Keywords 1: Decision support systems

Journal Pre-proof

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PII: S0377-2217(20)30847-X
DOI: https://doi.org/10.1016/j.ejor.2020.09.043
Reference: EOR 16789

To appear in: *European Journal of Operational Research*

Received date: 16 May 2020
Accepted date: 28 September 2020

Please cite this article as: Carmen Herrero , Antonio Villar , Group decisions from individual rankings: The Borda-Condorcet rule, *European Journal of Operational Research* (2020), doi: https://doi.org/10.1016/j.ejor.2020.09.043

Highlights:

- A protocol that yields cardinal evaluations out of ordinal information.
- Admits the presence of ties and non-comparable alternatives.
- Permits finding endogenously different “divisions” among the alternatives.
- Can be naturally extended to a multidimensional context
- Applicable to many different types of problems

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Group decisions from individual rankings: The Borda-Condorcet rule

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Abstract

This paper presents an evaluation protocol that transforms a collection of rankings, defined over a set of alternatives, into a complete, transitive, and cardinal assessment. It combines the ideas of Borda and Condorcet by computing the support that each alternative receives on average when confronted with any other. The protocol evaluates those alternatives in terms of pairwise comparisons but weighs the outcomes differently depending on how each alternative fares with respect to the others. The evaluation appears as the stable distribution of an iterative process in which each alternative competes randomly with any other, and results in a vector of positive numbers that tells us the relative support of the different options. We show that this protocol does not require linear orderings and can also be applied in the presence of incomplete rankings and when dealing with several issues simultaneously.

Keywords 1: Decision support systems

Keywords 2: Evaluation function, multiple rankings, Borda and Condorcet rules, stable distributions, incomplete rankings.

Acknowledgments: Thanks are due to M.A. Ballester, C. Ballester, S. Barberá, M. Baucells, J.L. García Lapresta, M. Mariotti, P. Manzini, J. Masso, M. Vorsatz, and three anonymous referees for helpful comments and suggestions. The first author acknowledges financial support from the Spanish *Ministerio de Economía, Industria y Competitividad* under project ECO2015-65820P, IMeRA, and Generalitat Valenciana, PROMETEO 2019/037. The second author recognizes financial support from the Spanish *Ministerio de Economía, Industria y Competitividad* under project ECO2015-65408-R (MINECO/FEDER).

1 Introduction

1.1 The problem

We consider here the problem of evaluating a series of alternatives that are ordered by a group of individuals (group decision making). Formally, this problem is similar to that faced by a single decision-maker who has to choose among a series of options by applying several criteria simultaneously (multi-criteria decision making). A committee having to evaluate a set of candidates for a position in a company is an example of the first type. An example of the second type is that of a firm having to decide on where to place a new factory depending on aspects such as the price of the land, location relative to the main roads, available surface, and nearby facilities. One can even think of the more complex case in which a group of individuals has to evaluate different alternatives applying several criteria simultaneously (group multi-criteria decision making).

A general way of formalizing the overall judgment is by recurring to an *evaluation function* (EF for short), that is, a mapping that associates to the individuals' rankings a vector of numbers, one for each alternative, so that a higher number means a better alternative. An EF is thus a procedure intended to solve those decision problems described above, by aggregating the judgment of a group of individuals or different criteria (or both), into numbers that permit one value the alternatives. This is a classical family of problems that appears in multiple contexts, from social choice theory to applied decision making.

Our informational inputs are the ranking of alternatives (i.e. ordinal and interpersonally non-comparable preferences). An EF can present different degrees of informational output. In particular: (A) It can merely pick up the best alternative (e.g., by giving value 1 to the best option and 0 to the rest). (B) It can provide a complete ranking of the alternatives, i.e., which one occupies the first position, which one the second, and so forth. (C) It may permit one to assess not only the ranking but also the "distance" between the different alternatives (i.e., a cardinal mapping that conveys information about how much better is an alternative than another). Trivially, (C) implies (B) and (B) implies (A). The most convenient type of EF depends on the nature of the problem we address. (A) might be enough in some situations, whereas (C) might be required in some others.

We shall focus here on the most demanding case, type (C), which looks for a cardinal evaluation that uses ordinal information as inputs. Indeed, the purpose of this paper is to present a cardinal evaluation function that combines the ideas of Borda and Condorcet by computing the support that each alternative receives on average when confronted with any other in a series of tournaments. We shall refer to this rule, not very imaginatively, as the Borda-Condorcet rule.

1.2 The Borda-Condorcet rule in perspective

In 1770 Borda proposed a method to socially rank alternatives assigning each one a score based on the place they occupy on the voters' list (Borda, 1784). In particular, each option or candidate was given, for each ballot, a number of points equal to the number of alternatives ranked below. Facing the same question, Condorcet offered a different solution, focusing on the support each alternative received vis-à-vis the others (Condorcet, 1785). Borda cared mostly for how much support an alternative had and Condorcet for how many individuals preferred that alternative when confronted with any other. Despite those differences, both procedures share some key features. In particular, they use the information of the complete rankings of the individuals to make the evaluation (as opposed to plurality voting, say) and take as primary inputs the numbers of individuals that prefer one alternative to another (the so-called Condorcet numbers that can also be used to determine the Borda scores). See Young 1994 for a discussion.

The controversy between Borda and Condorcet methods goes back to the XVIII century in the French Royal Academy of Sciences. It has continued ever since, as each method gives rise to a family of sensible and appealing procedures: scoring rules and Condorcet-consistent rules (see Moulin, 1988).

