# Alternative representations of the normal cone to the domain of supremum functions and subdifferential calculus* 

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#### Abstract

The first part of the paper provides new characterizations of the normal cone to the effective domain of the supremum of an arbitrary family of convex functions. These results are applied in the second part to give new formulas for the subdifferential of the supremum function, which use both the active and nonactive functions at the reference point. Only the data functions are involved in these characterizations, the active ones from one side, together with the nonactive functions multiplied by some appropriate parameters. In contrast with previous works in the literature, the main feature of our subdifferential characterization is that the normal cone to the effective domain of the supremum (or to finite-dimensional sections of this domain) does not appear. A new type of optimality conditions for convex optimization is established at the end of the paper.


Key words. Normal cone, supremum of convex functions, subdifferentials, convex optimization, optimality conditions.

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## 1 Introduction

Given the pointwise supremum $f:=\sup _{t \in T} f_{t}$ of a family of convex functions $f_{t}: X \rightarrow$ $\mathbb{R} \cup\{+\infty\}, t \in T, T$ being a non-empty and arbitrary, defined on a separated locally convex space $X$, many researchers have addressed the paradigmatic problem of characterizing the subdifferential of the supremum, $\partial f(x)$, at any point $x$ of the effective domain

[^0]of $f$. These characterizations are usually given in terms of the (approximate-) subdifferentials of the data functions, $\partial_{\varepsilon} f_{t}(x), t \in T, \varepsilon \geq 0$, and, in the most general cases, in terms also of the normal cone to the effective domain of $f$ or to finite-dimensional sections of it. The interest of this problem comes from the fact that many convex functions, such as the Fenchel conjugate, the sum, the composition with affine mappings, etc., can be expressed as the supremum of affine or convex functions. Therefore, getting formulas for the subdifferential of the supremum is expected to play a crucial role in convex optimization and variational analysis. Some remarkable contributions to the topic are: Brøndsted 11, Ioffe [14], Ioffe \& Levin [15], Ioffe \& Tikhomirov [16], Levin [17], Pschenichnyi [26], Rockafellar [27], Valadier [29], etc. In [28] the historical origins of the issue are traced out. More recently, in a series of papers (3), [4], [10, [11, etc.) new characterizations of the subdifferential supremum in different settings are provided, and some related calculus rules in convex analysis are derived as consequences.

If the functions $f_{t}, t \in T$, are proper convex and lower semicontinuous; that is, $\left\{f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$, and we additionally assume that the relative interior of the effective domain of $f$ is non-empty, i.e. $\operatorname{ri}(\operatorname{dom} f) \neq \emptyset$, in [11, Lemma 3] it is established that

$$
\begin{equation*}
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{\operatorname{dom} f}(x)\right), \quad \text { for all } x \in \operatorname{dom} f, \tag{1}
\end{equation*}
$$

where $\overline{\text { co }}$ stands for the $w^{*}$-closed convex hull, $\partial_{\varepsilon} f_{t}(x)$ is the $\varepsilon$-subdifferential of $f_{t}$ at $x$, and

$$
T_{\varepsilon}(x):=\left\{t \in T: f_{t}(x) \geq f(x)-\varepsilon\right\} .
$$

In [22, Theorem 4], formula (1) is also derived under different assumptions, namely if cone $(\operatorname{dom} f-x))$ is closed or $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} f-x)) \neq \emptyset$, where cone $(A)$ is the convex cone generated by $A$.

When these interiority/closedness assumptions are removed, the price that has to be paid is the need of involving the family

$$
\mathcal{F}(x):=\{L \subset X: L \text { is a finite-dimensional linear subspace such that } x \in L\} .
$$

In this very general framework, the following characterization is established in 11, Theorem 4]:

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x), \varepsilon>0} \overline{\operatorname{co}}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right) \text {, for all } x \in \operatorname{dom} f . \tag{2}
\end{equation*}
$$

Observe that $\operatorname{ri}(L \cap \operatorname{dom} f) \neq \emptyset$. The reader will find related formulas in [19].
In the so-called compact setting the following result, involving only the active functions at the reference point, is established in [4, Theorem 3.8] under the standard hypothesis (22); i.e., $T$ is compact and the mappings $t \mapsto f_{t}(z), z \in X$, are upper semicontinuous (usc, in brief):

$$
\partial f(x)=\bigcap_{L \in \mathcal{F}(x), \varepsilon>0} \overline{\overline{c o}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right) .
$$

One way to get rid of these normal cones is to impose additional assumptions as the finiteness and continuity of $f$ at $x$, in which case (2) gives rise to ([11, Corollary 10]; see, also, [30], for normed spaces):

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)
$$

Lemma 5 in [11] yields some characterizations of $\mathrm{N}_{\mathrm{dom}} f(x)$. Precisely,

$$
\begin{align*}
x^{*} \in \mathrm{~N}_{\operatorname{dom} f}(x) & \Leftrightarrow\left(x^{*},\left\langle x^{*}, x\right\rangle\right) \in\left[\overline{\mathrm{co}}\left(\cup_{t \in T} \operatorname{gph} f_{t}^{*}\right)\right]_{\infty} \\
& \Leftrightarrow\left(x^{*},\left\langle x^{*}, x\right\rangle\right) \in\left[\overline{\mathrm{co}}\left(\cup_{t \in T} \text { epi } f_{t}^{*}\right)\right]_{\infty}, \tag{3}
\end{align*}
$$

where $\operatorname{gph} f_{t}^{*}$ and epi $f_{t}^{*}$ represent the graph and the epigraph of the conjugate of $f_{t}$, respectively, and $[\cdot]_{\infty}$ defines the recession cone. In the linear case, i.e. if $f(x):=$ $\sup \left\{\left\langle a_{t}^{*}, x\right\rangle-b_{t}: t \in T\right\}$, with $a_{t}^{*} \in X^{*}$ and $b_{t} \in \mathbb{R}$, we get ([11, Corollary 7])

$$
x^{*} \in \mathrm{~N}_{\operatorname{dom} f}(x) \Leftrightarrow\left(x^{*},\left\langle x^{*}, x\right\rangle\right) \in\left[\overline{\operatorname{co}}\left((\theta, 0) \cup\left\{\left(a_{t}^{*}, b_{t}\right), t \in T\right\}\right)\right]_{\infty}
$$

where $\theta$ is the origin in $X^{*}$.
Another interesting problem in optimization consists of characterizing the normal cone to sublevel sets (see, e.g. [12, 2] and references therein). Observe, for instance, that if $g \in \Gamma_{0}(X)$ and $[g \leq 0]$ is the 0 -sublevel set, by taking $f:=\sup _{\alpha \geq 0}(\alpha g)=\mathrm{I}_{[g \leq 0]}$, we obtain that

$$
\mathrm{N}_{[g \leq 0]}(x)=\mathrm{N}_{\operatorname{dom} f}(x) .
$$

The main contribution of this paper consists of formulating alternative characterizations of $\partial f(x)$, relying exclusively on the data functions and not on any normal cone. In other words, the normal cone $\mathrm{N}_{L \cap \operatorname{dom}} f(x)$ does not appear explicitly in the new subdifferential formulas, and consequently, there is no need of intersecting over finite-dimensional subspaces $L$, as in previous quoted works. Extensions to the non-compact framework will be investigated in a forthcoming work, using different approaches including wellknown qualifications, like the strong CHIP, SECQ, linear regularity, Farkas-Minkowski, etc. (6, 7, 13, 18, 20]).

The structure of the paper is the following. After Section 2 devoted to notation and preliminary results, in Section 3 new characterizations of $\mathrm{N}_{\operatorname{dom} f}(x)$ are given in terms exclusively of $\partial_{\varepsilon} f_{t}(x), t \in T$, which are independent of $\varepsilon$ and much simpler than those in (3). The main result in this section is Theorem 6. Based on the results established in Section 3. Theorems 12 and 13 in Section 4 provide new formulas for the subdifferential of the supremum, $\partial f(x)$, involving both, the active functions at the reference point $x$, and also the rest of the functions but affected by a multiplying parameter. Finally, new optimality conditions for the convex optimization problem with infinitely many constraints are proposed.

## 2 Notation and preliminary results

Let $X$ be a (real) separated locally convex space (lcs, for short), whose topological dual space, $X^{*}$, is endowed with the $w^{*}$-topology; hence, $X^{* *}:=\left(X^{*}\right)^{*} \equiv X$. The spaces $X$ and $X^{*}$ are paired in duality by the bilinear form $\left(x^{*}, x\right) \in X^{*} \times X \mapsto\left\langle x^{*}, x\right\rangle:=x^{*}(x)$. The zero vectors in $X$ and $X^{*}$ are denoted by $\theta$. The basis of closed, convex and balanced neighborhoods of $\theta$, in both $X$ and $X^{*}$, called $\theta$-neighborhoods, is represented by $\mathcal{N}$. We use the notation $\overline{\mathbb{R}}:=\mathbb{R} \cup\{-\infty,+\infty\}$ and $\mathbb{R}_{\infty}:=\mathbb{R} \cup\{+\infty\}$, and adopt the conventions $(+\infty)+(-\infty)=(-\infty)+(+\infty)=+\infty, 0(+\infty):=+\infty$.

