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Approximating multiple integrals of continuous functions by δ-uniform curves

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Abstract
We present a method to approximate, with controlled and arbitrarily small error, multiple intregrals over the unit cube $[0, 1]^d$ by a single variable integral over [0, 1]. For this, we use the so called δ-uniform curves, which are a particular case of α-dense curves. Our main result improves and extends other existing methods on this subject.

Keywords Multiple integrals · Numerical integration · Numerical methods · δ-uniform curves · α-dense curves · Quasi-monte carlo methods

Mathematics Subject Classification Primary 26A42 · 65D30 · 82B80; Secondary 26A30 · 26A06

1 Introduction

To set the notation, we put $I := [0, 1]$ and $\left(\mathbb{R}^d, \|\cdot\|\right)$ is the Euclidean space. It is known that many mathematical problems, raised from engineering, physical sciences, etc., require to compute an integral of the form

$$\int_{I^d} f(x_1, \ldots, x_d) dx_1 \cdots dx_d, \tag{1.1}$$

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for a given continuous function $f : I^d \longrightarrow \mathbb{R}$. In general, the above integral can not be solved analytically and therefore, numerical methods have to be applied to approximate its value. For a general background and concrete references on this topic, see, for instance, the books [1,13,14,22].

In this paper we present a numerical method to approximate the integral (1.1) by a single variable one, that is,

$$\int_{I^d} f(x_1, \ldots, x_d) dx_1 \cdots dx_d \approx \int_0^1 g(t) dt, \qquad (1.2)$$

for a suitable continuous function $g : I \longrightarrow \mathbb{R}$, where the symbol $\approx$ will have a precise meaning (see Theorem 3.1). This *dimensionality reduction* is possible by using the so called δ-uniform curves, explained in detail in Sect. 2. It is worth to mention some important properties of such method, namely:

(1) Only the continuity of the function f is required.
(2) An explicit upper bound for the error, depending on the function f and the parameter δ of the used curve, is provided. If some numerical method is used to compute the single variable integral in (1.2), then the upper bound for the error is given in terms of f, δ and the error of the used method.
(3) From a deterministic point of view, we will obtain the value of the integral (1.1) with a prescribed approximation error.

In Sect. 4, to illustrate our results and the reliability of the proposed method, we provide some numerical examples. In particular, in Sect. 4.2, the *dimensionality reduction* suggested in (1.2) will be justified when a quasi-Monte Carlo method is used to compute these integrals.

2 α-dense curves and δ-uniform curves

Let (E, d) be a metric space and $\mathfrak{B}_E$ the class of non-empty and bounded subsets of E. In 1997 the concepts of α-dense curve and densifiable set were introduced in [16]:

Definition 2.1 Let $\alpha \geq 0$ and $D \in \mathfrak{B}_E$. A continuous mapping $\gamma : I \longrightarrow (E, d)$ is said to be an α-dense curve in D if it satisfies the following conditions:

(1) $\gamma(I) \subset D$.
(2) For any $x \in D$, there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

If for every $\alpha > 0$ there is an α-dense curve in D, then it is said to be densifiable.

Note that, given $D \in \mathfrak{B}_E$, there is always an α-dense curve in D for any $\alpha \geq \mathrm{Diam}(D)$, the diameter of D. Indeed, fixed a point $x_0 \in D$, the mapping $\gamma(t) := x_0$ for all $t \in I$ is an α-dense curve in D whenever $\alpha \geq \mathrm{Diam}(D)$. However not every subset of a metric space, even connected and compact , is densifiable:

Example 2.1 (see [18]) In the Euclidean plane, there is not any α-dense curve, for $0 < \alpha < 1$, in the set:

$$D := \{(x, \sin(1/x)) : x \in [-1, 0) \cup (0, 1]\} \cup \{[0, y] : y \in [-1, 1]\},$$

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for $0 < \alpha < 1$.

If D is a connected, compact and locally connected set, by the Hahn-Mazurkiewicz theorem (see [24]), there exists a continuous mapping $\gamma : I \longrightarrow (E, d)$ such that $\gamma(I) = D$ and, in particular, if $D := I^d$ then γ is called a *space-filling curve* (again, [24]). Since γ obviously satisfies the conditions of Definition 2.1 for $\alpha = 0$, such γ is a 0-dense curve in D. Therefore the α-dense curves generalize the space-filling curves (see also [15]).