Our proposal here is a compromise between the Borda and the Condorcet procedures. It takes into account both the "distance" between the alternatives in the individuals' preferences and the "number of individuals" who support them. This idea is reminiscent of those in Morales (1798, 1805), a Spanish thinker contemporary to Borda and Condorcet. He proposed the ranking of alternatives to be related to the "amount of opinion" of the citizens in favour of each of them. By that, he meant the sum of the number of times an alternative beats any other. Morales defended the Borda rule, but he was concerned for the conditions under which this rule declared best the Condorcet winner when there was one.

The evaluation procedure we propose adopts Condorcet's strategy to evaluate alternatives in terms of pairwise comparisons, but weighs the outcomes of those comparisons differently, depending on how each alternative performs relative to the others (a pinch of Borda count). In particular, the value of each alternative is determined by the relative frequency with which it beats another one when randomly matched in an indefinite iterative process. The evaluation of the alternatives appears as the stable distribution of this iterative process. It consists of a vector of positive numbers that tells us not only the order but also the relative value of the different alternatives. The Borda-

Condorcet rule is thus an evaluation function in which cardinality is generated endogenously, depending on the probabilities of each alternative beating some other, rather than through an external scoring system. Indeed, this new rule is neither Condorcet consistent nor a scoring rule.

Using stable distributions as a way of evaluating the relevance of different alternatives is far from new. We find this type of approach in a diversity of fields, as in the study of centrality in networks (e.g., Freeman, 1977, Wasserman & Faust, 1994, Newmann, 2003), the relevance of the journals in citation analysis (Pinsky & Narin, 1976, Liebowitz & Palmer, 1984, Palacios-Huerta & Volij, 2004, Albarrán et al., 2018), the ranking of teams in competitions (Keener, 1993, Slutzski & Volij, 2006, Anderson et al., 2009), or the measurement of population health and happiness (Pifarré & Dudel, 2019, Herrero & Villar, 2020, Ravin-Ripoll et al., 2020). See also Chebotarev & Shamis (1998), Saaty (2003), and Herrero & Villar (2018).

The procedure we propose goes back to Daniels (1969) and Moon & Pullman (1970), regarding the ranking of teams in round-robin tournaments, who refer to this type of process as the *fair-bets* protocol. Laslier (1997) provides an interpretation of the fair-bets protocol through the so-called ping-pong procedure, quite similar to ours. Slutsky & Volij (2006) go a step further and provide two axiomatic characterizations of that protocol, within the tournaments setting.

Previous contributions refer to the case where individual rankings are complete, linear (there are no ties), there is a single issue under consideration, and there are no alternatives fully dominated by others.

The BC rule proposed here is an evaluation protocol that shares the essence of those procedures but differs in some relevant respects. In particular:

(i) The BC rule allows for the presence of ties and incomplete preferences. Both aspects are important and singularize this evaluation protocol. The ability to cope with incomplete rankings, in particular, makes this evaluation function very general and attractive.

(ii) The BC rule can identify the presence of subsystems and allows evaluating them separately. Subsystems appear when alternatives can be gathered into different divisions or leagues. This happens when all individuals rank a subset of alternatives below any other.

(iii) The BC rule can be naturally extended to evaluation problems involving several issues simultaneously (group multi-criteria evaluations).

(iv) The BC rule is derived from the primitives of the model (individual rankings). It yields an intuitive evaluation function that involves both the Borda scores and the Condorcet numbers, making explicit how this rule combines the evaluation protocols of both thinkers.

Consider the following example that illustrates the relevance of some of those features. A new Art Gallery is deciding on how to allocate the available exhibition space between a certain number of promising young artists. To make the decision, it asks a series of specialists, museum directors, and art collectors to rank those young artists by relevance. Then the Art Gallery faces the problem of transforming those rankings into numbers that distribute the available space as a function of the artists' relevance. A type (C) evaluation is required. Note that one might reasonably expect the presence of ties and incomplete rankings in this context.

2. The BC rule: the reference model

Let A = {*1, 2, ..., m*} be a finite set of alternatives, with cardinal *m*, and *N* = {1, 2, ..., *n*} a finite set of individuals. Here individuals can be interpreted as "judges," decision-makers, or evaluators (e.g., members of a committee). By twisting the meaning of the words, we can also think of those "individuals" as the different criteria by which a single decision-maker compares a collection of alternatives.

Let us assume that, given a set of alternatives, A, each individual $h \in N$ provides a ranking of those alternatives, $R_h(A)$, which is a weak ordering (a complete and transitive binary relation). A *profile* is a collection of *n* of those rankings, one for each individual, denoted as $R(A) = [R_1(A), R_2(A), \dots, R_n(A)]$. Our task is to provide an evaluation of the alternatives in A based on the information in $R(A)$. More precisely, we look for an *evaluation function F* such that, for each possible profile, $R(A)$, yields a vector $F[R(A)] \in \mathbb{R}^m_+$ such that $F_i[R(A)] \geq F_j[R(A)]$ implies that alternative *i* is regarded as better than or equal to alternative *j*.