Given $k \geq 1$, we denote

$$
\begin{aligned}
\Delta_{k} & :=\left\{\left(\lambda_{1}, \cdots, \lambda_{k}\right) \geq 0: \lambda_{1}+\cdots+\lambda_{k}=1\right\}, \\
\Delta_{k}^{+} & :=\left\{\left(\lambda_{1}, \cdots, \lambda_{k}\right) \in \Delta_{k}: \lambda_{i}>0, i=1, \cdots, k\right\} .
\end{aligned}
$$

Given two sets $A$ and $B$ in $X$ (or in $X^{*}$ ), we define the Minkowski sum by

$$
\begin{equation*}
A+B:=\{a+b: a \in A, b \in B\}, \quad A+\emptyset=\emptyset+A=\emptyset, \tag{4}
\end{equation*}
$$

and, if $\Lambda \subset \mathbb{R}$,

$$
\Lambda A:=\{\lambda a: \lambda \in \Lambda, a \in A\}, \quad \Lambda \emptyset=\emptyset A=\emptyset,
$$

in particular, we write $\lambda A:=\{\lambda\} A, \lambda \in \mathbb{R}$.
By $\operatorname{co}(A)$ and cone $(A)$, we denote the convex and the conical convex hulls of the nonempty set $A$, respectively. In the topological side, $\operatorname{cl}(A)$ and $\bar{A}$ are indistinctly used for denoting the closure of $A$. When $A \subset X^{*}$, the closure is taken with respect to the $w^{*}$-topology, unless something else is explicitly stated.

Associated with a nonempty set $A \subset X$, we define the negative dual cone and the orthogonal subspace of $A$ as follows

$$
\begin{aligned}
& A^{-}:=\left\{x^{*} \in X^{*}:\left\langle x^{*}, x\right\rangle \leq 0 \text { for all } x \in A\right\}, \\
& A^{\perp}:=\left(-A^{-}\right) \cap A^{-}=\left\{x^{*} \in X^{*}:\left\langle x^{*}, x\right\rangle=0 \text { for all } x \in A\right\},
\end{aligned}
$$

respectively. Observe that $A^{-}=(\overline{\operatorname{cone}}(A))^{-}$. These concepts are defined similarly for sets in $X^{*}$. The so-called bipolar theorem establishes that

$$
\begin{equation*}
A^{--}:=\left(A^{-}\right)^{-}=\overline{\operatorname{cone}}(A) . \tag{5}
\end{equation*}
$$

If $A \subset X$, we define the normal cone to $A$ at $x$ by

$$
\mathrm{N}_{A}(x):= \begin{cases}(A-x)^{-}, & \text {if } x \in A, \\ \emptyset, & \text { if } x \in X \backslash A .\end{cases}
$$

If $A \neq \emptyset$ is convex and closed, $A_{\infty}$ represents its recession cone defined by

$$
A_{\infty}:=\{y \in X: x+\lambda y \in A \text { for some } x \in A \text { and all } \lambda \geq 0\} .
$$

Given a function $f: X \longrightarrow \overline{\mathbb{R}}$, its (effective) domain and epigraph are, respectively,

$$
\operatorname{dom} f:=\{x \in X: f(x)<+\infty\},
$$

and

$$
\operatorname{epi} f:=\{(x, \lambda) \in X \times \mathbb{R}: f(x) \leq \lambda\}
$$

We say that $f$ is proper when $\operatorname{dom} f \neq \emptyset$ and $f(x)>-\infty$ for all $x \in X$. The closed hull of $f$ is the function $\mathrm{cl} f: X \longrightarrow \overline{\mathbb{R}}$ whose epigraph is $\mathrm{cl}(\operatorname{epi} f)$. Moreover,

$$
\begin{equation*}
(\operatorname{cl} f)(x)=\liminf _{x^{\prime} \rightarrow x} f\left(x^{\prime}\right)=\sup _{V \in \mathcal{N}} \inf \left\{f\left(x^{\prime}\right): x^{\prime} \in x+V\right\} . \tag{6}
\end{equation*}
$$

The convex hull of $f$, co $f: X \longrightarrow \overline{\mathbb{R}}$, is the largest convex function which is dominated by $f$. Equivalently,

$$
\begin{align*}
(\cos f)(x) & =\inf \{\mu:(x, \mu) \in \operatorname{co}(\text { epi } f)\}  \tag{7}\\
& =\inf \left\{\sum_{i=1}^{k} \lambda_{i} f\left(x_{i}\right): \lambda \in \Delta_{k}^{+}, \sum_{i=1}^{k} \lambda_{i} x_{i}=x, k \in \mathbb{N}\right\} . \tag{8}
\end{align*}
$$

The closed convex hull of $f$ is the convex lower semicontinuous (lsc, in brief) function $\overline{\operatorname{co}} f: X \longrightarrow \overline{\mathbb{R}}$ such that

$$
\operatorname{epi}(\overline{c o} f)=\overline{\mathrm{co}}(\operatorname{epi} f) .
$$

Obviously, $\overline{\operatorname{co}} f \leq \operatorname{cl} f \leq f$.
Given $x \in X$ and $\varepsilon \in \mathbb{R}$, the $\varepsilon$-subdifferential (or the approximate subdifferential) of $f$ at $x$ is

$$
\begin{equation*}
\partial_{\varepsilon} f(x)=\left\{x^{*} \in X^{*}: f(y) \geq f(x)+\left\langle x^{*}, y-x\right\rangle-\varepsilon \text { for all } y \in X\right\}, \tag{9}
\end{equation*}
$$

when $x \in \operatorname{dom} f$, and $\partial_{\varepsilon} f(x):=\emptyset$ when $f(x) \notin \mathbb{R}$ or $\varepsilon<0$. The subdifferential of $f$ at $x$ is $\partial f(x):=\partial_{0} f(x)$.

We shall use the following relation (e.g., [31, Exercise 2.23])

$$
\begin{equation*}
\mathrm{N}_{\operatorname{dom} f}(x)=\left(\partial_{\varepsilon} f(x)\right)_{\infty}, \tag{10}
\end{equation*}
$$

where $f \in \Gamma_{0}(X), x \in \operatorname{dom} f$, and $\varepsilon>0$.
The Fenchel conjugate of $f$ is the function $f^{*}: X^{*} \longrightarrow \overline{\mathbb{R}}$ given by

$$
f^{*}\left(x^{*}\right):=\sup \left\{\left\langle x^{*}, x\right\rangle-f(x): x \in X\right\} .
$$

It is well-known that

$$
\begin{equation*}
\partial_{\varepsilon} f(x)=\left\{x^{*} \in X^{*}: f(x)+f^{*}\left(x^{*}\right) \leq\left\langle x^{*}, x\right\rangle+\varepsilon\right\}, \tag{11}
\end{equation*}
$$

for all $\varepsilon \geq 0$, and

$$
\begin{equation*}
\partial f(x)=\cap_{\varepsilon>0} \partial_{\varepsilon} f(x) \tag{12}
\end{equation*}
$$

The following lemma gives a slight extension of the last relation, which is used later on.
Lemma 1 Consider a function $f \in \Gamma_{0}(X)$ and suppose that $x$ is a minimizer of $f$. Then, for every $M \geq 0$,

$$
\begin{equation*}
\partial f(x)=\cap_{\varepsilon>0} \overline{\operatorname{co}}\left(\partial_{\varepsilon} f(x) \cup \varepsilon \partial_{\varepsilon+M} f(x)\right)=\cap_{\varepsilon>0} \overline{\operatorname{co}}\left(\partial_{\varepsilon} f(x) \cup \varepsilon \partial_{\varepsilon+M} f^{+}(x)\right), \tag{13}
\end{equation*}
$$

where $f^{+}:=\max \{f, 0\}$ is the positive part of $f$.
Proof. The inclusions

$$
\begin{equation*}
\partial f(x) \subset \cap_{\varepsilon>0} \overline{\operatorname{co}}\left(\partial_{\varepsilon} f(x) \cup \varepsilon \partial_{\varepsilon+M} f(x)\right) \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial f(x) \subset \cap_{\varepsilon>0} \overline{\operatorname{co}}\left(\partial_{\varepsilon} f(x) \cup \varepsilon \partial_{\varepsilon+M} f^{+}(x)\right) \tag{15}
\end{equation*}
$$

follow easily from (12). We only need to prove the opposite inclusion in (14), since that the same argument is valid for such an inclusion in (15). Take $x^{*}$ in the right-hand side of (14). Then, for each fixed $\varepsilon>0$,

$$
x^{*}=\lim _{i}\left(\lambda_{i} y_{i}^{*}+\left(1-\lambda_{i}\right) \varepsilon z_{i}^{*}\right)
$$

for some nets $\left(\lambda_{i}\right)_{i} \subset[0,1],\left(y_{i}^{*}\right)_{i} \subset \partial_{\varepsilon} f(x)$ and $\left(z_{i}^{*}\right)_{i} \subset \partial_{\varepsilon+M} f(x)$. Thus, for each $y \in \operatorname{dom} f$,

$$
\begin{aligned}
\left\langle x^{*}, y-x\right\rangle & =\lim _{i}\left\langle\lambda_{i} y_{i}^{*}+\left(1-\lambda_{i}\right) \varepsilon z_{i}^{*}, y-x\right\rangle \\
& \leq \limsup _{i}\left(\lambda_{i}(f(y)-f(x)+\varepsilon)+\left(1-\lambda_{i}\right) \varepsilon(f(y)-f(x)+\varepsilon+M)\right) \\
& \leq f(y)-f(x)+\varepsilon+\varepsilon(f(y)-f(x)+\varepsilon+M),
\end{aligned}
$$

as $f(y) \geq f(x)$. Finally, since the last inequality holds for all $\varepsilon>0$, when $\varepsilon \downarrow 0$ we obtain

$$
\left\langle x^{*}, y-x\right\rangle \leq f(y)-f(x) \text { for all } y \in X,
$$

which shows that $x^{*} \in \partial f(x)$.
The support and the indicator functions of $A \subset X$ are, respectively,

$$
\sigma_{A}\left(x^{*}\right):=\sup \left\{\left\langle x^{*}, x\right\rangle: x \in A\right\}, x^{*} \in X^{*},
$$

with $\sigma_{\emptyset} \equiv-\infty$, and

$$
\mathrm{I}_{A}(x):= \begin{cases}0 & \text { if } x \in A \\ +\infty & \text { if } x \in X \backslash A\end{cases}
$$