Example 2.2 The cosines curve. For each positive integer k, the mapping $\gamma_k : I \longrightarrow \mathbb{R}^d$ given by

$$\gamma_k(t) := \Big(t, \frac{1}{2}\big(1 - \cos(k\pi t)\big), \ldots, \frac{1}{2}\big(1 - \cos(k^{d-1}\pi t)\big)\Big) \quad \text{for all } t \in I,$$

is a $\frac{\sqrt{d-1}}{k}$-dense curve in I^d, as it is proved in [7, Proposition 9.5.4, p. 144]; see also Fig. 1.

Other examples of α-dense curves and their applications, can be found in [11,12, 16,17,27] and references therein.

As it was proposed by Wiener [28, pp. 16–17], the space-filling curves could be used to reduce Lebesgue integration in higher dimensions to Lebesgue integration in one dimension. Therefore, it seems reasonable to think that the α-dense curves could be used to reduce the integral (1.1) to a single variable one, because as we have pointed out above, these curves generalize the space-filling curves. In fact, this idea has been successfully used in [3,4,8,10,19–21] to obtain, under suitable conditions, the approximation stated in (1.2).

For instance, if $f : I^d \longrightarrow \mathbb{R}$ is positive and of class C^1, then the inequality

$$\left| \int_{I^d} f(x_1, \ldots, x_d)dx_1 \cdots dx_d - \frac{\pi}{2}\int_0^1 \left|\sin(k^d \pi t)\right| f\big(\gamma_k(t)\big)dt \right| \leq O\Big(\frac{1}{k}\Big), \quad (2.1)$$

is proved in [21], where $\gamma_k(t)$ is the $\frac{\sqrt{d-1}}{k}$-dense curve of Example 2.2. If f is Lipschitzian, with Lipschitz constant equal to L, then in [4] is stated that

$$\left| \int_{I^d} f(x_1, \ldots, x_d)dx_1 \cdots dx_d - \int_0^1 f\big(\gamma_k(t)\big)dt \right| \leq \frac{Ld}{k}, \quad (2.2)$$

for a suitable α-dense curve γ_k in I^d.

Our goal is to provide an approximation of type (1.2) requiring only the continuity of the function f. For this, we will need certain class of α-dense curves, introduced in [12], which have suitable properties related with the definite integrals. Before to give a formal definition of these curves, we recall the concept of δ-*mesh*. Given $\delta > 0$, and a cube $K \subset \mathbb{R}^d$, by a δ-mesh in K we mean a partition of K into subcubes of the form $[i_1\delta, (i_1 + 1)\delta] \times \cdots \times [i_d\delta, (i_d + 1)\delta]$, where $i_1, \ldots, i_d$ are integers.

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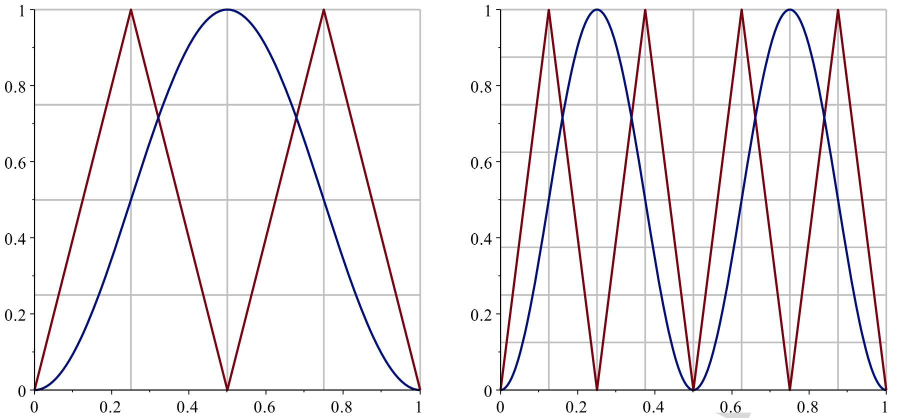