We denote by n_{ij} the number of individuals who prefer alternative *i* to alternative *j*, by n_{ji} the number of individuals who prefer *j* to *i*, and by $e_{ij} = e_{ji}$ the number of individuals who are indifferent between both options. By construction, $n = n_{ij} + n_{ji} + e_{ij}$. Let now:

$$c_{ij} = n_{ij} + \frac{e_{ij}}{2}$$

denote the number of people who prefer *i* over *j*, including one half of those who are indifferent. Those c_{ij} are the ***Condorcet numbers***, which yield the Condorcet ranking (if it exists), and can also be used to obtain the Borda score of each alternative (see Moulin, 1988). This is so because the sum over $j \neq i$ of all those c_{ij} corresponds, precisely, to the number of alternatives ranked below *i*. The ***Borda score*** of alternative *i* is thus given by:

$$B(i) = \sum_{j \neq i} c_{ij}$$

That is the number of individuals that prefer alternative *i* to any other, including one half of those who are indifferent.

Remark: *As we allow for indifferences in our setting, the notions of ranking, Condorcet numbers, or Borda scores are reinterpreted in this scenario.*

Consider now the following evaluation protocol. For a given problem (*A, N*) and a given profile $R(A)$, select a pair of alternatives arbitrarily, $(i, j) \in A$, and an individual $h \in N$ at random. If this individual prefers *i* to *j,* alternative *i* is declared the winner in this pairwise confrontation. Similarly, if *j* is preferred to *i* by the individual, the winner is *j*. When the individual is indifferent between both alternatives, we toss a coin, and each alternative is declared the winner with equal probability. Both the individual and the unchosen alternative go back to the pull before a new round starts (extraction with replacement). A new alternative is chosen to compete with the previous winner, according to the preferences of a new individual, also randomly selected, and the same protocol is implemented. Then, by running this process infinitely many times, we can evaluate each alternative by the fraction of times that it keeps the floor as the winning option in the long run.

It is immediate to see that the probability that alternative *i* beats alternative *j* in a given confrontation, conditional on *j* keeping the floor in the former one, is given by:

$$\frac{1}{m(n-1)} c_{ij}$$

That is, the Condorcet number (that measures how many times *i* beats *j*) times the probability that *i* be randomly chosen. Similarly, the probability that alternative *i* keeps the floor for one more round is given by:

$$\frac{1}{m(n-1)} B(i)$$

Let **BC** denote the ***Borda-Condorcet matrix***, defined as that matrix in which the off-diagonal elements correspond to the Condorcet numbers and the diagonal elements to the Borda scores. That is,

$$\mathbf{BC} = \begin{pmatrix} B(1) & c_{12} & \dots & c_{1m} \\ c_{21} & B(2) & \dots & c_{2m} \\ & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & B(m) \end{pmatrix}$$

The protocol presented above gives rise to a Markov chain, specified by a (column) stochastic matrix $\mathbf{M} = \frac{1}{m(n-1)}\mathbf{BC}$. The fraction of time that each alternative keeps the floor, w_i, is thus given by the stationary state of that dynamic process. Namely, the components of the positive eigenvector associated with the dominant eigenvalue of matrix **M**, according to the equation: $\mathbf{M}\boldsymbol{w} = \boldsymbol{w}$. The ith component of this vector **w** is given by:

$$w_i = \frac{1}{1 - B(i)} \sum_{j \neq i} c_{ij} w_j \qquad [1]$$

Note that the eigenvectors of matrix **M** coincide with those of the Borda-Condorcet matrix **BC** so that we can use matrices **M** and **BC** interchangeably. As eigenvectors have one degree of freedom, by imposing the condition $\sum_{i=1}^{m} w_i = 1$ we can interpret the values in equation [1] as the fraction of time that each alternative keeps competing against the others in the long run. Those values, therefore, measure the relevance of the alternatives in terms of the relative support accrued by the different options, and it is the evaluation we propose here. We call this mapping the *Borda-Condorcet* Evaluation Function, or the *BC rule*, for short.

It is well known that this dominant eigenvector always exists, and it is non-negative; besides, when **BC** is an irreducible matrix, we can ensure its uniqueness and strict positiveness (e.g. Berman & Plemmons, 1994).

The BC rule is an evaluation protocol that always provides a solution to any evaluation problem. This solution is unique and strictly positive whenever matrix **BC** is irreducible. Besides, it satisfies many of those properties that are standard in the field of social choice theory or multi-criteria decision making. In particular:

1. *Universal domain:* The evaluation function is well defined for all types of rankings.
2. *Anonymity and Neutrality:* Neither the names of the individuals nor those of the alternatives matter; only the rankings do.
3. *Weak Pareto Principle:* If all individuals strictly prefer alternative i to alternative j, then the rule gives a higher value to the first.
4. *Weak unanimity:* If all individuals rank alternative i first, then the rule attaches the highest value to this alternative.
5. *Independence of fully dominated alternatives:* If an alternative is ranked last by all individuals, then it receives a value of 0, and removing it from A does not change the relative evaluation of the remaining.
6. *Independence of generally unconcerned individuals:* If an individual is indifferent between all alternatives, it plays no role in the evaluation, and removing him/her from N does not change the outcome.

7. *Monotonicity:* If individuals change their preferences by moving one alternative up in the ranking, the new evaluation will never attach a smaller value to this alternative.
8. *Replication invariance:* Replicating all individuals, while keeping their preferences, does no change the valuation of the alternatives.

In summary: The Borda-Condorcet rule is a criterion that provides a complete and transitive cardinal evaluation of alternatives that combines some of the principles informing the Borda and Condorcet choice rules. Mind, though, that the ordering provided by the BC rule may disagree with those obtained by either Borda or Condorcet, as it implements a different evaluation criterion. Indeed, the BC rule is neither Condorcet consistent nor a scoring rule (we discuss this point in the Appendix).