It is known that, if $A$ is a closed convex set,

$$
\begin{equation*}
A_{\infty}=\left(\operatorname{dom} \sigma_{A}\right)^{-} \tag{16}
\end{equation*}
$$

or equivalently, by using (5),

$$
\begin{equation*}
\left(A_{\infty}\right)^{-}=\operatorname{cl}\left(\operatorname{dom} \sigma_{A}\right) \tag{17}
\end{equation*}
$$

Lemma 2 Consider nonempty sets $A$ and $A_{1}, \cdots, A_{k}$ in $X, k \geq 2$. Then

$$
\begin{equation*}
\left[\overline{\mathrm{co}}\left(A \cup\left(\cup_{i=1, \cdots, k} A_{k}\right)\right)\right]_{\infty}=\left(\overline{\mathrm{co}}\left(A \cup\left(A_{1}+\cdots+A_{k}\right)\right)\right)_{\infty} \tag{18}
\end{equation*}
$$

Proof. Obviously,

$$
\begin{aligned}
\operatorname{dom} \sigma_{A \cup\left(\cup_{i=1, \cdots, k} A_{k}\right)} & =\operatorname{dom}\left(\max \left\{\sigma_{A} ; \sigma_{A_{i}}, i=1, \cdots, k\right\}\right) \\
& =\operatorname{dom}\left(\max \left\{\sigma_{A}, \sigma_{A_{1}}+\cdots+\sigma_{A_{k}}\right\}\right) \\
& =\operatorname{dom}\left(\max \left\{\sigma_{A}, \sigma_{A_{1}+\cdots+A_{k}}\right\}\right) \\
& =\operatorname{dom}\left(\sigma_{A \cup\left(A_{1}+\cdots+A_{k}\right)}\right)
\end{aligned}
$$

and, by (17),

$$
\begin{aligned}
\left(\left[\overline{\operatorname{co}}\left(A \cup\left(\cup_{i=1, \cdots, k} A_{k}\right)\right)\right]_{\infty}\right)^{-} & =\operatorname{cl}\left(\operatorname{dom} \sigma_{\overline{\operatorname{co}}\left(A \cup\left(\cup_{i=1, \cdots, k} A_{k}\right)\right)}\right)=\operatorname{cl}\left(\operatorname{dom} \sigma_{A \cup\left(\cup_{i=1, \cdots, k} A_{k}\right)}\right) \\
& =\operatorname{cl}\left(\operatorname{dom}\left(\sigma_{A \cup\left(A_{1}+\cdots+A_{k}\right)}\right)\right)=\left[\left(\overline{\operatorname{co}}\left(A \cup\left(A_{1}+\cdots+A_{k}\right)\right)\right)_{\infty}\right]^{-}
\end{aligned}
$$

Thus, (18) follows from (5).
Lemma 3 Consider a family of nonempty sets $\left\{A_{t}, t \in T_{1} \cup T_{2}\right\} \subset X$, where $T_{1}$ and $T_{2}$ are disjoint nonempty sets. Then for every $m>0$ we have

$$
\begin{align*}
{\left[\overline{\operatorname{co}}\left(\cup_{t \in T_{1} \cup T_{2}} A_{t}\right)\right]_{\infty} } & =\left[\overline{\operatorname{co}}\left(\left(\cup_{t \in T_{1}} A_{t}\right) \cup\left(\cup_{t \in T_{2}} m A_{t}\right)\right)\right]_{\infty} \\
& =\left[\overline{\operatorname{co}}\left(\cup_{t_{1} \in T_{1}, t_{2} \in T_{2}}\left(A_{t_{1}}+m A_{t_{2}}\right)\right)\right]_{\infty} \tag{19}
\end{align*}
$$

Proof. Denote $T:=T_{1} \cup T_{2}$ and $A:=\overline{\mathrm{co}}\left(\cup_{t \in T} A_{t}\right)$. Then the functions $\varphi_{1}:=\sup _{t \in T_{1}} \sigma_{A_{t}}$, $\varphi_{2}:=\sup _{t \in T_{2}} \sigma_{m A_{t}}$ satisfy

$$
\begin{align*}
\varphi_{1}+\varphi_{2} & =\sup _{t_{1} \in T_{1}, t_{2} \in T_{2}}\left(\sigma_{A_{t_{1}}}+\sigma_{m A_{t_{2}}}\right) \\
& =\sup _{t_{1} \in T_{1}, t_{2} \in T_{2}} \sigma_{A_{t_{1}}+m A_{t_{2}}}=\sigma_{\cup_{t_{1} \in T_{1}, t_{2} \in T_{2}}\left(A_{t_{1}}+m A_{t_{2}}\right)} \tag{20}
\end{align*}
$$

and similarly

$$
\max \left\{\varphi_{1}, \varphi_{2}\right\}=\sigma_{\left(\cup_{t \in T_{1}} A_{t}\right) \cup\left(\cup_{t \in T_{2}} m A_{t}\right)}
$$

Since

$$
\operatorname{dom}\left(\varphi_{1}+\varphi_{2}\right)=\operatorname{dom}\left(\max \left\{\varphi_{1}, \varphi_{2}\right\}\right)=\operatorname{dom}\left(\max \left\{\varphi_{1}, m^{-1} \varphi_{2}\right\}\right)
$$

(17) yields

$$
\begin{aligned}
\left(\left[\overline{c o}\left(\cup_{t_{1} \in T_{1}, t_{2} \in T_{2}}\left(A_{t_{1}}+m A_{t_{2}}\right)\right)\right]_{\infty}\right)^{-} & =\left(\left[\overline{\operatorname{co}}\left(\left(\cup_{t \in T_{1}} A_{t}\right) \cup\left(\cup_{t \in T_{2}} m A_{t}\right)\right)\right]_{\infty}\right)^{-} \\
& =\left(\left[\overline{\mathrm{co}}\left(\left(\cup_{t \in T_{1}} A_{t}\right) \cup\left(\cup_{t \in T_{2}} A_{t}\right)\right)\right]_{\infty}\right)^{-},
\end{aligned}
$$

and we are done thanks to (5).
The following lemma provides the $\varepsilon$-subdifferential of the positive part of convex functions. It can be derived from [31, Corollary 2.8.11] but we prefer to give here a simple alternative proof based on [23, Lemma 1].

Lemma 4 Consider a function $f \in \Gamma_{0}(X)$ and let $x \in \operatorname{dom} f$. Then, for every $\varepsilon \geq 0$, we have

$$
\begin{equation*}
\partial_{\varepsilon} f^{+}(x)=\bigcup_{0 \leq \lambda \leq 1} \partial_{\varepsilon+\lambda f(x)-f^{+}(x)}(\lambda f)(x) \tag{21}
\end{equation*}
$$

where $(\lambda f)(z):=\lambda f(z), z \in X \quad$ (with $\left.0 f=\mathrm{I}_{\operatorname{dom} f}\right)$.
Proof. By [23, Lemma 1] we have that

$$
\left(f^{+}\right)^{*}=\operatorname{cl}\left(\inf _{\lambda \in[0,1]}\left(\lambda f+\mathrm{I}_{\mathrm{dom} f}\right)^{*}\right) \equiv \operatorname{cl}\left(\inf _{\lambda \in[0,1]}(\lambda f)^{*}\right) .
$$

We take $x^{*} \in \partial_{\varepsilon} f^{+}(x)$ and fix $n \geq 1$. Then, by (11),

$$
\begin{aligned}
\operatorname{cl}\left(\inf _{\lambda \in[0,1]}(\lambda f)^{*}\right)\left(x^{*}\right)+f^{+}(x) & =\left(f^{+}\right)^{*}\left(x^{*}\right)+f^{+}(x) \\
& \leq\left\langle x^{*}, x\right\rangle+\varepsilon<\left\langle x^{*}, x\right\rangle+\varepsilon+\frac{1}{n},
\end{aligned}
$$

and so, there exists a net $\left(x_{n, i}^{*}\right)_{i}$ which $\left(w^{*}-\right)$ converges to $x^{*}$ and satisfies

$$
\begin{aligned}
\lim _{i}\left(\inf _{\lambda \in[0,1]}(\lambda f)^{*}\right)\left(x_{n, i}^{*}\right)+f^{+}(x) & <\left\langle x^{*}, x\right\rangle+\varepsilon+\frac{1}{n} \\
& =\lim _{i}\left\langle x_{n, i}^{*}, x\right\rangle+\varepsilon+\frac{1}{n} .
\end{aligned}
$$

Without loss of generality, we may suppose that

$$
\inf _{\lambda \in[0,1]}(\lambda f)^{*}\left(x_{n, i}^{*}\right)+f^{+}(x)<\left\langle x_{n, i}^{*}, x\right\rangle+\varepsilon+\frac{1}{n} \text { for all } i
$$