Fig. 1 The graphs of $\gamma_{2^{-k+1}}$ of Example 2.2 (blue) and γ_{2^k} of Lemma 2.1 (red), for $k = 2$ (left) and $k = 3$ (right)

Definition 2.2 (see [12, Definition 3.1]) Given an integer N and $\delta := 1/N$, a continuous mapping $\gamma : I \longrightarrow I^d$ is said to be a δ-uniform curve in I^d if there exists a δ^d-mesh of I, $\mathcal{M}$, a δ-mesh of I^d, $\mathcal{N}$, and a biyective mapping $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ such that $\gamma(J) \subset \varphi(J)$ for every $J \in \mathcal{M}$.

From the above definition is clear that a δ-uniform curve in I^d is, in particular, a $\delta\sqrt{d}$-dense curve in I^d. Indeed, as the diameter of a subcube C of the δ-partition of I^d is $\delta\sqrt{d}$, given $x \in C$ as there is a subinteval J of I such that $\gamma(J) \subset C$, in particular we can take $t \in I$ such that $\|x - \gamma(t)\| \leq \delta\sqrt{d}$.

In [12, Lemma 3.1] was proved the existence of δ-uniform curves in I^d for arbitrarily small $\delta > 0$:

Lemma 2.1 *Given an integer $k \geq 1$, the mapping $\gamma_{2^k} : I \longrightarrow \mathbb{R}^d$ given by*

$$\gamma_{2^k}(t) := \left(t, \min\left\{|kt + 2i| : i \in \mathbb{Z}\right\}, \ldots, \min\left\{|k^{d-1}t + 2i| : i \in \mathbb{Z}\right\}\right) \quad \textit{for all } t \in I,$$

is a 2^{-k}-uniform curve in I^d.

We show in Fig. 1 the graph of some δ-uniform curves in I^2 given in the above theorem.

In Sect. 4, we provide an alternative expression of the mapping used in Lemma 2.1.

3 The main result

We start this section by recalling the notion of modulus of continuity (see, for instance, [26]):

Definition 3.1 Let (X, d) and (Y, d') be metric spaces and $h : (Y, d) \longrightarrow (Y', d')$. The modulus of continuity of h of order $\varepsilon > 0$ is given by

$$\omega(h; \varepsilon) := \sup\{d'(h(x), h(y)) : x, y \in X, d(x, y) \leq \varepsilon\}.$$

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Clearly, h is uniformly continuous on X if, and only if, $\lim_{\varepsilon\to 0^+}\omega(h;\varepsilon)=0$.

Our main result is the following:

Theorem 3.1 *Let $f : I^d \longrightarrow \mathbb{R}$ be continuous, and $\varepsilon > 0$. Then,*

$$\left|\int_{I^d} f(x_1,\ldots,x_d)dx_1\cdots dx_d - \int_0^1 f\big(\gamma_{2^k}(t)\big)dt\right| \le \omega\left(f;\frac{\sqrt{d}}{2^k}\right),$$

for every $k > 1$, being γ_{2^k} a 2^{-k}-uniform curve in I^d.

Proof Fixed $k > 1$, divide I into 2^{dk} subintervals of side-length 2^{-k}, put I_j for $j = 1,\ldots,2^{dk}$, and I^d into 2^k subcubes of side-length 2^{-k}, put C_j for $j = 1,\ldots,2^k$. That is,

$$I^d = \bigcup_{j=1}^{2^k} C_j, \quad I = \bigcup_{j=1}^{2^{dk}} I_j,$$

with C_j and I_j as above. Also, for each integer $m > k$ divide each subcube C_j into $2^{d(m-k)}$ subcubes, put $I^d_{i,j}$ for $i = 1,\ldots,2^{d(m-k)}$, of side-length 2^{-m}. So,

$$I^d = \bigcup_{j=1}^{2^k} C_j = \bigcup_{j=1}^{2^k}\bigcup_{i=1}^{2^{m-k}} I^d_{i,j}.$$

Then, we have two partitions of I^d, one of 2^k subcubes and other of 2^m subcubes, which we write as $\mathcal{C}$ and $\mathcal{I}^d$, respectively.