3. Reducible matrices

The uniqueness and strict positiveness of the eigenvector associated with the dominant eigenvalue of matrix **BC** may fail when the Borda-Condorcet matrix is reducible. Note that what a reducible matrix implies in this context is the existence of "subsystems," which are globally ranked (that is, one subsystem is better than another). This fact appears in the form of zeroes in the dominant eigenvector of matrix **BC**, which means that the group of alternatives with positive entries strictly dominates that with zeroes. Now observe that, if we consider the set of dominated alternatives on its own, we can find that not all of them are equally worthy. Indeed, that set may, in turn, consists of two or more different subsystems.

In short: whenever the Borda-Condorcet matrix is reducible, we can divide the alternatives into different "divisions" in such a way that only the alternatives within the same division are comparable. Importantly, those divisions are ranked; that is, anyone in a higher division dominates all alternatives in an inferior one (Moon & Pullman 1970 also noted this fact).

Let us illustrate this feature utilizing a real-life example. It refers to the comparison of the ten top UK universities in terms of four different criteria.

Example 1: Webometrics Ranking of top Universities in the UK

Since 2004, the Cybermetrics Lab presents a ranking of the performance of universities from all around the world. They consider four indicators and rank all universities independently in each of them (see https://www.webometrics.info/es/world for details). Those indicators are Presence (public knowledge shared in terms of the number of pages in the web), Visibility (impact of the web contents due to the external links), Transparency (citations of the top-cited researchers), and Excellence (number of papers among the top 10% most cited in each discipline). Each of those four indicators is assigned a different weight: 5% for presence, 50% for Visibility, 10% for Transparency, and 35% for Excellence.

We consider here the evaluation of the ten top universities in the UK, according to the Webometrics ranking published in January 2020. The web page provides the ranking of those Universities in each of the items, which appears in Table 1.

Table 1: Rankings of the top ten UK Universities according to four indicators

Overall ranking	Universities	(I) Presence	(II) Impact	(III) Openness	(IV) Excellence
1	Oxford	1	1	1	1
2	Cambridge	2	2	2	3
3	UCL	3	3	3	2
4	Edinburgh	4	4	7	7
5	Imperial Co	7	8	4	4
6	Manchester	5	5	6	6
7	King's Co	8	10	5	5
8	Leeds	10	6	9	9
9	Warwick	6	7	10	10
10	Nottingham	9	9	8	8

Source https://www.webometrics.info/es/world

As the Webometrics attaches different weights to the different indicators, we adopt those weights when building the Borda-Condorcet matrix to compare the ten selected universities. To do so, we build the **BC** matrix as follows: One hundred individuals are ranking the top 10 UK universities. Five of them agree on the ranking corresponding to the column (I) in Table 1. Fifty coincide on the ranking described by column (II), ten on the ranking of the column (III), and 35 on the ranking of the column (IV). The resulting BC matrix appears in Table 2.

Table 2: BC matrix of the top ten UK universities

	Oxf	Cam	UCL	Edin	Im Co	Man	King's	Leeds	War	Not
Oxf	900	100	100	100	100	100	100	100	100	100
Cam	0	765	65	100	100	100	100	100	100	100
UCL	0	35	735	100	100	100	100	100	100	100
Edin	0	0	0	465	55	55	55	100	100	100
ImCo	0	0	0	45	385	45	100	50	45	100
Man	0	0	0	45	55	455	55	100	100	100
King's	0	0	0	45	0	45	235	50	45	50
Leeds	0	0	0	0	50	0	50	245	95	50
War	0	0	0	0	55	0	55	5	170	55
Not	0	0	0	0	0	0	50	50	45	145

The first thing to observe is that the BC matrix is reducible. We obtain the following outcomes when normalizing those eigenvalues by setting the highest one to 100:

(1) The University of Oxford dominates all other universities, standing alone in the first place. The first eigenvector is (100, 0, 0, …, 0), meaning that Oxford is the first one, and, when comparing all ten universities together, there is no distinction among the remaining 9. That is, Oxford appears as belonging to a higher "division."

(2) When considering the evaluation of all the universities different from Oxford as an independent problem, we find again that the resulting **BC** matrix is reducible. The corresponding eigenvector is now (100, 53.85, 0, 0, …, 0), meaning that Cambridge and UCL stand above the rest, and, when comparing with these two universities, the remaining ones are indistinguishable. Thus, there is a second "division" consisting of the University of Cambridge and the University College London.

(3) Finally, if we now rank the remaining universities as a separate group, the resulting eigenvector is: (100, 62.32, 90, 90.15, 16.98, 12.56, 6.20). This group of universities conforms to the "third division" among the top ten.

The case of a reducible matrix illustrates well a key trait of the BC rule: the evaluation of each alternative is relative to the alternatives with which it is compared. This is obvious from the very beginning, but Example 1 puts it in perspective. Note that reducibility not only informs about the existence of different "divisions" within the set of alternatives compared but also provides an endogenous way of identifying them. That is a relevant aspect of many evaluation problems. This suggests an interesting application of the BC rule: shortlisting. That is, finding if there are one or several categories of alternatives and identifying them endogenously. This possibility might be especially relevant when individuals can evaluate alternatives with precision, but such an evaluation is very costly. Think, for instance, of the evaluation of complex research proposals by a scientific committee, or the selection of investment plans in a large city. A preliminary round in which individuals rank the alternatives based on some key traits may well identify the proper shortlist to which a more in-depth evaluation can be applied.