Hence, there exists some $\lambda_{n, i} \in[0,1]$ such that

$$
\left(\lambda_{n, i} f\right)^{*}\left(x_{n, i}^{*}\right)+f^{+}(x)<\left\langle x_{n, i}^{*}, x\right\rangle+\varepsilon+\frac{1}{n} \text { for all } i,
$$

and consequently, using a diagonal argument, we find $\left(x_{n, i(n)}^{*}\right)_{n}$ and $\left(\lambda_{n, i(n)}\right)_{n}$ such that $\left(x_{n, i(n)}^{*}\right)_{n}\left(w^{*}-\right)$ converges to $x^{*}$ and

$$
\left(\lambda_{n, i(n)} f\right)^{*}\left(x_{n, i(n)}^{*}\right)+f^{+}(x)<\left\langle x_{n, i(n)}^{*}, x\right\rangle+\varepsilon+\frac{1}{n} \text { for all } n \geq 1
$$

Equivalently, we have that

$$
\left(\lambda_{n, i(n)} f\right)^{*}\left(x_{n, i(n)}^{*}\right)+\left(\lambda_{n, i(n)} f\right)(x)<\left\langle x_{n, i(n)}^{*}, x\right\rangle+\left(\lambda_{n, i(n)} f\right)(x)-f^{+}(x)+\varepsilon+\frac{1}{n},
$$

so that $\left(\lambda_{n, i(n)} f\right)(x)-f^{+}(x)+\varepsilon+\frac{1}{n}>0$, because $\left\langle x_{n, i(n)}^{*}, x\right\rangle \leq\left(\lambda_{n, i(n)} f\right)^{*}\left(x_{n, i(n)}^{*}\right)+$ $\left(\lambda_{n, i(n)} f\right)(x)$, and $x_{n, i(n)}^{*} \in \partial_{\left(\lambda_{n, i(n)} f\right)(x)-f+(x)+\varepsilon+\frac{1}{n}}\left(\lambda_{n, i(n)} f\right)(x)$. Therefore, for each $y \in$ $X$,

$$
\begin{aligned}
\left\langle x_{n, i(n)}^{*}, y-x\right\rangle & \leq\left(\lambda_{n, i(n)} f\right)(y)-\left(\lambda_{n, i(n)} f\right)(x)+\left(\left(\lambda_{n, i(n)} f\right)(x)-f^{+}(x)+\varepsilon+\frac{1}{n}\right) \\
& =\left(\lambda_{n, i(n)} f\right)(y)-f^{+}(x)+\varepsilon+\frac{1}{n} .
\end{aligned}
$$

Since $\left(\lambda_{n, i(n)}\right)_{n}$, or a subnet of it, converges to some $\lambda \in[0,1]$, one gets by taking limits on the inequality, for all $y \in X$,

$$
\left\langle x^{*}, y-x\right\rangle \leq(\lambda f)(y)-f^{+}(x)+\varepsilon=(\lambda f)(y)-(\lambda f)(x)+\left((\lambda f)(x)-f^{+}(x)+\varepsilon\right) .
$$

In other words, $x^{*} \in \bigcup_{0 \leq \lambda \leq 1} \partial_{\varepsilon+(\lambda f)(x)-f^{+}(x)}(\lambda f)(x)$ showing that the inclusion " $\subset$ " holds.

To prove the converse inclusion " $\supset$ " pick $x^{*} \in \partial_{\varepsilon+\lambda f(x)-f^{+}(x)}(\lambda f)(x)$ for some $0 \leq$ $\lambda \leq 1$. Then, for every $y \in X$,

$$
\begin{aligned}
\left\langle x^{*}, y-x\right\rangle & \leq(\lambda f)(y)-(\lambda f)(x)+\varepsilon+\lambda f(x)-f^{+}(x) \\
& =(\lambda f)(y)-f^{+}(x)+\varepsilon \\
& \leq\left(\lambda f^{+}\right)(y)-f^{+}(x)+\varepsilon \\
& \leq f^{+}(y)-f^{+}(x)+\varepsilon,
\end{aligned}
$$

and $x^{*} \in \partial_{\varepsilon} f^{+}(x)$.

## 3 Normal cone to the domain

This section is devoted to give a representation of the normal cone to the effective domain of a supremum function by means of the $\varepsilon$-subdifferential of the data functions. This result will be a key tool to derive new formulas for the subdifferential of the supremum function in Section 4 .

We consider a nonempty family $\left\{f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$, where $X$ is a given lcs space, and the associated supremum function

$$
f=\sup _{t \in T} f_{t} .
$$

Our analysis is carried out in the following standard framework:
$T$ is Hausdorff compact and the mappings $t \mapsto f_{t}(z), z \in X$, are usc on $T$.
Given $x \in \operatorname{dom} f$ and $\varepsilon \geq 0$, remember that the $\varepsilon$-active set at $x$ is

$$
T_{\varepsilon}(x):=\left\{t \in T: f_{t}(x) \geq f(x)-\varepsilon\right\}, \quad T(x):=T_{0}(x) .
$$

We shall need the following lemma, which is valid for any family of convex functions, not necessarily lsc. We refer to [10, Page 854] for a finite-dimensional version of this result (without proof).

Lemma 5 Provided assumption (22) holds, we have that

$$
\begin{equation*}
\operatorname{dom} f=\cap_{t \in T} \operatorname{dom} f_{t} \tag{23}
\end{equation*}
$$

and, for every $x \in \operatorname{dom} f$,

$$
\begin{equation*}
\mathbb{R}_{+}(\operatorname{dom} f-x)=\cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right) \tag{24}
\end{equation*}
$$

Proof. Take $z \in \cap_{t \in T} \operatorname{dom} f_{t}$. For each $t \in T$ the upper semicontinuity assumption yields some $m_{t} \geq 0$ and a neighborhood $V_{t}$ of $t$ such that

$$
f_{s}(z) \leq m_{t} \text { for all } s \in V_{t}
$$

Consider $V_{t_{1}}, \cdots, V_{t_{k}}$ a finite covering of $T$. Then, for each $t \in T$,

$$
f_{t}(z) \leq \max \left\{m_{i}, i=1, \cdots, k\right\}<+\infty,
$$

and so $z \in \operatorname{dom} f$.
To prove the second statement we take $z \in \cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)$. If $z=\theta$, then we are obviously done. Otherwise, for each $t \in T$ there exist $\alpha_{t}, m_{t}>0$ and $z_{t} \in \operatorname{dom} f_{t}$ such that $z=\alpha_{t}\left(z_{t}-x\right)$ and

$$
f_{t}\left(z_{t}\right)<m_{t}
$$

By arguing as above, the upper semicontinuity assumption yields some neighborhood $V_{t}$ of $t$ such that

$$
f_{s}\left(z_{t}\right)<m_{t} \text { for all } s \in V_{t} .
$$

Consider $V_{t_{1}}, \cdots, V_{t_{k}}$, a finite covering of $T$, so that $z_{t_{i}}=\alpha_{t_{i}}^{-1} z+x \in \operatorname{dom} f_{s}$ for all $s \in V_{t_{i}}$. Next, for $\bar{\alpha}:=\max \left\{\alpha_{t_{i}}, i=1, \cdots, k\right\}$ we obtain that
$\bar{\alpha}^{-1} z=\bar{\alpha}^{-1}\left(\alpha_{t_{i}}\left(z_{t_{i}}-x\right)\right) \in \bar{\alpha}^{-1} \alpha_{t_{i}}\left(\cap_{s \in V_{t_{i}}}\left(\operatorname{dom} f_{s}-x\right)\right) \subset \cap_{s \in V_{t_{i}}}\left(\operatorname{dom} f_{s}-x\right), \quad i=1, \cdots, k$, where the last inclusion comes from the convexity of the set $\cap_{s \in V_{t_{i}}}\left(\operatorname{dom} f_{s}-x\right)$ and the
fact that $\theta \in \cap_{s \in V_{t_{i}}}\left(\operatorname{dom} f_{s}-x\right)$. Hence,

$$
\begin{aligned}
\bar{\alpha}^{-1} z & \in \cap_{i=1, \cdots, k} \cap_{s \in V_{t_{i}}}\left(\operatorname{dom} f_{s}-x\right) \\
& =\cap_{s \in V_{t_{i}}, i=1, \cdots, k}\left(\operatorname{dom} f_{s}-x\right)=\cap_{t \in T}\left(\operatorname{dom} f_{t}-x\right),
\end{aligned}
$$

and the first statement of the lemma leads us to

$$
z \in \bar{\alpha}\left(\cap_{t \in T}\left(\operatorname{dom} f_{t}-x\right)\right) \subset \mathbb{R}_{+}\left(\cap_{t \in T}\left(\operatorname{dom} f_{t}-x\right)\right)=\mathbb{R}_{+}(\operatorname{dom} f-x)
$$

We give now the main result of this section. When $T$ is a singleton, it reduces to (10). For instance, if we apply (10) to the supremum function $f$ we obtain that $\mathrm{N}_{\text {dom } f}(x)=\left(\partial_{\varepsilon} f(x)\right)_{\infty}$. Thus, one may think of using one of the known formulas of the $\varepsilon$-subdifferential of the supremum function $f$ like in [12, Theorem 2] (see, also, [24, Theorem 5], [25), but these formulas involve approximate subdifferentials $\partial_{\beta_{t}} f_{t}(x)$ with possibly very large parameters $\beta_{t}$, which are out of control.