By the continuity of f, we have

$$\int_{I^d} f(x_1,\ldots,x_d)dx_1\cdots dx_d = \lim_m \frac{1}{2^{dm}}\sum_{i_d=1}^{2^m}\cdots\sum_{i_1=1}^{2^m} f\big(x_{i_1},\ldots,x_{i_d}\big). \tag{3.1}$$

where each $(x_{i_1},\ldots,x_{i_d})$ is a point in the subcube $I^d_{i,j}$ (for instance, its center), for some $1 \le j \le 2^k$ and $1 \le i \le 2^{m-k}$.

Noticing the properties of γ_{2^k} and the partitions $\mathcal{C}$ and $\mathcal{I}^d$, by dividing each interval I_j into 2^m equal subintervals, for every integer $m > k$ we can take a set of points $\{t_1,\ldots,t_{2^{d(m-k)}}\} \subset I_j$, for some $1 \le j \le 2^{dk}$, such that

$$\{\gamma_{2^k}(t_1),\ldots,\gamma_{2^k}(t_{2^{d(k-m)}})\} \subset \gamma_{2^k}(I_j) \subset C_j = I^d_{1,j}\cup\cdots\cup I^d_{2^{m-k},j}.$$

Therefore, from the above considerations, for each $(x_{i_1},\ldots,x_{i_d}) \in I^d_j$ we can take $t_j \in I_j$ such that

$$\|(x_{i_1},\ldots,x_{i_d}) - \gamma_{2^k}(t_j)\| \le \frac{\sqrt{d}}{2^k}. \tag{3.2}$$

Also, as $f(\gamma_{2^k}(t))$ is continuous, its integral over I is given by the formula

$$\int_0^1 f\big(\gamma_{2^k}(t)\big)dt = \sum_{j=1}^{2^{dm}} f\big(\gamma_{2^k}(t_j)\big). \tag{3.3}$$

So, from (3.1), (3.2) and (3.2) we conclude

$$\left|\int_{I^d} f(x_1,\ldots,x_d)dx_1\cdots dx_d - \int_0^1 f\big(\gamma_{2^k}(t)\big)dt\right| \le \lim_m \frac{1}{2^{dm}}\sum_{j=1}^{2^{dm}} \omega\left(f;\frac{\sqrt{d}}{2^k}\right) = \omega\left(f;\frac{\sqrt{d}}{2^k}\right),$$

because of (3.2), $|f(x_{i_1},\ldots,x_{i_d})-f(\gamma_{2^k}(t_j))| \le \omega\left(f;\frac{\sqrt{d}}{2^k}\right)$ for each $(x_{i_1},\ldots,x_{i_d}) \in I_j^d$ and $t_j \in I$ satisfying (3.2). The proof is now complete. □

Let us note that, the uniformly continuity of f on I^d implies $\lim_k \omega\left(f;\sqrt{d}/2^k\right) = 0$ and hence

$$\lim_k \left|\int_{I^d} f(x_1,\ldots,x_d)dx_1\cdots dx_d - \int_0^1 f\big(\gamma_{2^k}(t)\big)dt\right| = 0.$$

Therefore, the approximation provided by the above result can be arbitrarily small.

On the other hand, in general we will need to approximate the single variable definite integral of Theorem 3.1, denoted by $f_k(t)$, by some numerical method, that is to say

$$\int_0^1 f_k(t)dt \simeq Q(f_k, N),$$

and we use the letter N in the notation because of, generally, such numerical methods evaluate the integrand $f_k(t)$ in a given number N of points of the interval I (see the references given in Sect. 1). By setting

$$E(f_k, N) := \left|\int_0^1 f_k(t)dt - Q(f_k, N)\right|,$$

we have the following consequence of Theorem 3.1:

Corollary 3.1 *Let $f : I^d \longrightarrow \mathbb{R}$ be continuous. Then, fixed $k \ge 1$ and $N \ge 1$ the inequality*

$$\left|\int_{I^d} f(x_1,\ldots,x_d)dx_1\cdots dx_d - Q\big(f_k, N\big)\right| \le \omega\left(f;\frac{\sqrt{d}}{2^k}\right) + E(f_k, N), \tag{3.4}$$

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is satisfied, with $f_k(t) := f(\gamma_{2^k}(t))$ *and* γ_{2^k} *as in Theorem 3.1.*