4. Incomplete rankings

There are cases in which some individuals might be unable to rank all alternatives (partial orderings). Let us see how we can introduce this case into our model.

Our evaluation protocol starts by selecting two alternatives at random, an individual also at random, and declaring the winner of that round that alternative preferred by the individual in question. Then, we randomly choose a new alternative to compete with the previous winner, based on another individual's preferences, and so on. Allowing for incomplete preferences implies that we can find three possible cases within each round: (1) the two chosen alternatives are comparable in the selected individual's ranking, and one is above the other (which determines that the preferred alternative keeps the floor). (2) The two chosen alternatives are comparable in the selected individual's ranking and turn out to be indifferent, in which case we toss a coin and luck decides the winner. (3) The chosen individual cannot compare the two alternatives; in this case, the former winning alternative still keeps the floor, and a new random extraction of alternative and individual occurs. If the non-comparability happens in the first round, we keep making random extractions of alternatives and individuals until we find two that are comparable.

This simple principle allows us to deal with incomplete rankings by slightly modifying the transition matrix. As before, let n_{ij} the number of individuals who strictly prefer alternative *i* to alternative *j*, let n_{ji} the number of individuals who strictly prefer *j* to *i*, let $e_{ij} = e_{ji}$ the number of individuals who are indifferent between both options, and let d_{ij} the number of individuals who are unable to compare alternatives i, j. Now $n = n_{ij} + n_{ji} + e_{ij} + d_{ij}$ and, as before, the probability that alternative *i* beats alternative *j* in a given confrontation, conditional on *j* keeping the floor in the former one, is proportional to the Condorcet number $c_{ij} = n_{ij} + e_{ij}/2$, whereas the probability that alternative *i* keeps the floor for one more round is proportional to the newly defined Borda score (which is no longer the sum of the Condorcet numbers):

$$\bar{B}(i) = \sum_{j \neq i} \left(c_{ij} + d_{ij} \right)$$

Again, to evaluate the overall likelihood that one alternative beats another in the long run, we build a matrix $\overline{\mathbf{BC}}$, where the off-diagonal elements are the Condorcet numbers, as before, and the diagonal elements are the newly defined Borda scores. More explicitly,

$$\overline{\mathbf{BC}} = \begin{pmatrix} \bar{B}(1) & \dots & c_{1j} & \dots & c_{1m} \\ \dots & & \dots & & \dots \\ c_{m1} & \dots & c_{mj} & \dots & \bar{B}(m) \end{pmatrix}$$

And we obtain:

$$\bar{w}_i = \frac{1}{1-\bar{B}(i)} \sum\nolimits_{j \neq i} c_{ij} \bar{w}_j \qquad [2]$$

Let us now illustrate how to deal with non-comparable alternatives with another real-life example.

Example 2: Ranking cities to pursue university studies

Think of a student that is considering where to engage in university studies. In looking for the right city, there are three aspects relevant to make a decision: the opinion of former students, living in a sustainable environment, and studying in a safe city.
The student relies on the information provided by three different rankings of cities:[1] the QS Best Student City ranking (education possibilities), the Arcadis Sustainable Cities Index, and The Economist Safest cities index.

To reduce the size of the problem, our student starts by focussing on the 10 top cities in the QS Best Student City ranking; then picks among those cities the ones that appear in at least one of the other two rankings in the top 20 positions. The result is a shortlist of 9 cities: London, New York, Paris, Seoul, Singapore, Sydney, Tokyo, Vienna, and Zurich. The ranking of those cities in the three issues appears in Table 4.

Notice that neither Sydney nor Tokyo appear in the Sustainability ranking, and neither Paris nor Vienna appear in the safety ranking. We do not have information on the situation of those cities concerning the others in those aspects.

Assuming that all three issues are equally important, we obtain the Borda-Condorcet matrix (Table 3) and then the corresponding eigenvector associated to the dominant eigenvalue (i.e., the vector that provides with the ranking of cities and their relative valuation, given in the column "BC valuation (1)" in Table 4).

Table 3: BC matrix of example 2

	Lon	NY	Paris	Seoul	Sing	Sydney	Tokio	Vienna	Zurich	BC val
London	20	3	2	2	2	1	1	2	3	39.4
NY	0	7	1	0	1	0	0	0	1	0.84
Paris	0	1	16	1	1	1	0	1	1	3.63
Seoul	1	3	1	12	1	0	0	1	1	4.85
Sing	1	2	1	2	14	1	0	1	2	7.57

[1] https://www.topuniversities.com/city-rankings/2019; https://www.arcadis.com/en/united-states/our-perspectives/sustainable-cities-index-2018/united-states/, https://safecities.economist.com/safe-cities-index-2019/.

Sydney	1	2	0	2	1	18	0	1	1	10.9
Tokio	1	2	1	2	2	2	23	1	2	100
Vienna	0	2	1	1	1	0	0	16	1	2.63
Zurich	0	2	1	2	1	1	0	1	12	3.03

As shown in the last column of Table 3, our student would conclude that Tokyo stands out of the remaining cities, followed by London and Sydney, then Singapore, Seoul, and Paris, then Zurich, Vienna, and New York.