Theorem 6 Consider $x \in \operatorname{dom} f$ and let $0<\varepsilon_{t} \leq 1, t \in T$, be such that

$$
\begin{equation*}
\inf _{t \in T}\left(\varepsilon_{t} f_{t}\right)(x)>-\infty . \tag{25}
\end{equation*}
$$

Then for every $\varepsilon>0$ we have that

$$
\begin{equation*}
\left[\overline{\operatorname{co}}\left(\bigcup_{t \in T} \partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty} \subset \mathrm{N}_{\operatorname{dom} f}(x), \tag{26}
\end{equation*}
$$

and, if the standard assumption (22) holds, then

$$
\mathrm{N}_{\operatorname{dom} f}(x)=\left[\overline{\mathrm{co}}\left(\bigcup_{t \in T} \partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty} .
$$

Proof. We fix $\varepsilon>0$ and denote

$$
E_{\varepsilon}:=\bigcup_{t \in T} \partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x) ;
$$

hence, $E_{\varepsilon} \neq \emptyset$ as $\left\{\varepsilon_{t} f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$.
To establish the inclusion (26), we take $x^{*} \in\left[\overline{\operatorname{co}}\left(E_{\varepsilon}\right)\right]_{\infty}$ and fix $x_{0}^{*} \in E_{\varepsilon}$. Then for every $\alpha>0$ we have that $x_{0}^{*}+\alpha x^{*} \in \overline{\operatorname{co}}\left(E_{\varepsilon}\right)$ and, so, there are nets $\left(\lambda_{j, 1}, \cdots, \lambda_{j, k_{j}}\right) \in \Delta_{k_{j}}^{+}$, $t_{j, 1}, \cdots, t_{j, k_{j}} \in T$, and $x_{j, 1}^{*} \in \partial_{\varepsilon}\left(\varepsilon_{t_{j, 1}} f_{t_{j, 1}}\right)(x), \cdots, x_{j, k_{j}}^{*} \in \partial_{\varepsilon}\left(\varepsilon_{t_{j, k_{j}}} f_{t_{j, k_{j}}}\right)(x)$ such that

$$
x_{0}^{*}+\alpha x^{*}=\lim _{j}\left(\lambda_{j, 1} x_{j, 1}^{*}+\cdots+\lambda_{j, k_{j}} x_{j, k_{j}}^{*}\right) .
$$

Hence, for every fixed $y \in \operatorname{dom} f$,

$$
\begin{aligned}
\left\langle x_{0}^{*}+\alpha x^{*}, y-x\right\rangle & =\lim _{j}\left\langle\lambda_{j, 1} x_{j, 1}^{*}+\cdots+\lambda_{j, k_{j}} x_{j, k_{j}}^{*}, y-x\right\rangle \\
& \leq \limsup _{j}\left(\sum_{i=1, \cdots, k_{j}} \lambda_{j, i}\left(\varepsilon_{t_{j, i}} f_{t_{j, i}}(y)-\varepsilon_{t_{j, i}} f_{t_{j, i}}(x)+\varepsilon\right)\right) \\
& \leq \lim \sup _{j}\left(\sum_{i=1, \cdots, k_{j}} \lambda_{j, i}\left(\varepsilon_{t_{j, i}} f^{+}(y)-\varepsilon_{t_{j, i}} f_{t_{j, i}}(x)+\varepsilon\right)\right) \\
& \leq f^{+}(y)-\inf \left\{\varepsilon_{t} f_{t}(x), t \in T\right\}+\varepsilon,
\end{aligned}
$$

and condition (25) ensures, by dividing by $\alpha$ and making $\alpha \uparrow+\infty$, that

$$
\left\langle x^{*}, y-x\right\rangle \leq 0,
$$

for all $y \in \operatorname{dom} f=\operatorname{dom} f^{+}$, that is, $x^{*} \in \mathrm{~N}_{\operatorname{dom} f}(x)$, as we wanted to prove.
Now we assume that the additional condition (22) holds. We have to prove the inclusion

$$
\begin{equation*}
\left(\left[\overline{\operatorname{co}}\left(E_{\varepsilon}\right)\right]_{\infty}\right)^{-} \subset\left(\mathrm{N}_{\operatorname{dom} f}(x)\right)^{-}, \tag{27}
\end{equation*}
$$

or equivalently, according to (17) and the fact that $\left(\mathrm{N}_{\operatorname{dom} f}(x)\right)^{-}=\operatorname{cl}\left(\mathbb{R}_{+}(\operatorname{dom} f-x)\right)$,

$$
\begin{equation*}
\operatorname{cl}\left(\operatorname{dom} \sigma_{E_{\varepsilon}}\right) \subset \operatorname{cl}\left(\mathbb{R}_{+}(\operatorname{dom} f-x)\right) . \tag{28}
\end{equation*}
$$

Take

$$
\begin{aligned}
z \in \operatorname{dom} \sigma_{E_{\varepsilon}} & =\operatorname{dom}\left(\sigma_{t \in T^{\partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)}}\right) \\
& =\operatorname{dom}\left(\sup _{t \in T} \sigma_{\partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)}\right)=\operatorname{dom}\left(\sup _{t \in T}\left(\varepsilon_{t} f_{t}\right)_{\varepsilon}^{\prime}(x ; \cdot)\right),
\end{aligned}
$$

where $\left(\varepsilon_{t} f_{t}\right)_{\varepsilon}^{\prime}(x ; \cdot)$ represents the $\varepsilon$-directional derivative of the function $\varepsilon_{t} f_{t} \in \Gamma_{0}(X)$ at $x$ (see [31, Theorem 2.4.11]). Then, by [31, Theorem 2.1.14],

$$
\begin{aligned}
z & \in \cap_{t \in T} \operatorname{dom}\left(\left(\varepsilon_{t} f_{t}\right)_{\varepsilon}^{\prime}(x ; \cdot)\right) \\
& =\cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom}\left(\varepsilon_{t} f_{t}\right)-x\right) \\
& =\cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right),
\end{aligned}
$$

and (24) gives rise to

$$
z \in \cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)=\mathbb{R}_{+}(\operatorname{dom} f-x) \subset \operatorname{cl}\left(\mathbb{R}_{+}(\operatorname{dom} f-x)\right) .
$$

Hence, (28) holds and (27) follows.
The following corollary, which is straightforward from Theorem 6, gives a practical example for the weighting parameters $\varepsilon_{t}, t \in T$, used in the above characterization of
$\mathrm{N}_{\mathrm{dom} f}(x)$.
Corollary 7 Consider $\varepsilon>0, x \in \operatorname{dom} f$ and denote

$$
\varepsilon_{t}:= \begin{cases}\frac{-\varepsilon}{2 f_{t}(x)-2 f(x)+\varepsilon}, & \text { if } t \in T \backslash T_{\varepsilon}(x),  \tag{29}\\ 1, & \text { if } t \in T_{\varepsilon}(x)\end{cases}
$$

Then $0<\varepsilon_{t}<1$ for all $t \in T_{\varepsilon}(x)$ and, provided that (22) holds,

$$
\mathrm{N}_{\operatorname{dom} f}(x)=\left[\overline{\mathrm{Co}}\left(\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T_{\varepsilon}(x)} \partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right)\right]_{\infty}
$$

Proof. We may assume that $f(x)=0$. Then, for every $t \in T \backslash T_{\varepsilon}(x)$, we have that $2 f_{t}(x)+\varepsilon<-\varepsilon$ and, so, $0<\varepsilon_{t}<1$. Also, for such $t \in T \backslash T_{\varepsilon}(x)$ we have that

$$
\varepsilon_{t} f_{t}(x)=\frac{-\varepsilon f_{t}(x)}{2 f_{t}(x)+\varepsilon}>-\varepsilon,
$$

so that $\inf _{t \in T}\left(\varepsilon_{t} f_{t}\right)(x) \geq-\varepsilon$ and condition (25) follows. Thus, the desired conclusion straightforwardly comes from Theorem 6.

The following result is a simple consequence of Theorem 6, giving a characterization of $\mathrm{N}_{\text {dom } f}(x)$ by means of the original functions $f_{t}$ 's and not the $\left(\varepsilon_{t} f_{t}\right)$ 's.

Corollary 8 Consider $x \in \operatorname{dom} f$ and assume that

$$
\begin{equation*}
\inf _{t \in T} f_{t}(x)>-\infty \tag{30}
\end{equation*}
$$

Then, under (22), for every $\varepsilon>0$ we have

$$
\begin{equation*}
\mathrm{N}_{\operatorname{dom} f}(x)=\left[\overline{\operatorname{co}}\left(\bigcup_{t \in T} \partial_{\varepsilon} f_{t}(x)\right)\right]_{\infty} \tag{31}
\end{equation*}
$$

Proof. Take $\varepsilon_{t}=1, t \in T$, in Theorem 6.
Condition (30) obviously holds when $T$ is finite. More generally, we have the following result.

Corollary 9 Consider $x \in \operatorname{dom} f$ and assume that (22) holds. If the mapping $t \mapsto f_{t}(x)$ is also lsc, then condition (30) fulfills. Consequently, (31) is satisfied.

Proof. Since $\left\{f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$, for each $t \in T$ we have that

$$
+\infty>f(x) \geq f_{t}(x)>f_{t}(x)-1
$$

and so there exists some neighborhood $V_{t}$ of $t$ such that

$$
f_{s}(x)>f_{t}(x)-1, \text { for all } s \in V_{t}
$$

But $T$ is compact, and so $T \subset \cup_{i=1}^{k} V_{t_{i}}$, for some $\left\{t_{1}, \cdots, t_{k}\right\} \subset T$. Hence, for each $t \in T$, we have that $t \in V_{t_{i 0}}$ for some $i \in\{1, \cdots, k\}$, so that

$$
f_{t}(x)>f_{t_{i_{0}}}(x)-1 \geq \min _{i \in\{1, \cdots, k\}} f_{t_{i}}(x)-1>-\infty .
$$

The following corollary shows that we can give different values to the parameter $\varepsilon$ and the formula in Theorem 6 is still valid. It is an extension to our current setting of [11, Lemma 11(ii)] dealing with finitely many functions.