Proof Noticing Theorem 3.1 and the definition of $E(f_k, N)$, we have

$$\begin{aligned} &\left|\int_{I^d} f(x_1,\dots,x_d)dx_1\cdots dx_d - Q(f_k,N)\right| \\ &\leq \left|\int_{I^d} f(x_1,\dots,x_d)dx_1\cdots dx_d - \int_0^1 f\big(\gamma_{2^k}(t)\big)dt\right| \\ &\quad + \left|\int_0^1 f\big(\gamma_{2^k}(t)\big)dt - Q(f_k,N)\right| \leq \omega\left(f;\frac{\sqrt{d}}{2^k}\right) + E(f_k,N), \end{aligned}$$

and so, the inequality (3.4) follows. □

For instance, a quasi-Monte Carlo method method can be used to approximate the single variable integral of Theorem 3.1 and then

$$Q(f_k, N) := \frac{1}{N}\sum_{i=1}^{N} f_k(t_i), \tag{3.5}$$

where $t_i \in I$, $i = 1, \dots, N$, are points of a so called *low discrepancy sequence*. As it is known, the price of this robust, simple and direct method is that it can be extremely slow. In particular, if $t_i := (2i-1)/2N$, for $i = 1, \dots, N$, the approximation error is given by (see, for instance, [22, Theorem 2.10])

$$E(f_k, N) = \omega\Big(f_k; \frac{1}{N}\Big). \tag{3.6}$$

Of course, the approximation method (3.5) can be used in multiple integrals, that is to say, the approximation

$$\int_{I^d} f(x_1,\dots,x_d)dx_1\cdots dx_d \simeq Q(f,N) := \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i), \tag{3.7}$$

holds, where $\mathbf{x}_i \in I^d$, $i = 1, \dots, N$, as above, are the points of a low discrepancy sequence. However, for multivariate functions the approximation error depends on the dimension d of the problem (see [6,9,13,22]), in the sense that for a fixed N the error approximation $E(f, N)$ increases with d (generally, in a exponential way). This phenomena is often called *curse of dimension*.

So, with the approximation proposed in Corollary 3.1 taking $Q(f_k, N)$ as in (3.5), in general, we obtain a more effective approximation than the approximation (3.7). In other words, the proposed *dimensionality reduction* is justified, at least, for quasi-Monte Carlo methods. This fact will be illustrated, by some numerical examples, in Sect. 4.2.

Certainly, the quasi-Monte Carlo methods are used in dimension $d \gg 2$ and some authors could consider formula (3.5) is not a quasi-Monte Carlo method in the strict

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sense. However, in order to do not introduce new notations and terminology, we will include approximations like (3.5) in the class of the quasi-Monte Carlo methods.

4 Some numerical examples

In this section we provide some numerical examples to illustrate our results, and more specifically, the method proposed in Corollary 3.1.

In order to state a more practical formula for the α-dense curve of Lemma 2.1, let us note that fixed the integers k and d, as $\min\left\{\left|2^{k(j-1)}t + 2i\right| : i \in \mathbb{Z}\right\} \in I$ for every $t \in I$ and $j = 2, \ldots, d$, we have

$$-1 \leq 2^{k(j-1)}t + 2i \leq 1,$$

or equivalently,

$$i \in \left[\frac{-1 - 2^{k(j-1)}t}{2}, \frac{1 - 2^{k(j-1)}t}{2}\right].$$

Now, as the above interval has length 1 we find one or two possible values for i. Specifically, the possible values for i are $i := \left\lfloor \frac{-2^{k(j-1)}t}{2} \right\rceil$ if $2^{k(j-1)}t$ is not an odd integer, and $i := \left\lfloor \frac{\pm 1 - 2^{k(j-1)}t}{2} \right\rceil$ otherwise, where $\lfloor a \rceil$ stands for the nearest integer to a. Therefore, we can use the following expression for the j-th coordinates of $\gamma_{2^k}(t)$, for $j = 2, \ldots, d$

$$\min\left\{\left|2^{k(j-1)}t + 2\left\lfloor \frac{-1 - 2^{k(j-1)}t}{2} \right\rceil\right|, \left|2^{k(j-1)}t + 2\left\lfloor \frac{-1 + 2^{k(j-1)}t}{2} \right\rceil\right|, \right.$$
$$\left. \left|2^{k(j-1)}t + 2\left\lfloor \frac{-2^{k(j-1)}t}{2} \right\rceil\right|\right\}, \tag{4.1}$$

which can be computed very quickly with any suitable software.