It is important to note that non-comparability does not imply putting non-comparable alternatives at the bottom of individual rankings. To show this, we present in Table 4 a new BC matrix, assuming that Sydney and Tokyo are indifferent and last in Sustainability and, similarly, Paris and Vienna are also placed at the bottom in Safety and indifferent between them. Applying the BC rule to those preferences yields a different ranking (last column of Table 4) in which London would be the best option, followed by Tokyo and Singapore, then Seoul, Sydney, Zurich, Paris, Vienna, and New York. The reason for this discrepancy is that non-comparable alternatives have fewer pairwise confrontations, rather than losing all of them. Table 5 presents a summary of the results, as well as the order of the cities in the different issues.

Table 4: BC matrix of example 2 treating non-comparabilities as worst options

	Lon	NY	Paris	Seoul	Sin	Sydney	Tokio	Vienna	Zurich	BC val
Lon	20	3	3	2	2	2	2	3	3	100
NY	0	7	2	0	1	1	1	1	1	9.04
Paris	0	1	8,5	1	1	2	1	1,5	1	12.11
Seoul	1	3	2	12	1	1	1	2	1	25
Sing	1	2	2	2	14	2	1	2	2	34.61
Sydney	1	2	1	2	1	10,5	0,5	2	1	21.15
Tokio	1	2	2	2	2	2,5	15,5	2	2	44.23
Vienna	0	2	1,5	1	1	1	1	8,5	1	11.34
Zurich	0	2	2	2	1	2	1	2	12	19.23

Table 5: Ranking of the cities according to the issues, BC valuation, and overall rankings

	Sustainability	Safety	Education	Overall ranking (1) (noncomparability)	Overall ranking (2) Noncomparable elements are at the bottom

London	1	5	1	2	1
New York	6	6	8	9	9
Paris	7		3	6	7
Seoul	5	4	6	5	4
Singapore	2	2	9	4	3
Sydney		3	5	3	5
Tokyo		1	2	1	2
Vienna	3		7	8	8
Zurich	4	7	4	7	6

Source: own elaboration and https://www.topuniversities.com/city-rankings/2019; https://www.arcadis.com/en/united-states/our-perspectives/sustainable-cities-index-2018/united-states/, https://safecities.economist.com/safe-cities-index-2019/

Incomplete rankings appear when the individuals are not able to compare all the alternatives accurately. It often happens when there are many alternatives to be compared, and the individuals divide into several teams, each of which evaluates a subset to those options. A case in point is what usually happens when selecting candidates in the job market: not all professors interview all the candidates. Then one has to integrate those partial evaluations into a single one. The BC rule is a procedure that can cope with this situation.

5. Multi-issue evaluation

We now consider the case in which a group of individuals has to evaluate a collection of alternatives regarding different issues or dimensions simultaneously (group multi-criteria decision making). Our protocol can also accommodate this situation.

Assume, for the time being, that all dimensions are equally important and let us see how the BC rule behaves in this context. We start by selecting two alternatives at random, an issue at random, and an individual also at random; we declare the winner relative to the issue that alternative preferred by the individual. Then, a new alternative is randomly chosen to compete with the previous winner, based on another issue, randomly selected, according to another individual's ranking in this new issue, and so forth. So now, alternatives compete within the corresponding dimensions, and issues are also randomly chosen, as alternatives and individuals.

Let us admit, for the sake of generality, that multidimensional evaluation may also involve incomplete rankings. For a given issue, *k,* let n_{ij}^{k} the number of individuals who strictly prefer alternative *i* to alternative *j*, regarding issue *k.* Let n_{ji}^{k} the number of individuals who strictly prefer *j* to *i*, and let e_{ij}^{k} the number of individuals who are indifferent between both options, and let d_{ij}^k the number of individuals who are unable to compare alternatives i, j in issue *k.* Now, if there are *K* issues, for any *k,* we have $n = n_{ij}^k + n_{ji}^k + e_{ij}^k + d_{ij}^k$, for each *k*, and the probability that alternative *i* beats alternative *j* in a given confrontation, conditional on *j* keeping the floor in the former one, is proportional to the average Condorcet number,

$$\gamma_{ij} = \frac{1}{K}\sum\nolimits_k c_{ij}^k$$

where c_{ij}^k is the Condorcet number corresponding to dimension k (that is, $c_{ij}^k = n_{ij}^k + e_{ij}^k/2$).

Now, the probability that alternative *i* keeps the floor for one more round is proportional to the average Borda score, that is:

$$\beta(i) = \frac{1}{K}\sum\nolimits_k \bar{B}^k(i)$$

where $\bar{B}^k(i) = \sum_{j\neq i}\left(n_{ij}^k + \left(e_{ij}^k/2\right) + d_{ij}^k\right)$

Again, to evaluate the overall likelihood that one alternative beats another in the long run, we build the corresponding Borda-Condorcet matrix **BC***, where the (i, j) entry is the average Condorcet number, γ_{ij}, and the diagonal entries are given by the average Borda scores, $\beta(i)$. It is easy to check that this matrix corresponds to the average of the involved single-dimensional matrices:

$$\mathbf{BC}^* = \frac{1}{K}\sum_k \mathbf{BC}^k \qquad [3]$$

where $\mathbf{BC}^k$ is the matrix associated with the kth issue, considered in isolation. The evaluation function for the multidimensional case is, therefore, a natural and intuitive extension of the single-dimensional case.

