\varphi^{CC}(E,c)+\varphi^{CC}\left(E''-E,c-\varphi^{CC}(E,c)\right)$. Nevertheless,

$$\begin{aligned}\varphi^{CC}\left(E'',c\right)&=\varphi^{CC}(2400,(750,1500,2250,3000))\\&=(192,226,736,1246)\neq\\&\neq(147,147,633,1173)\\&\quad+(4.16,4.16,46.66,245)=\\&=\varphi^{CC}(2100,(750,1500,2250,3000))\\&\quad+\varphi^{CC}(300,(603,1353,1617,1827)).\end{aligned}$$

To conclude this proof we show that φ^{CC} fails to be consistent. Reconsider the instance described in Example 1, in which $E=2100$ and $c=(750,1500,2250,3000)$. In this case $\varphi^{CC}(E,c)=(147,147,633,1173)$. Note that $\varphi_1^{CC}(E,c)+\varphi_4^{CC}(E,c)=1320$. Now consider the two-agent problem $\left(E',c'\right)=(1320,(750,3000))$ involving creditors 1 and 4. Consistency implies that for each $i\in\{1,4\}$, $\varphi_i^{CC}(E,c)=\varphi_i^{CC}\left(E',c'\right)$. Nevertheless,

$\varphi_1^{CC}(E,c) \neq 232.32 = \varphi_1^{CC}(E',c')$; and $\varphi_4^{CC}(E,c) \neq 1087.68 = \varphi_4^{CC}(E',c')$. ■

For the sake of completeness, one might be interested in exploring the solution reversing the roles of the CEA and the CEL in the Compromise Convex solution. In such a case we obtain a new solution that is described as

$$\varphi^{RCC}(E,c) = \varphi^{CEL}\left(\theta^{\mathbb{P}}E, \theta^{\mathbb{P}}c\right) + \varphi^{CEA}\left(\left(1-\theta^{\mathbb{P}}\right)E, \left(1-\theta^{\mathbb{P}}\right)c\right).$$

In this case, it is immediate that by using $\sigma^{\mathbb{P}}$ instead of $\theta^{\mathbb{P}}$ in the CC solution, we obtain the same proposal; that is,

$$\varphi^{RCC}(E,c) = \varphi^{CEA}\left(\sigma^{\mathbb{P}}E, \sigma^{\mathbb{P}}c\right) + \varphi^{CEL}\left(\left(1-\sigma^{\mathbb{P}}\right)E, \left(1-\sigma^{\mathbb{P}}\right)c\right).$$

The normative analysis of this Reverse Compromise Convex solution runs in an analogous way as the one we did with the CC solution.

5. Concluding remarks

Several solutions for claims problems share a common primitive description. Any given problem is split into two different subproblems, named the main and the secondary problems respectively. Then, each subproblem is solved according to some equity guidelines, and the solution to the initial problem comes from the addition of the solutions to the two subproblems.

A common oversight exhibited by all these solutions comes from the fact that the way in which the two subproblems are generated follows a rigid formula, and thus it is not sensitive enough to the specific data of the problem. Our approach comes from considering an endogenous way to split the initial problem so that the magnitude of each subproblem captures the difficulty of finding a consensus about how to split the available amount of resource among the agents. This approach allows to provide a new interpretation to the Proportional solution as an agents' compromise from piece-wise mixing solutions.

An alternative solution we propose is the Compromise Convex solution providing an equitable distribution of the resources, which is sensitive to the characteristics of the particular problem being solved.

Appendix. Standard properties of solutions for claims problems

For the sake of completeness, this appendix is devoted to describe the main properties used along the literature to justify the adoption of certain solutions. These properties appear in alphabetical order.

Let us consider a given solution for claims problem, say φ. Solution φ satisfies:

(a) ***Anonymity*** if for each problem (E,c), agent i and any permutation $\pi : \mathcal{N} \to \mathcal{N}$, $\varphi_i(E,c) = \varphi_{\pi(i)}(E, \pi(c))$, where $\pi(c) = \left(c_{\pi(1)}, \ldots, c_{\pi(i)}, \ldots, c_{\pi(n)}\right)$.

(b) ***Claim Monotonicity*** if for any two problems (E,c) and (E,c') such that $c_i < c_i'$, whereas $c_j = c_j'$ for each $j \neq i$, $\varphi_i(E,c) \leq \varphi_i(E,c')$.

(c) ***Composition Down*** if for each problem (E,c), and any positive scalar $0 \leq E' < E$, $\varphi(E',c) = \varphi(E', \varphi(E,c))$.

(d) ***Composition Up*** if for any two problems (E,c) and (E',c) with $E < E'$, $\varphi(E',c) = \varphi(E,c) + \varphi(E'-E, c-\varphi(E,c))$.

(e) ***Consistency*** if for each problem (E,c) and any subset of agents $\mathcal{S} \subseteq \mathcal{N}$,

$$(\varphi(E,c))_{\mathcal{S}} = \varphi\left(\sum_{i\in\mathcal{S}} \varphi_i(E,c), c_{\mathcal{S}}\right).$$

(f) ***Continuity*** if for any converging succession of problems $\left\{\left(E^t,c^t\right)\right\} \to (E,c)$, the succession of solutions $\left\{\varphi\left(E^t,c^t\right)\right\}$ converges to $\varphi(E,c)$.

(g) ***Invariance under Claims Truncation*** if for each problem (E,c), $\varphi(E,c) = \varphi(E,\bar{c})$, where $\bar{c}_i = \min\{c_i, E\}$.

(h) ***Minimal Rights First*** if for each problem (E,c) and any agent i, $\varphi_i(E,c) = m_i(E,c) + \varphi_i\left(E - \sum_{j=1}^n m_j(E,c), c - m(E,c)\right)$, where for each agent i, his minimal right is defined as $m_i(E,c) = \max\left\{0, E - \sum_{j\neq i} c_j\right\}$.

(i) ***Order Preservation*** if for each problem (E,c) and any two agents, say i and j, $c_i \leq c_j$ implies that $\varphi_i(E,c) \leq \varphi_j(E,c)$.

(j) ***Population Monotonicity*** if for any set of agents $\mathcal{N}$, agent $i \notin \mathcal{N}$ and problems (E,c) and (E,c^+), where $c^+ = (c, c_i)$, $\varphi_j(E,c^+) \leq \varphi_j(E,c)$ for all $j \in \mathcal{N}$.

(k) ***Resource Monotonicity*** if for any two problems (E,c) and (E',c) such that $E < E'$, $\varphi_i(E,c) \leq \varphi_i(E',c)$ for each agent i.

(l) ***Self-duality*** if for each problem (E,c), $\varphi(E,c) = c - \varphi(C-E,c)$.

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