As the computational complexity of the computation of the minimum given in (4.1) is an $O(3)$, the computational complexity of the computation of $\gamma_{2^k}((2i-1)/N)$ is an $O(3(d-1)) = O(d)$. In particular, if we use a quasi-Monte Carlo method (3.5) to approximate the single variable integral of Theorem 3.1 the time complexity is an $O(dN)$.

The multiple integrals ι_j of the below examples have been computed with the software ®Maple. The used notation is the following:

- ε is the absolute error, that is, $\varepsilon := |\iota_j - \tilde{\iota}_j|$, where $\tilde{\iota}_j$ is the approximation of ι_j obtained with the indicated method in each case.
- C.T. is the computational time, in seconds.

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4.1 Comparison with other α-dense curves

In this section we compare the efficiency, to approximate multiple definite integrals, of the δ-uniform curve of Lemma 2.1 with other α-dense curves. For this goal, we consider the integrals

$$\begin{aligned} \iota_1 &:= \int_{I^2} \frac{xy}{1+xy} dxdy, \quad \iota_2 := \int_{I^2} y \sin\left(2\pi(x+y)\right) dxdy, \\ \iota_3 &:= \int_{I^2} \exp(|x-y|) dxdy, \quad \iota_4 := \int_{I^3} xyzdxdydz \\ \iota_5 &:= \int_{I^3} \exp(-(x+y+z)^2) dxdydz, \\ \iota_6 &:= \int_{I^3} (x+y-z)^2 (x\sin(2\pi y) + y\cos(2\pi x) + xyz)^2 dxdydz, \end{aligned} \tag{4.2}$$

and approximate them by the single variable integral

$$\int_0^1 g(t)dt, \tag{4.3}$$

$g(t)$ being the composition of f (the integrand of the above integrals) with the following α-dense curves:

- Cosines curve. In this case, $g(t) := \left|\sin\left(2^{8d}\pi t\right)\right| f(\gamma_{2^8}(t))$ where $\gamma_{2^8}(t)$ is the $\frac{\sqrt{d-1}}{2^8}$-dense curve in I^d of Example 2.2 (see [21]). This curve is used only for positive and of class C^1 integrands.
- Hilbert curve. We take $g(t) := f(\gamma_{2^8}(t))$, $\gamma_{2^8}(t)$ being the 8-th approximation of the Hilbert space-filling curve (see [2,24]). As we have pointed out in Sect. 2, some space filling curves can be used to reduce multiple integrals to single variable integrals.
- δ-Uniform curve. Here, $g(t) := f(\gamma_{2^8}(t))$, $\gamma_{2^8}(t)$ being the δ-uniform curve of Lemma 2.1, with $\delta := 2^{-8}$.

Also, the integral (4.3) is approximated by the quasi-Monte Carlo method proposed in (3.5)–(3.6).

$$\frac{1}{N}\sum_{i=1}^{N} g\left(\frac{2i-1}{2N}\right).$$

We show in Table 1 the obtained results, for the indicated values of N.

As we can see, in most cases the (absolute) error approximation ε is smaller for approximations obtained with the δ-uniform curve than those obtained with the cosines and Hilbert curves. Also, for $N = 5000$ the C.T. of the δ-uniform curve is significantly smaller than the C.T. of the cosines curve.