We have assumed so far that all dimensions are equally important. Hence, they are randomly selected with the same probability. When this is not the case, different dimensions could be selected with different probabilities, and each dimension will have to be weighted by a particular coefficient α_k. That is,

$$\mathbf{BC}^* = \sum_k \alpha_k \mathbf{BC}^k \qquad [3']$$

with $\alpha_1 + \alpha_2 + \cdots + \alpha_K = 1$.

We now illustrate the multidimensional case with a health example:

Example 3: Alternative treatments for Relapsing-remitting Multiple Sclerosis (RRMS)

On December 2015, a group of six European clinical neurologists confronted the traditional drug for treating Multiple Sclerosis, *Cladribine* (C), with five new drugs: Dimethyl fumarate (D), Natalizumab (N), Alemtuzumab (A), Terifunomide (T), and Fyngolimod (F). The comparison was based on a model initially constructed by the Merck KGaA staff regarding the benefit-safety balance of the different drugs (Vermerch et al., 2019).

The experts considered seven possible benefits (1: Relapse rate; 2: Reductions in T2 lesions; 3: Reductions in T1 Gd lesions; 3: Reductions in EDSS 3 months; 4: Reduction in EDSS 6 months; 5: Ease of use; 7: Durability), as well as 11 unfavorable effects (1: AR infections; 2: AR GI effects; 3: Liver functions; 4: Malignancy; 5: Autoimmune disease; 6: Lymphopenia; 7: AV block; 8: Bradycardia; 9: Serious infection; 10: Herpetic Infections; 11: PML). They also determined that the relative importance of Benefits with respect to Unfavorable effects should be 60/40. Table 6 presents the weights and rankings of the drugs for the different effects.

Table 6: Benefits, Adverse effects, weights, and order of the drugs.

Benefits	**Weights**	**Best**					**Worst**
1	9.1	A	N	C	F	D	T
2	8.1	A	D	N	F	C	T
3	7	D	NA	C	F	T	
4	8.6	A	N	D	C	T	F
5	10.1	A	N	C	F	T	D
6	4	C	T	D	F	N	A
7	2.5	AD	N	CFT			
Safety	**Weights**	**Worst**					**Best**
1	2.8	N	A	F	T	D	C
2	3	A	T	D	F	C	N
3	2.5	T	F	D	C	NA	
4	4	F	D	CA	N	T	
5	6	A	C	DNFT			
6	6.5	A	F	C	D	NT	
7	2	F	DCANT				
8	1.5	F	A	C	DNT		

9	7	A	C	N	T	D	F
10	5	A	F	N	C	T	D
11	10.1	N	F	D	CAT		

Source: Vermerch et al., 2019

Tables 7 and 8 present the BC matrix corresponding to Benefits and Adverse effects, respectively.

Table 7: BC matrix of Benefits

	C	D	N	A	T	F
C	215.25	41.9	44.15	45.4	38.4	45.4
D	7.5	171.5	42.9	45.4	30.3	45.4
N	5.25	6.5	128.15	41.3	25.7	49.4
A	4	4	8.1	72.1	20.45	35.55
T	11	19.1	23.7	28.95	116.8	34.05
F	4	4	0	13.85	15.35	37.2

Table 8: BC Matrix of Adverse effects

	C	D	N	A	T	F
C	151.9	27	29.65	32.8	25.05	37.4
D	23.4	133.9	18.9	39.85	14.35	37.4
N	20.75	31.5	128.5	35.25	13.5	27.5
A	17.6	10.55	15.15	71.95	8.55	20.1
T	25.35	36.05	36.9	41.85	175.05	34.9
F	13	13	22.9	30.3	15.5	94.7

Finally, the BC matrix of this problem, given in Table 9, is six times the benefits matrix plus four times the adverse effects matrix. The last column gives the valuation of the drugs, setting 100 for the standard treatment.

Table 9: BC matrix of the drugs problem

	C	D	N	A	T	F	BC valuation
C	1899.1	359.4	383.5	403.6	330.6	422	100
D	138,6	1564.6	333	431.8	239.2	422	190.48
N	114.5	165	1282.9	388.8	208.2	406.4	64.97
A	94.4	66.2	109.2	720.4	156.9	293.7	29.9
T	167.4	258.8	289.8	341.1	1401	343.9	87.67
F	76	76	91.6	204.3	154.1	602	25.21

As a consequence, the best drug is Dymethil fumarate. Yet, this does not automatically imply that it should substitute Cladribine as a new standard because one has to take into account the related costs. The cardinality of the evaluation permits applying cost-benefit analysis techniques to make a decision. From those values, it follows that as long as the cost of D does not exceed 1.9 times the cost of C, the substitution pays.

6. Back to basics: voting

The BC rule is an evaluation protocol that transforms rankings into cardinal evaluations by making use of both the Condorcet numbers and the Borda scores, which can be applied under very general circumstances. The cardinality feature makes the BC rule close to Borda's method, even though the BC rule is not a scoring rule. The BC rule derives from the application of a Markov chain to a competitive process based on pairwise comparisons that are performed forever. From this viewpoint, the procedure is close to Condorcet's idea. The novelty of the BC rule is that it links rather precisely the evaluation process to the principles of Borda and Condorcet and that it can be applied to situations involving indifferences, incomplete rankings, multiple issues, and different "divisions."