Corollary 10 Assume that hypothesis (222) holds. Given $\left.\left.x \in \operatorname{dom} f,\left(\varepsilon_{t}\right)_{t} \subset\right] 0,1\right]$ satisfying (25), and $\delta_{t}>0, t \in T$, such that $0<\inf _{t \in T} \delta_{t} \leq \sup _{t \in T} \delta_{t}<+\infty$, we have

$$
\mathrm{N}_{\mathrm{dom} f}(x)=\left[\overline{\operatorname{co}}\left(\cup_{t \in T} \partial_{\delta_{t}}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty} .
$$

Proof. We denote $\bar{\delta}:=\inf _{t \in T} \delta_{t}$ and $\hat{\delta}:=\sup _{t \in T} \delta_{t}$. Then

$$
\left[\overline{\mathrm{co}}\left(\cup_{t \in T} \partial_{\bar{\delta}}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty} \subset\left[\overline{\mathrm{co}}\left(\cup_{t \in T_{1}} \partial_{\delta_{t}}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty} \subset\left[\overline{\mathrm{co}}\left(\cup_{t \in T_{1}} \partial_{\hat{\delta}}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right]_{\infty},
$$

and we are done thanks to Theorem 6.
The following corollary provides another representation of $\mathrm{N}_{\operatorname{dom}} f(x)$, using the positive part of the $f_{t}$ 's instead of the weighted functions $\varepsilon_{t} f_{t}, t \in T \backslash T_{\varepsilon}(x)$.

Corollary 11 Given $x \in \operatorname{dom} f$ and $\varepsilon>0$, under hypothesis (22) we have that

$$
\begin{equation*}
\mathrm{N}_{\operatorname{dom} f}(x)=\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}^{+}(x)\right)\right)\right]_{\infty} \tag{32}
\end{equation*}
$$

consequently,

$$
\begin{equation*}
\mathrm{N}_{\mathrm{dom} f}(x)=\left[\overline{\mathrm{co}}\left(\left(\underset{t \in T_{\varepsilon}^{+}(x)}{ } \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\underset{\substack{t \in T \backslash T_{\varepsilon}^{+}(x) \\ 0 \leq \lambda \leq 1}}{ } \partial_{\varepsilon+\lambda f_{t}(x)}\left(\lambda f_{t}\right)(x)\right)\right)\right]_{\infty}, \tag{33}
\end{equation*}
$$

where

$$
T_{\varepsilon}^{+}(x):=T_{\varepsilon}(x) \cup\left\{t \in T: f_{t}(x) \geq 0\right\} .
$$

Proof. We may assume that $f(x)=0$. First, observe that

$$
\left\{f_{t}, t \in T_{\varepsilon}^{+}(x) ; f_{t}^{+}, t \in T \backslash T_{\varepsilon}^{+}(x)\right\} \subset \Gamma_{0}(X)
$$

and satisfies

$$
\min \left\{\inf _{t \in T_{\varepsilon}^{+}(x)} f_{t}(x), \inf _{t \in T \backslash T_{\varepsilon}^{+}(x)} f_{t}^{+}(x)\right\} \geq-\varepsilon ;
$$

that is, (25) follows and, so, the first statement in Theorem 6 implies that

$$
\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T_{\varepsilon}^{+}(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T_{\varepsilon}^{+}(x)} \partial_{\varepsilon} f_{t}^{+}(x)\right)\right)\right]_{\infty} \subset \mathrm{N}_{\operatorname{dom} f}(x) .
$$

To establish the opposite inclusion we argue as in the proof of the second statement in Theorem 6. We introduce the nonempty set

$$
D_{\varepsilon}:=\left(\bigcup_{t \in T_{\varepsilon}^{+}(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T_{\varepsilon}^{+}(x)} \partial_{\varepsilon} f_{t}^{+}(x)\right)
$$

and proceed by showing that

$$
\operatorname{cl}\left(\operatorname{dom} \sigma_{D_{\varepsilon}}\right) \subset \operatorname{cl}\left(\mathbb{R}_{+}(\operatorname{dom} f-x)\right) .
$$

Take

$$
\begin{aligned}
z \in \operatorname{dom} \sigma_{D_{\varepsilon}} & =\operatorname{dom}\left(\max \left\{\sup _{t \in T_{\varepsilon}^{+}(x)} \sigma_{\partial_{\varepsilon} f_{t}(x)}, \sup _{t \in T \backslash T_{\varepsilon}^{+}(x)} \sigma_{\partial_{\varepsilon} f_{t}^{+}(x)}\right\}\right) \\
& =\operatorname{dom}\left(\max \left\{\sup _{t \in T_{\varepsilon}^{+}(x)}\left(f_{t}\right)_{\varepsilon}^{\prime}(x ; \cdot), \sup _{t \in T \backslash T_{\varepsilon}^{+}(x)}\left(f_{t}^{+}\right)_{\varepsilon}^{\prime}(x ; \cdot)\right\}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
z & \in\left(\cap_{t \in T_{\varepsilon}^{+}(x)} \operatorname{dom}\left(\left(f_{t}\right)_{\varepsilon}^{\prime}(x ; \cdot)\right)\right) \cap\left(\cap_{T \backslash T_{\varepsilon}^{+}(x)} \operatorname{dom}\left(\left(f_{t}^{+}\right)_{\varepsilon}^{\prime}(x ; \cdot)\right)\right) \\
& =\left(\cap_{t \in T_{\varepsilon}^{+}(x)} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)\right) \cap\left(\cap_{T \backslash T_{\varepsilon}^{+}(x)} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}^{+}-x\right)\right) \\
& =\left(\cap_{t \in T_{\varepsilon}^{+}(x)} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)\right) \cap\left(\cap_{T \backslash T_{\varepsilon}^{+}(x)} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)\right) \\
& =\cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right),
\end{aligned}
$$

and (24) gives rise to

$$
z \in \cap_{t \in T} \mathbb{R}_{+}\left(\operatorname{dom} f_{t}-x\right)=\mathbb{R}_{+}(\operatorname{dom} f-x) \subset \operatorname{cl}\left(\mathbb{R}_{+}(\operatorname{dom} f-x)\right),
$$

as required.
Finally, the last statement of the corollary follows from Lemma 4 .
Remark 1 Let us note that formula (33) can be simplified, observing for instance that for all $t \in T \backslash T_{\varepsilon}^{+}(x)$,

$$
\begin{gathered}
\partial_{\varepsilon}\left(0 f_{t}\right)(x)=\mathrm{N}_{\text {dom }}^{\varepsilon} f_{t}(x), \\
\partial_{\varepsilon+\lambda f_{t}(x)}\left(\lambda f_{t}\right)(x)=\lambda \partial_{\lambda^{-1} \varepsilon+f_{t}(x)} f_{t}(x), \text { for all } 0<\lambda \leq 1 .
\end{gathered}
$$

## 4 Alternative representations of the subdifferential

We are dealing again with a nonempty family $\left\{f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$ and its supremum function $f:=\sup _{t \in T} f_{t}$. As we remembered in the introduction, in the general case, when no assumption is made neither on $T$ nor on the mappings $t \mapsto f_{t}(z), z \in X$, the subdifferential of $f$ at a point $x \in \operatorname{dom} f$ is given by ([1])

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x), \varepsilon>0} \overline{\mathrm{co}}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right), \tag{34}
\end{equation*}
$$

where

$$
\mathcal{F}(x):=\{L \subset X: L \text { finite-dimensional subspace with } x \in L\} .
$$

We consider in this section the same family $\left\{f_{t}, t \in T\right\}$ satisfying the standard hypothesis (22); i.e., $T$ is Hausdorff compact and the mappings $t \mapsto f_{t}(z), z \in X$, are usc. In such a case, instead of (34) we have the following more precise characterization of the subdifferential of $f$ (see [4, Theorem 3.8]),

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x), \varepsilon>0} \overline{\operatorname{co}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right), \tag{35}
\end{equation*}
$$

where we use the active set $T(x)$ instead of $T_{\varepsilon}(x)$.
Our objective in this section is to give alternative representations to (35) for $\partial f(x)$, which are free of $\mathrm{N}_{L \cap \operatorname{dom} f}(x), L \in \mathcal{F}(x)$. The main tools will be the characterizations of the normal cone to $\operatorname{dom} f$ provided in the previous section.

The general characterization of $\partial f(x)$ is given in Theorem 13, but we prefer to establish first a preliminary version of it, which is valid in the relevant case when $f$ attains its minimum at $x$.