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Table 1 Approximations of the integrals ι_j, $j = 1, \ldots 6$, given in (4.2) by some types of α-dense curves

Integral	N	Cosines curve		Hilbert curve		δ-Uniform curve	
		ε	C.T.	ε	$C.T.$	ε	C.T.
ι_1	500	0.0059350	0.63	0.0001943	0.31	0.0000104	0.09
	1500	0.0059441	2.19	0.0000306	0.83	7E-07	3.00
	5000	0.0059582	13.10	0.0000190	2.73	8E-07	0.12
ι_2	500	–	–	0.0010566	0.32	0	0.09
	1500	–	–	0.0002043	0.85	0	0.38
	5000	–	–	0	2.90	0	3.02
ι_3	500	–	–	0.0004331	0.51	0.0000147	0.10
	1500	–	–	0.0000031	0.82	0.0000059	0.65
	5000	–	–	0.0000302	2.96	8E-07	2.482
ι_4	500	0.0000097	0.52	0.0004668	0.32	0.0000918	0.06
	1500	0.0000109	1.47	0.0001659	0.84	0.0000051	0.19
	5000	0.0000412	6.36	0.0000030	2.84	0.0000014	0.71
ι_5	500	0.0208630	1.06	0.0003278	0.37	0.0000695	0.15
	1500	0.0209796	2.47	0.0005819	1.45	7E-07	0.53
	5000	0.0207839	43.63	0.0000460	3.85	0.0000010	4.72
ι_6	500	0.0885380	3.20	0.0127937	0.37	0.0001390	0.21
	1500	0.0547469	20.04	0.0022514	2.201	0.0000821	1.13
	5000	0.0546944	83.19	0.0006028	11.05	0.0000317	12.81

4.2 Dimensionality reduction in quasi-Monte Carlo methods

The purpose of this example is to justify the proposed *dimensionality reduction* commented in Sect. 3, when a quasi-Monte Carlo method is used, to approximate the multiple integral (1.2) by δ-uniform curves. For this we consider the integrals

$$\begin{aligned} \iota_1 &:= \int_{I^d} 2^d \prod_{i=1}^{d} x_i \, dx_1 \cdots dx_d, \\ \iota_2 &:= \int_{I^d} \left(\prod_{i=1}^{d} |1 - x_i| \right) \left(\sum_{i=1}^{d} (-1) x_i^i \right) dx_1 \cdots dx_d \\ \iota_3 &:= \int_{I^d} \exp\left(-\frac{1}{2} \left(\sum_{i=1}^{d} x_i \right)^2 \right) dx_1 \cdots dx_d, \quad \iota_4 := \int_{I^d} \prod_{i=1}^{d} \left| x_i - \frac{1}{3} \right| dx_1 \cdots dx_d, \end{aligned} \tag{4.4}$$

and approximate them the quasi-Monte Carlo method

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Table 2 Comparison of the quasi-Monte Carlo method with and without the use of a δ-uniform curve for the integrals given in (4.4)

Integral	Dimension	Formula (4.5)		Formula (4.6)	
		ε	C .T.	ε	C .T.
ι_1	3	0.0012381	14.75	0.0001278	9.29
	5	0.0047808	23.05	0.0011278	14.18
	10	0.0604033	23.87	0.0102027	14.57
ι_2	3	0.0000280	27.93	0.00000534	22.94
	5	0.0000534	46.38	0.00003780	35.15
	10	0.0000136	66.01	0.00000524	46.19
ι_3	3	0.0002463	16.16	0.0000959	11.68
	5	0.0001885	24.46	0.0000240	16.00
	10	0.0001460	41.67	0.0000589	24.83
ι_4	3	0.3453033	24.02	0.3452953	19.41
	5	0.1613389	31.95	0.1613396	23.74
	10	0.0214609	52.15	0.0214607	35.65

$$\int_{I^d} f(x_1, \ldots, x_d)dx_1 \cdots dx_d \simeq \frac{1}{N}\sum_{i=0}^{N} f\big(\mathbf{x}_i\big), \tag{4.5}$$

and the single variable integral of Corollary 3.1 is approximated also by the quasi-Monte Carlo method

$$\int_0^1 f\big(\gamma_{2^7}(t)\big)dt \simeq \frac{1}{N}\sum_{i=0}^{N} f\big(\gamma_{2^7}(t_i)\big), \tag{4.6}$$

respectively, where $\gamma_{2^7}(t)$ is the 2^{-7}-uniform curve of Lemma 2.1, and the points $\mathbf{x}_i$ and t_i are described below. We show the obtained results in Table 2. In each case, we take $N := 10^4$.