One may wonder what the BC-rule implies in terms of voting problems, which was at the basis of the initial discussion between those authors. As in the case of Borda and Condorcet, this is a voting procedure that uses all the information on the ranking of the alternatives and not only on the preferred options (see Dasgupta & Maskin 2020). The use of that information was the motivation of Borda and Condorcet to propose alternatives to plurality voting. Rank-voting and score-voting are some of those enriched voting procedures that are actually implemented, some of them enjoying very good properties, as range voting or Kemeny voting (see Kemeny 1959, Maskin 2018).

From this point of view, the BC rule is a procedure that: (a) values the candidates using all the information of the rankings of the voters; (b) takes into account both how much support each candidate obtains (as in Borda); and (c) also takes into account how many individuals support the candidate (as in Condorcet). Indeed, the Borda-Condorcet matrix corresponds to the sum of the vote matrix (Young, 1994), a matrix with the Condorcet numbers in the off-diagonal elements and zeroes in the diagonal, and a diagonal matrix with the Borda scores. The outcome of this voting process gives to each candidate a value that is proportional to the probability of beating any other candidates in a random matching.

A related problem is the allocation of seats in a Parliament to different parties, again based on citizens' preferences. When individuals can only submit their top option, the percentage of votes each party obtains becomes the basis for parties' representation. Using richer scenarios, namely, allowing citizens to submit their preferences in full, would allow them to obtain a political representation more in line with the true preferences of the citizens. Here, the BC rule offers an interesting solution that always exists.

Consider the following example that illustrates how different voting procedures allocate parliamentary seats between parties.

Example 4: Allocating parliamentary seats in an election

There are four parties, A, B, C, and D, competing in an election in which to allocate 135 parliamentary seats. Table 10 describes the voters' preferences, ordering parties from top to bottom (note that there is no Condorcet winner here).

Table 10: Voters ranking of four parties

30%	40%	30%
A	D,C	B,A
B,C	B	D,C
D	A	

Table 11 shows the results concerning the distribution of the seats for different voting procedures: plurality, approval voting (assuming that citizens only approve their best-preferred options), Borda, and the BC rule. Apportionment was done using the Hamilton method (Young, 1994). The BC matrix appears in Table 12.

Table 11: Allocation of seats in Parliament according to different voting schemes

	Plurality	**Approval**	**Borda**	**BC rule**
Party A	61	62	37	39
Party B	20	31	36	37
Party C	27	21	36	36
Party D	27	21	26	23

Table 12: BC matrix of Example 3

	A	B	C	D
A	165	45	60	60
B	55	160	45	60

C	40	55	160	65
D	40	40	35	115

The interest of the BC rule in this context is twofold. On the one hand, it uses the information on the voters' preferences in full (contrary to plurality or approval voting). On the other hand, obtains endogenously the cardinality required to assign the seats (as opposed to score voting rules). Notice that no Condorcet consistent rule (as it happens with the Kemeny rule) provides with a cardinal valuation of the relative support of the parties, coming only from ordinal information on the voters' preferences.

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APPENDIX:
The BC rule is neither Condorcet consistent nor a scoring rule

Let $A = \{a, b, c, d\}$, and take the profiles R and R' described in Table A.1. Profile R involves 21 individuals whose rankings are described by columns, grouping all those individuals with identical rankings. Profile R' involves 100 individuals, and their valuations are described similarly.

Table A.1

Profile R				Profile R'		
3	**5**	**7**	**6**	**70**	**25**	**5**
a	*a*	*b*	*c*	*a*	*b*	*c*
b	*c*	*d*	*b*	*b*	*c*	*b*
c	*b*	*c*	*d*	*c*	*d*	*a*
d	*d*	*a*	*a*	*d*	*a*	*b*

For profile R, the Condorcet ranking is: $c > b > d > a$. The Borda ranking is $b > c > a > d$, which in this case coincides with the BC ranking.

For profile R', alternative a is the Condorcet winner, whereas the Borda count yields the ranking: $b > a > c > d$. The BC order is $a > b > c > d$, which selects as the best option the Condorcet winner.

It might be tempting to think that the BC rule provides an endogenous scoring system for the alternatives so that it can be regarded as a member of the family of scoring rules that extend the Borda criterion (see Young 1975). This is not the case: the BC evaluation function is not a member of the family of scoring rules, as shown in the next example.

Consider a problem involving three alternatives, $\{a, b, c\}$, and suppose that there exists a scoring system that provides the same order as our procedure. As there are only three alternatives, it suffices to consider two scores, $0 \leq \alpha \leq \beta$, with $\beta > 0$ (the implicit assumption is that the third score is equal to 0). Consider the profiles R, R' given in Table A.2.

Table A.2

Profile R							Profile R'					
30	**3**	**27**	**10**	**10**	**1**		**6**	**5**	**6**	**2**	**6**	**3**
a	*a*	*b*	*b*	*c*	*c*		*a*	*a*	*b*	*c*	*c*	*c*
b	*c*	*a*	*c*	*a*	*b*		*b*	*c*	*c*	*a*	*a*	*b*
c	*b*	*c*	*a*	*b*	*a*		*c*	*b*	*a*	*b*	*b*	*a*

For profile R, the BC order is $a > b > c$. To get this order out of some scores, we need $\beta < \left(\frac{3}{2}\right)\alpha$. The BC order for profile R' is $b > a > c$. To get this order out of some scores, $\beta > \left(\frac{3}{2}\right)\alpha$, which contradicts the above requirement. That is, there is no scoring system compatible with the evaluation provided by the BC rule.