Theorem 12 Assume that hypothesis (22) fulfils. Consider $x \in \operatorname{dom} f$ and let $0<$ $\rho_{t} \leq 1, t \in T$, be such that

$$
\inf _{t \in T}\left(\rho_{t} f_{t}\right)(x)>-\infty
$$

Then we have that

$$
\begin{equation*}
\partial f(x) \subset \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) \tag{36}
\end{equation*}
$$

Moreover, if $f$ attains its minimum at $x$, then (36) becomes an equality.
Proof. Fix $x \in \operatorname{dom} f$ and assume, without loss of generality, that $f(x)=0$. Fix $\varepsilon>0$, $U \in \mathcal{N}$, and pick $L \in \mathcal{F}(x)$ such that $L^{\perp} \subset U$. Observe that the family $\left\{f_{t}, t \in T ; \mathrm{I}_{L}\right\} \subset$ $\Gamma_{0}(X)$ also satisfies hypothesis (22) as we can assign to the function $\mathrm{I}_{L}$ an (isolated) index not belonging to $T$. Therefore, by applying Theorem 6 to the family $\left\{f_{t}, t \in T ; \mathrm{I}_{L}\right\}$, we
obtain that

$$
\begin{aligned}
\mathrm{N}_{L \cap \operatorname{dom} f}(x) & =\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x) \cup L^{\perp}\right) \cup\left(\left(\bigcup_{t \in T \backslash T(x)} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right)\right)\right]_{\infty} \\
& =\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x) \cup L^{\perp}\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right)\right]_{\infty},
\end{aligned}
$$

where the last equality is a consequence of Lemma 3, Moreover, applying Lemma 2, we have that

$$
\begin{equation*}
\mathrm{N}_{L \cap \operatorname{dom} f}(x)=\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)+L^{\perp}\right)\right)\right]_{\infty} \tag{37}
\end{equation*}
$$

Next, by combining this relation and (35),

$$
\begin{aligned}
\partial f(x) & \subset \overline{\operatorname{co}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f(x)}\right) \\
& =\overline{\operatorname{co}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\underset{t \in T \backslash T(x)}{\bigcup} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)+L^{\perp}\right)\right)\right]_{\infty}\right) \\
& \subset \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)+L^{\perp}\right)\right) \\
& \subset \operatorname{co}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\varepsilon_{t} f_{t}\right)(x)\right)\right)+2 U .
\end{aligned}
$$

Consequently, (36) follows by intersecting over $\varepsilon>0$ and $U \in \mathcal{N}$.
We proceed now by showing the opposite inclusion in (36) when $x$ is a minimizer of $f$. By the current assumption, we choose an $M>0$ such that

$$
\inf _{t \in T}\left(\rho_{t} f_{t}\right)(x) \geq-M
$$

and take $x^{*}$ in the right-hand side of (36); that is, for each fixed $\varepsilon>0$,

$$
\begin{equation*}
x^{*} \in \overline{\mathrm{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) . \tag{38}
\end{equation*}
$$

Observe that, if $z^{*} \in \partial_{\varepsilon} f_{t}(x), t \in T(x)$, then for every $z \in X$

$$
\left\langle z^{*}, z-x\right\rangle \leq f_{t}(z)-f_{t}(x)+\varepsilon \leq f(z)+\varepsilon
$$

and so $z^{*} \in \partial_{\varepsilon} f(x)$. Moreover, if $z^{*} \in \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x), t \in T \backslash T(x)$, then for every $z \in X$

$$
\begin{aligned}
\left\langle z^{*}, z-x\right\rangle & \leq \rho_{t} f_{t}(z)-\rho_{t} f_{t}(x)+\varepsilon \\
& \leq \rho_{t} f(z)-M+\varepsilon \leq f(z)+M+\varepsilon
\end{aligned}
$$

since that $x$ is a minimizer of $f$ and $f(z) \geq f(x)=0$. Hence, $z^{*} \in \partial_{\varepsilon+M} f_{t}(x)$. Consequently, taking into account Lemma 1 and that $\varepsilon>0$ was arbitrarily chosen, (38) leads us to

$$
x^{*} \in \cap_{\varepsilon>0} \overline{\operatorname{co}}\left(\partial_{\varepsilon} f(x) \cup \varepsilon \partial_{\varepsilon+M} f(x)\right)=\partial f(x),
$$

that is, the opposite inclusion in (36) is also true.
Example 1 The inclusion in (36) may be strict when $x$ is not a minimizer of $f$. Consider the family

$$
\left\{f_{t}, t \in T ; h\right\},
$$

where $h$ is the constant function $h \equiv f(x)-1$, and denote by $g$ the associated supremum function. Then, by the lower semicontinuity of the $f_{t}$ 's, the functions $f$ and $g$ coincide in a neighborhood of $x$, entailing

$$
\partial f(x)=\partial g(x)
$$

If (36) would be an equality, with any weighting parameter associated to $h$, then by taking into account that $\partial_{\varepsilon} h(x)=\{\theta\}$ we would have

$$
\partial g(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x) \bigcup\{\theta\}\right)\right),
$$

but this implies that

$$
\theta \in \partial g(x)=\partial f(x),
$$

which contradicts our assumption that $x$ is not a minimizer of $f$.
Theorem 13 Assume that hypothesis (22) fulfils. Consider $x \in \operatorname{dom} f$ and let $0<$ $\rho_{t} \leq 1, t \in T$, be such that

$$
\inf _{t \in T}\left(\rho_{t} f_{t}\right)(x)>-\infty
$$

Then we have that

$$
\begin{equation*}
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\overline{\operatorname{co}}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) \tag{39}
\end{equation*}
$$

(with $\bigcup_{\emptyset}=\{\theta\}$ when $T(x)=T$ ).
Remark 2 (before the proof) Observe that the operator co determines that (39) can be equivalently written as

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}[0, \varepsilon] \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) .
$$

Proof. Take $x \in \operatorname{dom} f$ such that $f(x)=0$ (without loss of generality). Fix $\varepsilon>0$, $U \in \mathcal{N}$, and pick $L \in \mathcal{F}(x)$ such that $L^{\perp} \subset U$. By arguing as in the beginning of the proof of (36), and taking into account Lemma 2, we obtain that

$$
\begin{align*}
\mathrm{N}_{L \cap \operatorname{dom} f}(x) & =\left[\overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)+L^{\perp}\right)\right]_{\infty}  \tag{40}\\
& \subset\left[\overline{\mathrm{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)+L^{\perp}\right)\right]_{\infty}
\end{align*}
$$

(Observe the difference between (40) and (377).)

Due to the lower semicontinuity of the $\left(\rho_{t} f_{t}\right)$ 's, the sets $\partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)$ are nonempty and we have

$$
\theta \in \bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)+L^{\perp}
$$

Thus, using again (35),

$$
\begin{aligned}
\partial f(x) & \subset \overline{\operatorname{co}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right) \\
& \subset \overline{\operatorname{co}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)+L^{\perp}\right) \\
& \subset \operatorname{co}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)+\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)+2 U .
\end{aligned}
$$

Therefore the first inclusion " $\subset$ " in (39) follows by intersecting over $\varepsilon>0$ and $U \in \mathcal{N}$.
Conversely, to show the inclusion " $\supset$ " in (39), we take $x^{*}$ in the right-hand side of (39) and choose an $M>0$ such that

$$
\inf _{t \in T}\left(\rho_{t} f\right)(x)>-M
$$

Thus, for each $\varepsilon>0$ we obtain (similarly to the last part of the proof of Theorem 12)

$$
\begin{aligned}
x^{*} & \in \overline{\mathrm{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) \\
& \subset \overline{\mathrm{co}}\left(\partial_{\varepsilon} f(x)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{M+\varepsilon}\left(\rho_{t} f\right)(x)\right)\right)
\end{aligned}
$$

(Observe that, compared to the proof of Theorem 12, here we maintain the parameters $\rho_{t}, t \in T \backslash T(x)$, because $x$ is not necessarily a minimizer of $f$.)

Writing, due to last relation,

$$
\begin{aligned}
x^{*} & \in \operatorname{cl}\left(\operatorname{co}\left(\partial_{\varepsilon} f(x)+\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{M+\varepsilon}\left(\rho_{t} f\right)(x)\right)\right) \\
& =\operatorname{cl}\left(\partial_{\varepsilon} f(x)+\operatorname{co}\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{M+\varepsilon}\left(\rho_{t} f\right)(x)\right)\right)
\end{aligned}
$$

leading to the existence of nets $\left(y_{i}^{*}\right)_{i} \subset \partial_{\varepsilon} f(x),\left(\lambda_{i, k}\right)_{i} \subset[0,1], \Sigma_{k=1, \cdots, k_{i}} \lambda_{i, k} \leq 1$, $\left(t_{i, k}\right)_{i} \subset T \backslash T(x)$, and $\left(z_{i, k}^{*}\right)_{i} \subset \partial_{M+\varepsilon}\left(\rho_{t_{i, k}} f\right)(x), k=1, \cdots, k_{i}, k_{i} \geq 1$, such that

$$
x^{*}=\lim _{i}\left(y_{i}^{*}+\sum_{k=1, \cdots, k_{i}} \varepsilon \lambda_{i, k} z_{i, k}^{*}\right)
$$

Consequently, since $f(x)=0$, for each $y \in \operatorname{dom} f$

$$
\begin{aligned}
\left\langle x^{*}, y-x\right\rangle & =\lim _{i}\left\langle y_{i}^{*}+\varepsilon \sum_{k=1, \cdots, k_{i}} \lambda_{i, k} z_{i, k}^{*}, y-x\right\rangle \\
& \leq \limsup _{i}\left((f(y)-f(x)+\varepsilon)+\varepsilon\left(\sum_{k=1, \cdots, k_{i}} \lambda_{i, k}\left(\rho_{t_{i, k}} f(y)-\rho_{t_{i, k}} f(x)+M+\varepsilon\right)\right)\right) \\
& =\limsup _{i}\left(f(y)+\varepsilon+\varepsilon \sum_{k=1, \cdots, k_{i}} \lambda_{i, k}\left(\rho_{t_{i, k}} f(y)+M+\varepsilon\right)\right) \\
& \leq f(y)+\varepsilon+\varepsilon\left(f^{+}(y)+M+\varepsilon\right),
\end{aligned}
$$

and $x^{*} \in \partial_{\varepsilon} f(x)$, by taking $\varepsilon \downarrow 0$.
Corollary 14 Assume that hypothesis (22) fulfills. Then for every $x \in \operatorname{dom} f$ we have that

