Comment on "Emergence of the Cotton tensor for describing gravity"

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Recently, it has been claimed by J. Harada [Phys. Rev. D **103**, L121502 (2021) that the Cotton tensor can describe the effects of gravity beyond general relativity. In this short comment, we show that the Cotton tensor is already present in general relativity. Moreover, we show that Harada's proposal is equivalent to general relativity concerning the equation of motion both for the sources and for the trace-free part of the curvature. In essence, what remains completely free in Harada's theory are the field equations relating the part of the curvature which is locally determined by the energy-momentum distribution.

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Harada has recently obtained [1] the following field equations for gravity:

$$C_{\nu\rho\sigma} = 16\pi G \nabla_\mu T^\mu{}_{\nu\rho\sigma}, \tag{1}$$

where $C_{\nu\rho\sigma}$ stands for the Cotton tensor, defined as

$$C_{\nu\rho\sigma} = \nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho} - \frac{1}{6}(g_{\nu\sigma}\nabla_\rho R - g_{\nu\rho}\nabla_\sigma R) \tag{2}$$

and

$$\begin{aligned} T^{\mu\nu\rho\sigma} \equiv & \frac{1}{2}(g^{\mu\nu}T^{\rho\sigma} - g^{\nu\rho}T^{\mu\sigma} - g^{\mu\sigma}T^{\nu\rho} + g^{\nu\sigma}T^{\mu\rho}) \\ & - \frac{1}{6}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma})T, \end{aligned} \tag{3}$$

where $T^{\mu\nu}$ is the energy-momentum tensor.

As noted by the author, energy-momentum conservation is guaranteed, as expressed by

$$g^{\nu\sigma}C_{\nu\rho\sigma} = \nabla_\mu G^\mu{}_\rho = 16\pi G \nabla_\mu T^\mu{}_\rho = 0, \tag{4}$$

where $G_{\mu\nu}$ stands for the Einstein tensor.

First of all, note that Eq. (1) can be stated as

$$C_{\nu\rho\sigma} = 16\pi G\left(T_{\nu[\sigma;\rho]} - \frac{1}{3}g_{\nu[\sigma}T_{,\rho]}\right), \tag{5}$$

by direct substitution of Eq. (3) into Eq. (1).

Let us now turn to general relativity. The well-known decomposition of the Riemann tensor into its components that are irreducible with respect to the Lorentz group is

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma} + \frac{R}{12}g_{\mu\nu\rho\sigma}, \tag{6}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, $E_{\mu\nu\rho\sigma} \equiv g_{\mu\nu\lambda[\rho}S^\lambda{}_{\sigma]}$, $S^\lambda{}_\sigma \equiv R^\lambda{}_\sigma - \frac{R}{4}\delta^\lambda{}_\sigma$ and $g_{\mu\nu\rho\sigma} \equiv 2g_{\mu[\rho}g_{\sigma]\nu}$.

If we insert Eq. (6) into the Bianchi identities, written as

$$\nabla_\sigma {}^*R^{*\mu\nu\rho\sigma} = 0, \tag{7}$$

($*$ denotes the Hodge dual), we get

$$\nabla_\sigma C^{\mu\nu\rho\sigma} = \nabla_\sigma E^{\mu\nu\rho\sigma} - \frac{g^{\mu\nu\rho\sigma}}{12}R_{,\sigma}. \tag{8}$$

Finally, using Einstein's field equations,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{9}$$

we arrive to [2]

$$C_{\nu\rho\sigma} = 16\pi G\left(T_{\nu[\sigma;\rho]} - \frac{1}{3}g_{\nu[\sigma}T_{,\rho]}\right), \tag{10}$$

which are identical to Harada's field equations.

At this point, some comments are in order: (i) The equations of motion for the sources are $\nabla_\mu T^\mu{}_\nu = 0$ in both Harada and Einstein's cases. (ii) The field equations for the Cotton tensor giving that part of the curvature at a point that depends on the energy-momentum distribution at other points are given by Eq. (5) in both Harada and Einstein's cases, and importantly, (iii) the part of the curvature which is locally determined by the energy-momentum distribution by Einstein's equations does not have an equivalent in Harada's approach. Therefore, there are no equations relating the part of the curvature which is locally

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determined by the energy-momentum distribution in Harada's model.

Regarding this last point, we make the following clarification. In Ref. [1], the author *assumes* Einstein's field equations (without cosmological constant) to be valid. Specifically, the author states that any solution of the Einstein equations satisfies the new field equations. Even more, he states that Eq. (1) has other solutions which are not solutions of the Einstein equations, and therefore, the new field equations have more information than general relativity. We strongly disagree with the author regarding this last fact. As we have shown before, the new field equations are identical to that of general relativity with the exception of the absence of a field equation relating the curvature which is locally determined by $T^{\mu\nu}$. Therefore, in this sense, the new field equations have less information than those of general relativity.

We would like to conclude highlighting some points:

(i) The emergence of the Cotton tensor is already present in general relativity. In fact, it can be seen from Eq. (10) that the nonlocal part of the vacuum equations of general relativity can be stated as $C_{\mu\nu\rho} = 0$, as in Harada's theory. However, Einstein's gravity *also* demands the local vacuum equations, $R_{\mu\nu} = 0$, to be satisfied. Therefore, the spherically symmetric solution reported by Harada in [1] is not allowed in Einstein's theory essentially due to the local vacuum equations (Ricci flatness).

(ii) Although Harada's attempt of finding some gravitational field equations to be written *a la Maxwell* is very elegant, the rhs of Harada's gravitational field equations [Eq. (1)] is nonlocal, and consequently, no Newtonian limit can be rigorously extracted at the level of the field equations (of course, this can be done by working with a particular solution, as worked out in [1]). In fact, the Newtonian limit of general relativity is reached by using Einstein's equations because only ${R^i}_{0j0} = \partial^i \partial_j \phi$ ($i = 1, 2, 3$ and ϕ is the Newtonian potential) survives in this limit (contracting these equations, we arrive to $R_{00} = \Delta\phi$). Therefore, the nonlocal Maxwell-like equations for the Weyl tensor are not useful to study the aforementioned limit.

(iii) Concerning the search for Maxwell-like gravitational field equations, it must be noted that Harada's equations (which I remark again are exactly the same as Einstein's equations for the evolution of the Weyl or Cotton tensors) can be explicitly written in Maxwell-like form as

$$\nabla_\sigma C^{\mu\nu\rho\sigma} = 16\pi G J^{\mu\nu\rho}, \tag{11}$$

where the current, J, can be formed by using Eq. (10) (an explicit expression can be found in [3]).

As we have shown, Harada's theory is equivalent to general relativity concerning the equation of motion both for the sources and for the trace-free part of the curvature. In essence, what remains completely free in Harada's theory are the field equations relating the part of the curvature which is locally determined by the energy-momentum distribution.

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[1] J. Harada, Phys. Rev. D **103**, L121502 (2021).

[2] W. Kundt and M. Trmper, Republication of: Contributions to the theory of gravitational radiation fields. Exact solutions of the field equations of the general theory of relativity V., Gen. Relativ. Gravit. **48**, 44 (2016).

[3] S. W. Hawking and G. F. R. Ellis, in *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973), See Eq. 4.28, pp. 85.