Fixed an integer $b \geq 2$ we recall that the *radical inverse function* (see, for instance, [9, Example 2.4]) is defined as

$$\phi_b(m) := \sum_{i=1}^{\infty} \frac{m_i}{b^i} \quad \text{for each integer } m := \sum_{i=1}^{\infty} m_i b^{i-1},$$

where each $m_i \in \{0, \ldots, m-1\}$ are the "coordinates" of the integer m in base b. From the above function, we can define the Halton sequence (see, for instance, [9, Example 2.4])

$$\mathbf{x}_i := \big(\phi_{p_1}(i), \ldots, \phi_{p_d}(i)\big), \quad \text{for } i = 0, 1, 2, \ldots$$

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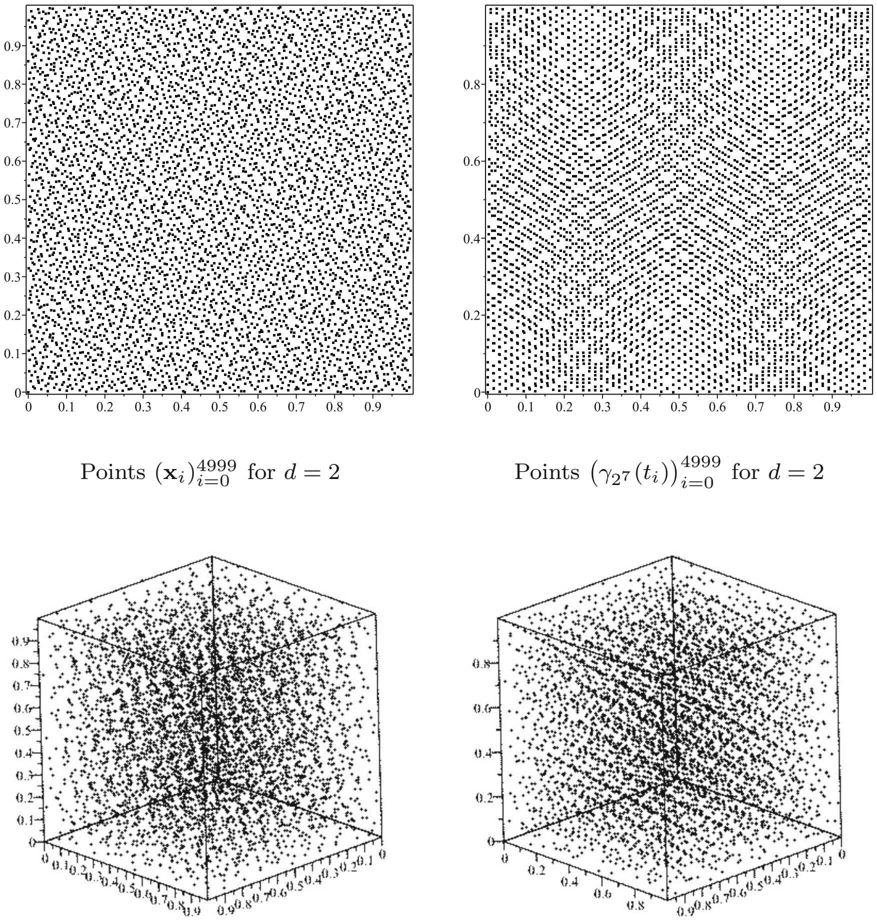

Points $(\mathbf{x}_i)_{i=0}^{4999}$ for $d = 2$ Points $\left(\gamma_{2^7}(t_i)\right)_{i=0}^{4999}$ for $d = 2$

Points $(\mathbf{x}_i)_{i=0}^{4999}$ for $d = 3$ Points $\left(\gamma_{2^7}(t_i)\right)_{i=0}^{4999}$ for $d = 3$

Fig. 2 The points $(\mathbf{x}_i)_{i=0}^{N}$ and the transformed by $\gamma_{2^7}(t)$ of the points $(t)_{i=0}^{N}$, for the indicated values of N and d

where p_j is the j-th prime, for $j = 1 \ldots, d$. In Fig. 2 we show some points of Halton sequence and the sequence $\left(\gamma_{2^7}(t_i)\right)_{i \geq 0}$, with $t_i := \phi_{p_2}(i)$.

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