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\overline{c o}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right)
$$

where $\rho_{t}=\rho_{t}(\varepsilon)$ is defined as

$$
\rho_{t}:=\frac{\varepsilon}{2 f(x)-2 f_{t}(x)+\varepsilon}, \quad t \in T \backslash T(x) .
$$

In particular, if $f$ attains its minimum at $x$, then we also have that

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\overline{c o}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(x)\right)\right) .
$$

Proof. It suffices to apply Theorems 12 and 13 by replacing the parameters $\rho_{t}, t \in T$, there by

$$
\hat{\rho}_{t}:=1, \text { if } t \in T(x), \hat{\rho}_{t}:=\frac{\varepsilon}{2 f(x)-2 f_{t}(x)+\varepsilon}, t \in T \backslash T(x) .
$$

Indeed, for all $t \in T$ we have that $0<\rho_{t}<1$ and

$$
\hat{\rho}_{t} f_{t}(x) \geq \min \left\{0, \inf _{t \in T \backslash T(x)} \frac{\varepsilon f_{t}(x)}{2 f(x)-2 f_{t}(x)+\varepsilon}\right\} \geq-\frac{\varepsilon}{2} .
$$

Corollary 15 Assume that (22) fulfills. If $x \in \operatorname{dom} f$ is such that

$$
\inf _{t \in T} f_{t}(x)>-\infty
$$

then we have

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)+\left(\bigcup_{t \in T \backslash T(x)}\{0, \varepsilon\} \partial_{\varepsilon} f_{t}(x)\right)\right),
$$

and, when additionally $f$ attains its minimum at $x$,

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right) \cup\left(\bigcup_{t \in T \backslash T(x)} \varepsilon \partial_{\varepsilon} f_{t}(x)\right)\right)
$$

An obvious consequence of Theorem 13 is the following extension of the Brøndsted formula in [1], and the formula given in [10, Proposition 6.3] (in finite dimensions and under the continuity of the $f_{t}$ 's).

Corollary 16 Assume that (22) fulfills. If $x \in \operatorname{dom} f$ is such that $T(x)=T$, then

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\overline{\mathrm{co}}}\left(\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right)
$$

Proof. It is immediate from Theorem [13, since that $\inf _{t \in T} f_{t}(x)=\inf _{t \in T(x)} f_{t}(x)=0$.
We close the paper by deriving new optimality conditions for the following convex optimization problem with infinitely many constraints,

$$
(\mathcal{P}): \quad \operatorname{Inf} g(x), \quad \text { subject to } f_{t}(x) \leq 0, t \in T
$$

where $T$ is an arbitrary (possibly, infinite) set. We refer e.g. to [8], 9], 21], etc., and references therein, for theory, algorithms and applications of this model. We have the following result in the continuous framework; i.e., $T$ is a Hausdorff compact set and the family $\left\{g ; f_{t}, t \in T\right\} \subset \Gamma_{0}(X)$ satisfies condition (22). See, also, [3, 5] for optimality conditions for $(\mathcal{P})$ in different frameworks.

Corollary 17 Let $\bar{x}$ be an optimal solution of $(\mathcal{P})$, and take $0<\rho_{t} \leq 1, t \in T \backslash T(\bar{x})$, such that

$$
\inf _{t \in T \backslash T(\bar{x})}\left(\rho_{t} f_{t}\right)(\bar{x})>-\infty
$$

Then, for every $\varepsilon>0$ and every $U \in \mathcal{N}$, there are associated $t_{1}, \cdots, t_{m} \in T(\bar{x})$, $t_{m+1}, \cdots, t_{m+n} \in T \backslash T(\bar{x})$, and $\left(\lambda_{0}, \lambda_{1}, \cdots, \lambda_{m}, \lambda_{m+1}, \cdots, \lambda_{m+n}\right) \in \Delta_{m+n+1}, m, n \geq 1$, such that

$$
0_{n} \in \lambda_{0} \partial_{\varepsilon} g(\bar{x})+\sum_{i=1}^{m} \lambda_{i} \partial_{\varepsilon} f_{t_{i}}(\bar{x})+\sum_{i=m+1}^{m+n} \varepsilon \lambda_{i} \partial_{\varepsilon}\left(\rho_{t_{i}} f_{t_{i}}\right)+U
$$

Moreover, $\lambda_{0}>0$ when the Slater condition is satisfied; that is, there is some $x_{0} \in X$ such that

$$
f_{t}\left(x_{0}\right)<0 \quad \text { for all } t \in T
$$

Remark 3 (before the proof) Observe that some of the multipliers $\lambda_{0}, \lambda_{1}, \cdots, \lambda_{m}$, $\lambda_{m+1}, \cdots, \lambda_{m+n}$ can be zero, but their sum is one. Note that, due to the hypothesis (22), $T(\bar{x}) \neq \emptyset$ but $T \backslash T(\bar{x})$ can be empty. In the last case, the last relation collapses to

$$
0_{n} \in \lambda_{0} \partial_{\varepsilon} g(\bar{x})+\sum_{i=1}^{m} \lambda_{i} \partial_{\varepsilon} f_{t_{i}}(\bar{x})+U
$$

Proof. It is easy to see that $\bar{x}$ is a global minimum of the supremum function $f: X \rightarrow$ $\mathbb{R} \cup\{+\infty\}$, defined as

$$
f(x):=\sup \left\{g(x)-g(\bar{x}), f_{t}(x), t \in T\right\}
$$

that is, $0_{n} \in \partial f(\bar{x})$. Then, by Theorem 12 ,

$$
0_{n} \in \partial f(\bar{x})=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left(\left(\partial_{\varepsilon} g(\bar{x}) \cup\left(\bigcup_{t \in T(\bar{x})} \partial_{\varepsilon} f_{t}(\bar{x})\right)\right) \cup\left(\bigcup_{t \in T \backslash T(\bar{x})} \varepsilon \partial_{\varepsilon}\left(\rho_{t} f_{t}\right)(\bar{x})\right)\right)
$$

leading us to the conclusion of the first statement of the corollary.
Now, we suppose that the Slater condition holds; that is, due to (22), the supremum function $h:=\sup _{t \in T} f_{t}$ satisfies

$$
h\left(x_{0}\right)<0
$$

for some $x_{0} \in X$. Let us proceed by contradiction, assuming that $\lambda_{0}=0$; that is, since $g \in \Gamma_{0}(X)$ and so $\partial_{\varepsilon} g(\bar{x}) \neq \emptyset$,

$$
\begin{equation*}
0_{n} \in \sum_{i=1}^{m} \lambda_{i} \partial_{\varepsilon} f_{t_{i}}(\bar{x})+\sum_{i=m+1}^{m+n} \varepsilon \lambda_{i} \partial_{\varepsilon}\left(\rho_{t_{i}} f_{t_{i}}\right)+U \tag{41}
\end{equation*}
$$

Observe that

$$
\partial_{\varepsilon} f_{t_{i}}(\bar{x}) \subset \partial_{\varepsilon} h(\bar{x}), \quad i=1, \cdots, m
$$

since that $f_{t_{i}} \leq h$ and $f_{t_{i}}(\bar{x})=h(\bar{x})=0, i=1, \cdots, m$. Moreover, if $M>0$ is such that $\inf _{t \in T \backslash T(\bar{x})}\left(\rho_{t} f_{t}\right)(\bar{x})>-M$, then for every $x^{*} \in \partial_{\varepsilon}\left(\rho_{t_{i}} f_{t_{i}}\right)(\bar{x}), t_{i} \in T \backslash T(\bar{x})$, we have for all $y \in \operatorname{dom} h$

$$
\begin{aligned}
\left\langle x^{*}, y-\bar{x}\right\rangle & \leq\left(\rho_{t_{i}} f_{t_{i}}\right)(y)-\left(\rho_{t_{i}} f_{t_{i}}\right)(\bar{x})+\varepsilon \\
& \leq\left(\rho_{t_{i}} h\right)(y)-\inf _{t \in T \backslash T(\bar{x})}\left(\rho_{t} f_{t}\right)(\bar{x})+\varepsilon \\
& \leq h^{+}(y)+M+\varepsilon
\end{aligned}
$$

that is, $x^{*} \in \partial_{M+\varepsilon} h^{+}(\bar{x})$. Consequently, (41) reads

$$
0_{n} \in \sum_{i=1}^{m} \lambda_{i} \partial_{\varepsilon} h(\bar{x})+\varepsilon \sum_{i=m+1}^{m+n} \lambda_{i} \partial_{M+\varepsilon} h^{+}(\bar{x})+U \subset \operatorname{co}\left(\partial_{\varepsilon} h(\bar{x}), \varepsilon \partial_{M+\varepsilon} h^{+}(\bar{x})\right)+U,
$$

and so, according to Lemma 1,

$$
0_{n} \in \cap_{\varepsilon>0} \overline{\overline{\operatorname{co}}}\left(\partial_{\varepsilon} h(\bar{x}), \varepsilon \partial_{M+\varepsilon} h^{+}(\bar{x})\right)=\partial h(\bar{x})
$$

This is a contradiction because $0=h(\bar{x}) \leq h\left(x_{0}\right)<0$.

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